Article

The Zero-mass Renormalization Group Differential Equations and Limit Cycles in Non-smooth Initial Value Problems

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Abstract

In the present paper, using the equation transform in fractal space, we point out the zero-mass renormalization group equations. Under limit cycles in the non-smooth initial value, we devote to the analytical technique of the local fractional Fourier series for treating zero-mass renormalization group equations, and investigate local fractional Fourier series solutions.

Keywords: Zero-mass, renormalization group equations, non-smooth initial value, limit cycles, local fractional Fourier series.

1. Introduction

To study the photon propagator, Gell-Mann and Low used a method which was since become known as the renormalization group approach [1], and the renormalization group theory (RGT) was invented by Stueckelberg and Petermann [2]. The RGT was an elegant mathematical expression. The Gell-Mann-Low equation was known as the β-function equation, or the RG differential equation [4-5]. Wilson suggested RGT determine the renormalized coupling constants of strong interactions, the zero-mass RG differential equation was given [3]

$$\frac{dx}{dt}=\psi(0,x) \tag{1.1}$$

and the two coupled zero-mass RG differential equations with time-independent forces arrived at the expression[3]

$$\begin{cases}\dfrac{dx}{dt}=\psi_1(x,y)\\ \dfrac{dy}{dt}=\psi_2(x,y)\end{cases} \tag{1.2}$$

where x and y are the momentum-dependent coupling constants. The RG differential equations have been reviewed and discussed in[6-9].

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A limit cycle is an intriguing alternative to a fixed point, that is, suppose that there are at least two renormalized coupling constants in strong interactions, there is an intriguing alternative to a fixed point [10-13]. A limit cycle is a renormalization trajectory which is a closed orbit and functions x and y satisfies [3]

$$x(t+\tau)=x(t) \tag{1.3}$$

and

$$y(t+\tau)=y(t) \tag{1.4}$$

where τ is a constant giving the period of the limit cycle, and where x and y are continuous functions.

However, the above results are under smooth initial values. As is well known, fractal curves are everywhere continuous but nowhere differentiable, and we cannot employ the convenient calculus to describe the motions in fractal time–space [14-17]. Local fractional calculus, which was revealed as one of useful tools to deal with everywhere continuous but nowhere differentiable functions in areas ranging from fundamental science to engineering in fractal space, was successfully applied in the local fractional Laplace and Fourier problems [16-20], local fractional Fourier series [16, 17, 21], local fractional short time transform [16, 17], local fractional wavelet transform [16, 17], fractal signal [20], fast Yang-Fourier transform [22].

In this paper our aim is to investigate the zero-mass renormalization group differential equations under limit cycles in non-smooth initial value using the local fractional Fourier series in fractal space. This paper is organized as follows. In Section 2, we investigate the fundamentals of local fractional calculus and local fractional Fourier series. In Section 3, we present the equation transforms in fractal space. In section 4, we study the expression solution with Mittag-Leffler functions in fractal space. In section 5, we discuss the fractal characteristics of limit cycles. Finally, section 6 is conclusions.

2. Preliminaries

2.1 Local fractional continuity

Definition 1 If there exists [16, 17]

$$\left|f(x)-f(x_0)\right|<\varepsilon^{\alpha} \tag{2.1}$$

with $\left|x-x_0\right|<\delta$,for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in$, now $f(x)$ is called local fractional continuous at $x=x_0$, denote by $\lim_{x\to x_0} f(x)=f(x_0)$. Then $f(x)$ is called local fractional continuous on the interval (a,b), denoted by

$$f(x)\in C_{\alpha}(a,b), \tag{2.2}$$

where α is fractal dimension with $0<\alpha\le 1$.

Definition 2 A function $f(x)$ is called a non-differentiable function of exponent α, $0<\alpha\le 1$, which satisfies Hölder function of exponent α, then for $x,y\in X$ such that [16, 17,21-21]

$$\left|f(x)-f(y)\right|\le C\left|x-y\right|^{\alpha}. \tag{2.3}$$

Definition 3 A function $f(x)$ is called to be continuous of order α, $0<\alpha\le 1$, or shortly α continuous, when we have that [16, 17, 21]

$$f(x)-f(x_0)=o\left((x-x_0)^{\alpha}\right). \tag{2.4}$$

Remark 1. Compared with (2.4), (2.1) is standard definition of local fractional continuity. Here (2.3) is unified local fractional continuity.

2.2 Local fractional calculus

Definition 4 Let $f(x)\in C_{\alpha}(a,b)$. Local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined as [16-21]

$$f^{(\alpha)}(x_0)=\frac{d^{\alpha}f(x)}{dx^{\alpha}}\Big|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{(x-x_0)^{\alpha}}, \tag{2.5}$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(1+\alpha)\Delta\left(f(x)-f(x_0)\right)$. For any $x\in(a,b)$, there exists

$$f^{(\alpha)}(x)=D_x^{(\alpha)}f(x),$$

denoted by

$$f(x)\in D_x^{(\alpha)}(a,b).$$

Remark 2. The following rules are valid [16-17]:

(1) $\dfrac{d^{\alpha}x^{k\alpha}}{dx^{\alpha}}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}$;

(2) $\dfrac{d^{\alpha}E_{\alpha}\left(kx^{\alpha}\right)}{dx^{\alpha}}=kE_{\alpha}\left(kx^{\alpha}\right), k$ is a constant.

Remark 3. If $y(x)=(f\circ u)(x)$ where $u(x)=g(x)$, then we have [17]

$$\frac{d^{\alpha}y(x)}{dx^{\alpha}}=f^{(\alpha)}(g(x))\left(g^{(1)}(x)\right)^{\alpha}.$$

where there are $f^{(\alpha)}(g(x))$ and $g^{(1)}(x)$.

If $y(x)=(f\circ u)(x)$ where $u(x)=g(x)$, then

$$\frac{d^{\alpha}y(x)}{dx^{\alpha}}=f^{(1)}(g(x))g^{(\alpha)}(x) \tag{2.7}$$

where there exist $f^{(1)}(g(x))$ and $g^{(\alpha)}(x)$.

Definition 5 Let $f(x)\in C_{\alpha}(a,b)$. Local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is given [16-21, 32-36]

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha , \tag{2.8} $$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, ...\}$ and $[t_j, t_{j+1}]$, $j = 0,..., N-1$, $t_0 = a, t_N = b$, is a partition of the interval $[a,b]$. For convenience, we assume that

$$ {}_aI_a^{(\alpha)} f(x) = 0 \ \text{ if } \ a = b \ \text{ and } \ {}_aI_b^{(\alpha)} f(x) = -{}_bI_a^{(\alpha)} f(x) \ \text{ if } \ a < b . $$

For any $x \in (a,b)$, we get

$$ {}_aI_x^{(\alpha)} f(x), $$

denoted by

$$ f(x) \in I_x^{(\alpha)}(a,b). $$

Remark 4. If $f(x) \in D_x^{(\alpha)}(a,b)$, or $I_x^{(\alpha)}(a,b)$, we have that

$$ f(x) \in C_\alpha(a,b). $$

Remark 5. The following relations are valid [16-17]:

$$ \frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(x^\alpha)(dx)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha); \tag{2.9} $$

$$ \frac{1}{\Gamma(1+\alpha)} \int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha} \right); \tag{2.10} $$

2.3 Fractional-order complex mathematics reviews

Definition 6 Fractional-order complex number is defined by [16-17]

$$ I^\alpha = x^\alpha + i^\alpha y^\alpha, \ x, y \in \ , 0 < \alpha \le 1, \tag{2.11} $$

where its conjugate of complex number shows that

$$ \overline{I^\alpha} = x^\alpha - i^\alpha y^\alpha , \tag{2.22} $$

and where the fractional modulus is derived as

$$ \left| I^\alpha \right| = I^\alpha \overline{I^\alpha} = \overline{I^\alpha} I^\alpha = \sqrt{x^{2\alpha} + y^{2\alpha}} . \tag{2.23} $$

Definition 7 Complex Mittag-Leffler function in fractal space is defined by[16-17]

$$ E_\alpha(z^\alpha) := \sum_{k=0}^{\infty} \frac{z^{\alpha k}}{\Gamma(1+k\alpha)}, \tag{2.24} $$

for $z \in C$ (complex number set) and $0 < \alpha \le 1$.

The following rules hold:

$$ E_\alpha(z_1^\alpha) E_\alpha(z_2^\alpha) = E_\alpha((z_1 + z_2)^\alpha); \tag{2.25} $$

$$ E_\alpha(z_1^\alpha) E_\alpha(-z_2^\alpha) = E_\alpha((z_1 - z_2)^\alpha); \tag{2.26} $$

$$ E_\alpha(i^\alpha z_1^\alpha) E_\alpha(i^\alpha z_2^\alpha) = E_\alpha\left(i^\alpha (z_1^\alpha + z_2^\alpha)^\alpha\right). \tag{2.27} $$

When $z^\alpha = i^\alpha x^\alpha$, the complex Mittag-Leffler function is computed by

$$E_\alpha\left(i^\alpha x^\alpha\right)=\cos_\alpha x^\alpha+i^\alpha \sin_\alpha x^\alpha \tag{2.28}$$

with $\cos_\alpha x^\alpha:=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2\alpha k}}{\Gamma(1+2\alpha k)}$ and $\sin_\alpha x^\alpha:=\sum_{k=0}^{\infty}(-1)^k\frac{x^{\alpha(2k+1)}}{\Gamma\left[1+\alpha(2k+1)\right]}$, for $x\in$ and $0<\alpha\leq 1$, we have that

$$E_\alpha\left(i^\alpha x^\alpha\right)E_\alpha\left(i^\alpha y^\alpha\right)=E_\alpha\left(i^\alpha\left(x+y\right)^\alpha\right) \tag{2.29}$$

and

$$E_\alpha\left(i^\alpha x^\alpha\right)E_\alpha\left(-i^\alpha y^\alpha\right)=E_\alpha\left(i^\alpha\left(x-y\right)^\alpha\right). \tag{2.30}$$

2.4 Local fractional Fourier series with the Mittag-Leffler function in fractal space

Definition 8 Suppose that $f(x)\in C_\alpha(-\infty,\infty)$ and $f(x)$ be l-periodic. For $k\in$, complex generalized Mittag-Leffler form of local fractional Fourier series of $f(x)$ is defined [16-17]

$$f(x)=\sum_{k=-\infty}^{\infty}C_k E_\alpha\left(i^\alpha\left(2\pi\right)^\alpha\left(\frac{kx}{l}\right)^\alpha\right), \tag{2.31}$$

where the local fractional Fourier coefficients is

$$C_k=\frac{1}{l^\alpha}\int_0^l f(x)E_\alpha\left(-i^\alpha\left(2\pi\right)^\alpha\left(\frac{kx}{l}\right)^\alpha\right)(dx)^\alpha \text{ with } k\in\ . \tag{2.32}$$

The above generalized forms of local fractional series are valid and are also derived from the generalized Hilbert space [16-17].

The weights of the Mittag-Leffler functions are written in the form [22]

$$C_k=\frac{\frac{1}{l^\alpha}\int_{t_0}^{l+t_0} f(x)E_\alpha\left(-i^\alpha\left(2\pi\right)^\alpha\left(\frac{kx}{l}\right)^\alpha\right)(dx)^\alpha}{\frac{1}{l^\alpha}\int_{t_0}^{l+t_0} E_\alpha\left(-i^\alpha\left(2\pi\right)^\alpha\left(\frac{kx}{l}\right)^\alpha\right)\overline{E_\alpha\left(-i^\alpha\left(2\pi\right)^\alpha\left(\frac{kx}{l}\right)^\alpha\right)}(dx)^\alpha}. \tag{2.33}$$

Above is generalized to calculate local fractional Fourier series.

3. Fractal complex transform

Recently, the fractional complex transform, which is the technique for convert the fractional derivatives into classical derivatives, derived from Modified Riemann-Liouville Derivative [22-24] was introduced in [25]. This method inspires me to introduce the fractal complex transform in fractal space.

Proposition 1 Suppose that there is a relation

$$\begin{cases} X = \dfrac{(px)^{\alpha}}{\Gamma(1+\alpha)} \\ Y = \dfrac{(qy)^{\alpha}}{\Gamma(1+\alpha)} \end{cases} \tag{3.1}$$

where q and q are constants and $0 < \alpha \le 1$, then there exists an equation transformation pair

$$p^{\alpha}\frac{dU_1(X)}{dX} + q^{\alpha}\frac{dU_2(Y)}{dY} = 0 \quad \Leftrightarrow \quad \frac{d^{\alpha}U_1(x)}{dx^{\alpha}} + \frac{d^{\alpha}U_2(y)}{dy^{\alpha}} = 0 \tag{3.2}$$

where there exist the relations

$$\frac{dU_1(X)}{dX}, \frac{dU_2(X)}{dX}, \quad \frac{d^{\alpha}U_1(x)}{dx^{\alpha}}, \frac{d^{\alpha}U_2(y)}{dy^{\alpha}}.$$

Proof. Take the basic properties of the local fractional derivative, we arrive at the following relations

$$\frac{d^{\alpha}U_1}{dx^{\alpha}} = \frac{dU_1}{dX}\frac{d^{\alpha}X}{dx^{\alpha}} = p^{\alpha}\frac{dU_1}{dX} \tag{3.3}$$

and

$$\frac{d^{\alpha}U_2}{dy^{\alpha}} = \frac{dU_2}{dY}\frac{d^{\alpha}Y}{dy^{\alpha}} = q^{\alpha}\frac{dU_2}{dY} \tag{3.4}$$

where $\dfrac{d^{\alpha}X}{dx^{\alpha}} = p^{\alpha}$ and $\dfrac{d^{\alpha}Y}{dx^{\alpha}} = q^{\alpha}$.

Using the above relations, we come to the expression

$$p^{\alpha}\frac{dU_1}{dX} + q^{\alpha}\frac{dU_2}{dY} = 0. \tag{3.5}$$

As a direct result, we have the result as follows:

Proposition 2 Let

$$\begin{cases} X = \dfrac{(qx)^{\alpha}}{\Gamma(1+\alpha)} \\ Y = \dfrac{(qy)^{\alpha}}{\Gamma(1+\alpha)} \end{cases}, \tag{3.6}$$

then we have the following equation transformation pair

$$\frac{dU_1(X)}{dX} + \frac{dU_2(Y)}{dY} = 0 \quad \Leftrightarrow \quad \frac{d^{\alpha}U_1(x)}{dx^{\alpha}} + \frac{d^{\alpha}U_2(y)}{dy^{\alpha}} = 0, \tag{3.7}$$

where there exist the relations

$$\frac{dU_1(X)}{dX}, \frac{dU_2(X)}{dX}, \quad \frac{d^{\alpha}U_1(x)}{dx^{\alpha}}, \frac{d^{\alpha}U_2(y)}{dy^{\alpha}}.$$

Remark 4. Using the above equations, we can convert classical derivatives into the local fractional derivatives each other.

4. The zero-mass renormalization group differential equations and limit cycles in non-smooth initial values

Here, we investigate the limit cycles in non-smooth initial values as follows:

$$\left|x(t)-x(t_0)\right|<\varepsilon^{\alpha} \tag{4.1}$$

and

$$\left|y(t)-y(t_0)\right|<\varepsilon^{\alpha}, \tag{4.2}$$

where $x(t)$ and $y(t)$ are local fractional continuous at $t=t_0$.
The given limit cycles in non-smooth initial values satisfy

$$x(t+\tau)=x(t) \tag{4.3}$$

and

$$y(t+\tau)=y(t), \tag{4.4}$$

where τ is a constant giving the period of the limit cycle, and where x and y are local fractional continuous functions.

Take the relation

$$t=\frac{\xi^{\alpha}}{\Gamma(1+\alpha)}. \tag{4.5}$$

Substituting the relations

$$\frac{d^{\alpha}x}{d\xi^{\alpha}}=\frac{dx}{dt} \tag{4.6}$$

and

$$\frac{d^{\alpha}y}{d\xi^{\alpha}}=\frac{dy}{dt} \tag{4.7}$$

into

$$\begin{cases}\dfrac{dx}{dt}=\psi_1(x,y)\\ \dfrac{dy}{dt}=\psi_2(x,y)\end{cases}, \tag{4.8}$$

we arrive at the two coupled zero-mass RG differential equations in fractal space

$$\frac{d^{\alpha}x}{d\xi^{\alpha}}=\psi_1\left(x(\xi),y(\xi)\right)=\psi_1(\xi) \tag{4.9}$$

and

$$\frac{d^{\alpha}y}{d\xi^{\alpha}}=\psi_2\left(x(\xi),y(\xi)\right)=\psi_2(\xi) \tag{4.10}$$

where the limit cycles in non-smooth initial values become

$$x(\xi+\tau)=x(\xi) \tag{4.11}$$

and

$$y(\xi+\tau)=y(\xi). \tag{4.12}$$

Here, we give the local fractional Fourier series of $x(\xi)$, which is written as

$$x(\xi)=\sum_{k=-\infty}^{\infty} C_{k,\xi} E_\alpha\left(i^\alpha (2\pi)^\alpha \left(\frac{k\xi}{\tau}\right)^\alpha\right) \tag{4.13}$$

where

$$C_{k,\xi}=\frac{1}{\tau^\alpha}\int_0^\tau x(\xi) E_\alpha\left(-i^\alpha (2\pi)^\alpha \left(\frac{k\xi}{\tau}\right)^\alpha\right)(d\xi)^\alpha . \tag{4.14}$$

Successively, we derived as

$$\begin{aligned}
&\frac{d^\alpha x}{d\xi^\alpha}\\
&=\frac{d^\alpha}{d\xi^\alpha}\left(\sum_{k=-\infty}^{\infty} C_{k,\xi} E_\alpha\left(i^\alpha (2\pi)^\alpha \left(\frac{k\xi}{\tau}\right)^\alpha\right)\right)\\
&=\sum_{k=-\infty}^{\infty} (2\pi i)^\alpha \left(\frac{k}{\tau}\right)^\alpha C_{k,\xi} E_\alpha\left(i^\alpha (2\pi)^\alpha \left(\frac{k\xi}{\tau}\right)^\alpha\right)\\
&=\psi_1(\xi)
\end{aligned} \tag{4.15}$$

Hence, from (4.15) we arrive at the relation

$$\begin{aligned}
&(2\pi i)^\alpha \left(\frac{k}{\tau}\right)^\alpha C_{k,\xi}\\
&=\frac{\frac{1}{\tau^\alpha}\int_0^\tau \psi_1(\xi) E_\alpha\left(-i^\alpha (2\pi)^\alpha \left(\frac{k\xi}{\tau}\right)^\alpha\right)(d\xi)^\alpha}{\frac{1}{\tau^\alpha}\int_0^\tau E_\alpha\left(-i^\alpha (2\pi)^\alpha \left(\frac{k\xi}{\tau}\right)^\alpha\right)\overline{E_\alpha\left(-i^\alpha (2\pi)^\alpha \left(\frac{k\xi}{\tau}\right)^\alpha\right)}(d\xi)^\alpha}\\
&=\frac{1}{\tau^\alpha}\int_0^\tau \psi_1(\xi) E_\alpha\left(-i^\alpha (2\pi)^\alpha \left(\frac{k\xi}{\tau}\right)^\alpha\right)(d\xi)^\alpha
\end{aligned} \tag{4.16}$$

and therefore

$$C_{k,\xi}=\frac{1}{(2\pi k i)^\alpha}\int_0^\tau \psi_1(\xi) E_\alpha\left(-i^\alpha (2\pi)^\alpha \left(\frac{k\xi}{\tau}\right)^\alpha\right)(d\xi)^\alpha . \tag{4.17}$$

So, we obtain that

$$\begin{aligned}&\frac{1}{\tau^{\alpha}}\int_0^{\tau} x(\xi)E_{\alpha}\left(-i^{\alpha}(2\pi)^{\alpha}\left(\frac{k\xi}{\tau}\right)^{\alpha}\right)(d\xi)^{\alpha}\\&=\frac{1}{(2\pi ki)^{\alpha}}\int_0^{\tau}\psi_1(\xi)E_{\alpha}\left(-i^{\alpha}(2\pi)^{\alpha}\left(\frac{k\xi}{\tau}\right)^{\alpha}\right)(d\xi)^{\alpha}\end{aligned} \tag{4.18}$$

and

$$\frac{x(\xi)}{\tau^{\alpha}}=\frac{\psi_1(\xi)}{(2\pi ki)^{\alpha}},\xi\in(0,\tau),k\in Z\setminus 0. \tag{4.19}$$

From (4.19), we give

$$x(\xi)\left(\frac{2\pi ki}{\tau}\right)^{\alpha}=\psi_1(\xi),\xi\in(0,\tau),k\in Z\setminus 0. \tag{4.20}$$

In like manner, we come to the equality

$$y(\xi)\left(\frac{2\pi ki}{\tau}\right)^{\alpha}=\psi_2(\xi),\xi\in(0,\tau),k\in Z\setminus 0. \tag{4.21}$$

5. Fractal characteristics of limit cycles

Since the function $\psi_1(\xi)$ is local fractional integral, we have a constant $M_1>0$ such that

$$\left|\psi_1(\xi)\right|\le M_1. \tag{5.1}$$

Hence we arrive at the relation

$$x(\xi)\le\left(\frac{\tau}{2\pi ki}\right)^{\alpha}M_1,\xi\in(0,\tau),k\in Z\setminus 0 \tag{5.2}$$

and therefore

$$\left|x(\xi)\right|\le\left(\frac{\tau}{2\pi k}\right)^{\alpha}M_1,\xi\in(0,\tau),k\in Z\setminus 0 \tag{5.3}$$

And further, we come to the equality

$$\left|x(\xi)\right| \quad \tau^{\alpha}M_1 \qquad (5.3)$$

Hence, we obtain fractal dimension of the limit cycle $x(\xi)$ is α when we take into account the fractal dimension [27-31].

From (4-20) we directly deduce that

$$\left|x(\xi)\right|\left(\frac{2\pi|k|}{\tau}\right)^{\alpha}=\left|\psi_1(\xi)\right|,\xi\in(0,\tau),k\in Z\setminus 0 \tag{5.4}$$

and therefore

$$\left|x(\xi)\right| = \left|\psi_1(\xi)\right| \left(\frac{\tau}{2\pi|k|}\right)^{\alpha}, \xi \in (0,\tau), k \in Z \setminus 0 \tag{5.5}$$

where α is fractal dimension when consider the definition of the generalized fractal dimension[31].

In like manner, we get fractal dimension of the limit cycle $x(\xi)$ is α.

6. Conclusions

In this paper, we discuss the zero-mass renormalization group differential equations and fractal characteristics of limit cycles using mathematical technology of local fractional Fourier series, which is derived from local fractional calculus. This focus is relationship between fractal dimension and renormalization group. In addition, we discuss the limit cycles in non-smooth initial value problems by using fractal dimension theory, and directly obtain the realization of renormalization group and fractal dimension.

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