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Exact Vacuum Solutions of Five-Dimensional Bianchi Type-I Space-Time in *f(R)* Theory of Gravity

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Abstract

In the present paper, we have discussed the well-known phenomenon of the universe expansion in the context of $f(R)$ theory of gravity in five dimensional Bianchi type-I space time using the vacuum field equations on the lines of M. Sharif and M. Farasat Shamir (2009). In particular, we have obtained two exact solutions by using the variation law of Hubble parameter. These solutions correspond to two models of the universe in V_5. The first solution $(n \neq 0)$ gives a singular model while the second $(n = 0)$ provides a non-singular model. The physical behaviour of these five dimensional Bianchi type-I space time models has also been discussed. Moreover, the function $f(R)$ of the Ricci scalar has been evaluated for both the models. The work of M. Sharif and M. Farasat Shamir (2009) regarding exact solutions of the four dimensional Bianchi type-I space-time in $f(R)$ theory of gravity emerge as special case of the work carried out in the present paper.

Keywords: *f (R)* theory of gravity, five dimensional, Bianchi type-I, vacuum field equation.

1. Introduction

Issues like the expansion of the universe, dark energy and dark matter are the most interesting questions in modern day cosmology. The cosmological constant is considered as one of the candidate responsible for the dark energy. Modified theories of gravity provide an alternative approach to study the universe. The $f(R)$ theory of gravity is one of the modified theories which is considered most suitable due to cosmologically important $f(R)$ models. The problem of dark matter can also be addressed by using viable $f(R)$ gravity models.

Another problem in general relativity proposed by Einstein is the occurrence of singularity. We observed that this type of singularity in Einstein's theory of general relativity can be removed in the context of modified theories of gravity. Many authors have shown that the

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singularity free cosmological models in modified $f(R)$ gravity [Kanti et al.(1999), Nojiri and Odintsov (2008), Bamba et al.(2008)].

S. N. Pandey (2008) has developed a higher order theory of gravitation based on a Lagrangian density consisting of a polynomial of scalar curvature R to obtain gravitational wave equations conformally flat. Therefore the study of the solutions of the field equations in modified theories of gravity is an important source for the researchers in the field of general relativity to solve the problems like dark energy and singularity.
The most commonly explored exact solutions in $f(R)$ gravity are the spherically symmetric solutions. Recently S. N. Pandey and B. K. Sinha (2009) have studied spherically symmetric metric in the field equations of higher order theory of gravitation. Many authors have studied spherically symmetric space times in $f(R)$ gravity [Multam¨aki and Vilja (2006), Capozziello et al. (2007), Hollenstein and Lobo (2008)].

The concept of cylindrical symmetry can also be used to study the exact solutions of the field equations in $f(R)$ theory of gravity. Some of the authors have studied cylindrically symmetric solutions in metric $f(R)$ theory of gravity [Azadi et al. (2008)].

M. Sharif and M. Farasat Shamir (2009) have explored static plane symmetric vacuum solutions in $f(R)$ theory of gravity using the assumption of constant scalar curvature which may be zero or non-zero. Recently, the accelerating expansion of the universe has been studied by them using four dimensional Bianchi type-I space time and have obtained exact vacuum solutions of Bianchi type-I space time in $f(R)$ theory of gravity using metric approach.

We observed that the work of M. Sharif and M. Farasat Shamir (2009) regarding the accelerating expansion of the Universe can further be extended to higher five dimensional Bianchi type-I space-time and therefore an attempt has been made in the present paper.

The paper is organized as follows: In section 2, we give a brief introduction about the five dimensional field equations in the context of $f(R)$ gravity. Sections-3 is used to find exact vacuum solutions of five dimensional Bianchi type-I space-time and made the analysis of singularity of these solutions. Section-4 and 5 are dealt with five dimensional models of the universe for $n \neq 0$ and $n = 0$ respectively and in the last section-6, we summarize and conclude the results.

2. Five dimensional field equations in $f(R)$ theory of gravity

The $f(R)$ theory of gravity is nothing but the modification of general theory of relativity proposed by Einstein. Therefore, this theory is the generalization of Einstein's general theory of relativity. In the $f(R)$ theory of gravity there are two approaches to obtain the solutions of modified Einstein's field equations. The first approach is known as metric approach and second one is called Palatini formalism. The $f(R)$ gravity theory is modified by replacing R with $f(R)$ in the standard Einstein's Hilbert action and $f(R)$ is a general function of the Ricci scalar. If we consider R in place of $f(R)$ then the action of standard Einstein's Hilbert can be obtained.

In recent years superstring and other field theories provoked great interest among theoretical physicists in studying physics of higher dimension therefore in this direction most recent efforts have been directed at studying theories in which the dimensions of the space-time are greater than (3+1) of the order which we observe. Also many authors have studied higher dimensional space-times in general relativity as well as non-symmetric unified field theory [Adhao (1994), Thengane (2000), Ambatkar (2002) and Jumale (2006), etc.]. With this motivation, in our present paper we have extended the four dimensional work of M. Sharif and M. Farasat Shamir (2009) in $f(R)$ theory of gravity regarding the accelerating universe to higher five dimensional Bianchi type-I space-time. Therefore, in this paper, we propose to solve the Einstein's modified field equations of $f(R)$ theory of gravity using the metric approach with the help of the vacuum field equations in five dimensional Bianchi type-I space-time on the lines of M. Sharif and M. Farasat Shamir (2009). The corresponding field equations of $f(R)$ theory of gravity in V_5 are given by

$$F(R)R_{ij} - \frac{1}{2} f(R)g_{ij} - \nabla_i \nabla_j F(R) + g_{ij} \Box F(R) = kT_{ij}, \quad (i, j = 1,2,3,4,5) \tag{1}$$

where $$F(R) \equiv \frac{df(R)}{dR}, \qquad \Box \equiv \nabla^i \nabla_i \tag{2}$$

with ∇_i the covariant derivative and T_{ij} is the standard matter energy momentum tensor. These are the fourth order partial differential equations in the metric tensor. The fourth order is due to the last two terms on the left hand side of the equation. If we consider $f(R) = R$, these equations of $f(R)$ theory of gravity reduce to the five dimensional field equations of Einstein's general theory of relativity.

The corresponding five dimensional vacuum field equation (1) are given by

$$F(R)R_{ij} - \frac{1}{2} f(R)g_{ij} - \nabla_i \nabla_j F(R) + g_{ij} \Box F(R) = 0, \quad (i, j = 1,2,3,4,5). \tag{3}$$

After contraction of the field equation (1), we get

$$F(R)R - \frac{5}{2} f(R) + 4 \Box F(R) = kT. \tag{4}$$

In vacuum this field equation (4) reduces to

$$F(R)R-\frac{5}{2}f(R)+4\,\square\,F(R)=0\,. \tag{5}$$

This yields a relationship between $f(R)$ and $F(R)$ which can be used to simplify the field equations and to evaluate $f(R)$.

3. Exact vacuum solutions of five dimensional Bianchi type-I space-time

In this section we propose to find exact vacuum solutions of five dimensional Bianchi type-I space time in $f(R)$ gravity.

The line element of the five dimensional Bianchi type-I space time is given by

$$ds^2=dt^2-A^2(t)dx^2-B^2(t)(dy^2)-C^2(t)(dz^2+du^2) \tag{6}$$

where A, B and C are cosmic scale factors. The corresponding Ricci scalar becomes

$$R=-2\left[\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{2\ddot{C}}{C}+\frac{\dot{A}\dot{B}}{AB}+\frac{2\dot{B}\dot{C}}{BC}+\frac{2\dot{A}\dot{C}}{AC}+\frac{\dot{C}^2}{C^2}\right], \tag{7}$$

where dot denotes the derivative with respect to t.

We define the average scale factor a as

$$a=(ABC^2)^{\frac{1}{4}} \tag{8}$$

and the volume scale factor is defined as

$$V=a^4=ABC^2\,. \tag{9}$$

The generalized mean Hubble parameter H is defined by

$$H=\frac{1}{4}\sum_{i=1}^{4}H_i\,, \tag{10}$$

where $H_1=\frac{\dot{A}}{A}$, $H_2=\frac{\dot{B}}{B}$, $H_3=H_4=\frac{\dot{C}}{C}$ are the directional Hubble parameters in the directions of x, y, z and u axis respectively. Using equations (8), (9) and (10) , we obtain

$$H = \frac{1}{4}\frac{\dot{V}}{V} = \frac{1}{4}\sum_{i=1}^{4} H_i = \frac{\dot{a}}{a}. \tag{11}$$

From equation (5) we have

$$f(R) = \frac{2}{5}[4\,\Box F(R) + F(R)R]. \tag{12}$$

Putting this value of $f(R)$ in the vacuum field equations (3), we obtain

$$\frac{F(R)R_{ij} - \nabla_i \nabla_j F(R)}{g_{ij}} = \frac{1}{5}[F(R)R - \Box F(R)]. \tag{13}$$

Since the metric (6) depends only on t, one can view equation (13) as the set of differential equations for $F(t)$, A, B and C. It follows from equation (13) that the combination

$$A_i = \frac{F(R)R_{ii} - \nabla_i \nabla_i F(R)}{g_{ii}}, \tag{14}$$

is independent of the index i and hence $A_i - A_j = 0$ for all i and j. Consequently $A_5 - A_1 = 0$ gives

$$-\frac{\ddot{B}}{B} - \frac{2\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{2\dot{A}\dot{C}}{AC} + \frac{\dot{A}\dot{F}}{AF} - \frac{\ddot{F}}{F} = 0, \tag{15}$$

Also, $A_5 - A_2 = 0$, $A_5 - A_3 = 0$ and $A_5 - A_4 = 0$ give respectively

$$-\frac{\ddot{A}}{A} - \frac{2\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{2\dot{B}\dot{C}}{BC} + \frac{\dot{B}\dot{F}}{BF} - \frac{\ddot{F}}{F} = 0, \tag{16}$$

$$-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + \frac{\dot{C}^2}{C^2} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{F}}{CF} - \frac{\ddot{F}}{F} = 0, \tag{17}$$

$$-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + \frac{\dot{C}^2}{C^2} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{F}}{CF} - \frac{\ddot{F}}{F} = 0. \tag{18}$$

Thus we obtain three independent non-linear differential equations with four unknowns namely A, B, C and F.

Now (15)-16), (16) - (17) and (14)- (17) we get respectively

$$-\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{2\dot{C}}{C}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)+\frac{\dot{F}}{F}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=0, \quad (19)$$

$$-\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}-\frac{\dot{C}^2}{C^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}-\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{F}}{F}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=0, \quad (20)$$

$$-\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}-\frac{\dot{C}^2}{C^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{A}\dot{C}}{AC}-\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{F}}{F}\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)=0. \quad (21)$$

After integration the above equations imply that

$$\frac{B}{A}=d_1\exp[c_1\int\frac{dt}{a^4F}], \quad (22)$$

$$\frac{C}{B}=d_2\exp[c_2\int\frac{dt}{a^4F}], \quad (23)$$

$$\frac{A}{C}=d_3\exp[c_3\int\frac{dt}{a^4F}] \quad (24)$$

where c_1, c_2, c_3 and d_1, d_2, d_3 are constants of integration which satisfied the relation

$$c_1+c_2+2c_3=0 \quad \text{and} \quad d_1d_2d_3=1.$$

From equations (22), (23) and (24), the metric functions are obtained explicitly as

$$A=ap_1\exp[q_1\int\frac{dt}{a^4F}], \quad (25)$$

$$B=ap_2\exp[q_2\int\frac{dt}{a^4F}], \quad (26)$$

$$C=ap_3\exp[q_3\int\frac{dt}{a^4F}] \quad (27)$$

where $p_1=(d_1^{-3}d_2^{-2})^{1/4}, \quad p_2=(d_1d_2^{-2})^{1/4}, \quad p_3=(d_1d_2^{2})^{1/4},$ (28)

and

$$q_1 = -\frac{3c_1 + 2c_2}{4}, \quad q_2 = \frac{c_1 - 2c_2}{4}, \quad q_3 = \frac{c_1 + 2c_2}{4}. \tag{29}$$

We have pointed out that p_1, p_2, p_3 and q_1, q_2, q_3 are related by

$$p_1 p_2 p_3 = 1, \quad q_1 + q_2 + 2q_3 = 0. \tag{30}$$

Now we use the power law assumption to solve the integral part in the above equations as

$$F \,\alpha\, a^m \tag{31}$$

where m is an arbitrary constant.

The equation (31) implies that

$$F = ka^m \tag{32}$$

where k is the constant of proportionality and m is any integer.

The deceleration parameter q in cosmology is the measure of the cosmic acceleration of the universe expansion and is defined as

$$q = -a\ddot{a}/\dot{a}^2. \tag{33}$$

The sign of q plays an important role to identify the behaviour of the universe. The positive deceleration parameter corresponds to a decelerating model while the negative value provides inflation.

The well-known relation between the average Hubble parameter H and average scale factor a given as

$$H = la^{-n} \tag{34}$$

where $l > 0$ and $n \geq 0$.

From equation (11) and (34), we have

$$\dot{a} = la^{1-n} \tag{35}$$

and consequently the deceleration parameter becomes

$$q = n - 1 \tag{36}$$

which is a constant. After integrating equation (35), we have

$$a = (nlt + k_1)^{1/n}, \qquad n \neq 0 \tag{37}$$

and

$$a = k_2 \exp(lt), \qquad n = 0, \tag{38}$$

k_1 and k_2 are constants of integration.

Thus we have two values of the average scale factors which correspond to two different models of the universe.

From the scalar R given in the equation (7), we can check the singularity of the solutions. If we consider $m = -3$ as a special case then from equation (32) we have

$$F = ka^{-3}$$

and

$$R = -\frac{2}{a^2 k^2}[4k^2(\ddot{a}\, a + \frac{3}{2}\dot{a}^2) - 4(q_1 q_3 + q_2 q_3 + \frac{q_1 q_2}{4} + \frac{3}{2} q_3^2)] \tag{40}$$

which shows that singularity occurs at $a = 0$.

4. Five dimensional Model of the Universe when $n \neq 0$

In this section we study the five dimensional model of the universe for $n \neq 0$.

When $n \neq 0$ then we have $a = (nlt + k_1)^{1/n}$ and for $m = -3$ as a special case, F becomes

$$F = k(nlt + k_1)^{-3/n} \tag{41}$$

For this value of F equations (25), (26) and (27), imply that

$$A = p_1 (nlt + k_1)^{1/n} \exp[\frac{q_1 (nlt + k_1)^{\frac{n-1}{n}}}{kl(n-1)}], \qquad n \neq 1 \tag{42}$$

$$B = p_2 (nlt + k_1)^{1/n} \exp[\frac{q_2 (nlt + k_1)^{\frac{n-1}{n}}}{kl(n-1)}], \qquad n \neq 1 \tag{43}$$

$$C = p_3(nlt+k_1)^{1/n}\exp[\frac{q_3(nlt+k_1)^{\frac{n-1}{n}}}{kl(n-1)}], \quad n \neq 1. \tag{44}$$

The directional Hubble parameters $H_i (i = 1,2,3,4)$ take the form

$$H_i = \frac{l}{nlt+k_1} + \frac{q_i}{k(nlt+k_1)^{1/n}}. \tag{45}$$

The mean generalized Hubble parameter becomes

$$H = \frac{l}{nlt+k_1} \tag{46}$$

And the volume scale factor becomes

$$V = (nlt+k_1)^{4/n}. \tag{47}$$

From the equation (12), the function $f(R)$, found as

$$f(R) = \frac{2k}{5}(nlt+k_1)^{-\frac{3}{n}}R + \frac{24}{5}kl^2(n-1)(nlt+k_1)^{-\frac{3}{n}-2}. \tag{48}$$

From equation (40) Ricci scalar R becomes

$$R = -2[2l^2(5-2n)(nlt+k_1)^{-2} - \frac{(q_1q_2+2q_2q_3+2q_1q_3+q_3^2)}{k^2}(nlt+k_1)^{-2/n}], \tag{49}$$

which clearly indicates that $f(R)$ cannot be explicitly written in terms of R. However, by inserting this value of R, $f(R)$ can be written as a function of t, which is true as R depends upon t. For a special case when $n = 1/2$, $f(R)$ turns out to be

$$f(R) = \frac{2k}{5}[\frac{-16l^2 \pm \sqrt{256l^2 + 8(q_1q_2+2q_2q_3+2q_1q_3+q_3^2)R}}{2R}]^{-3}R$$
$$-\frac{12kl^2}{5}[\frac{-16l^2 \pm \sqrt{256l^2 + 8(q_1q_2+2q_2q_3+2q_1q_3+q_3^2)R}}{2R}]^{-4}. \tag{50}$$

This gives $f(R)$ only as a function of R.

5. Five dimensional Model of the Universe when $n = 0$

In this section, we study the five dimensional model of the universe for $n = 0$. For $n = 0$ the average scale factor for the model of the universe is $a = k_2 \exp(lt)$ and hence F becomes

$$F = \frac{k}{k_2^3} \exp(-3lt). \tag{51}$$

For this value of F equations (25), (26) and (27), imply that

$$A = p_1 k_2 \exp(lt) \exp[-\frac{q_1 \exp(-lt)}{klk_2}], \tag{52}$$

$$B = p_2 k_2 \exp(lt) \exp[-\frac{q_2 \exp(-lt)}{klk_2}], \tag{53}$$

$$C = p_3 k_2 \exp(lt) \exp[-\frac{q_3 \exp(-lt)}{klk_2}]. \tag{54}$$

The mean generalized Hubble parameter becomes

$$H = l. \tag{55}$$

And the volume scale factor becomes

$$V = k_2^4 \exp(4lt). \tag{56}$$

From the equation (12), the function $f(R)$, found as

$$f(R) = \frac{2k}{5k_2^3} \exp(-3lt)(R - 12l^2). \tag{57}$$

From equation (40) Ricci scalar R becomes

$$R = -2[10l^2 - \frac{q_1q_2 + 2q_2q_3 + 2q_1q_3 + q_3}{k^2 k_2^2 \exp(2lt)}]. \tag{58}$$

Here we can get the general function $f(R)$ in terms of R

$$f(R) = \frac{2k^4}{5k_2} (\frac{1}{q_1q_2 + 2q_2q_3 + 2q_1q_3 + q_3^2})^{3/2} (R - 12l^2)(R + 20l^2)^{3/2}. \tag{59}$$

which corresponds to the general function $f(R)$

$$f(R) = \sum a_n R^n \,, \tag{60}$$

where n may take the values from negative or positive.

6. Concluding Remark

In this paper we have investigated two exact vacuum solutions of the five dimensional Bianchi type I space time in $f(R)$ theory of gravity by using the variation law of Hubble parameter to discuss the well-known phenomenon of the universe expansion on the lines of M. Sharif and M. Farasat Shamir (2009). These five dimensional solutions correspond to two models of the universe (i.e., $n \neq 0$ and $n = 0$). The first solution gives a singular model with power law expansion and positive deceleration parameter while the second solution gives a non-singular model with exponential expansion and negative deceleration parameter. The functions $f(R)$ are evaluated for both models.

The physical behavior of these five dimensional models is observed as under :

i. For $n \neq 0$ i.e, Singular model of the universe

For this model average scale factor $a = (nlt + k_1)^{1/n}$.

This model has point singularity at $t \equiv t_s = -k_1 / nl$.
The physical parameters H_1, H_2, H_3, H_4 and H are all infinite at this point.

The volume scale factor V vanishes at this point.

The function of the Ricci scalar, $f(R)$, is also infinite.

The metric functions A, B and C vanish at this point of singularity.

In this way we can conclude from these observations that the five dimensional model of the universe starts its expansion with zero volume at $t \equiv t_s$ and it continues to expand for $0 < n < 1$.

ii. For $n = 0$ i.e, non-singular five dimensional model of the universe

For this model average scale factor $a = k_2 \exp(lt)$

This five dimensional model of the universe is non-singular because exponential function is never zero and hence there does not exist any physical singularity for this five dimensional model of the universe.

The physical parameters H_1, H_2, H_3 and H_4 are all finite for all finite values of t.

The mean generalized Hubble parameter H is constant.

The function of the Ricci scalar, $f(R)$, is also finite.

The metric functions A, B and C do not vanish for this model.

The volume scale factor increases exponentially with time which indicates that the five dimensional model of the universe starts its expansion with zero volume from infinite past.

We observed that all the five dimensional results obtained here are more general to that of M. Sharif and M. Farasat Shamir (2009) in four dimension. Therefore, the work of M. Sharif and M. Farasat Shamir (2009) is a particular case of our carried out in the present paper. The work of M. Sharif and M. Farasat Shamir (2009) can be reproduced by reducing the dimension.

Acknowledgement: We are thankful to Professor S. N. Pandey from India for his constant inspiration.

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