Article

Six-dimensional Static Plane Symmetric Vacuum Solutions in f(R) Gravity

D. P. Teltumbade[&], J. K. Jumale[*] & K. D. Thengane[@]

[&]Govertment Science College, Gadchiroli, India
[*]Department of Physics, R. S. Bidkar College,Hinganghat, Wardha, India
[@]Principal, N.S. Science & Arts College, Bhadravati, Dist. Chandrapur, India

Abstract

In the paper [2], we have obtained five dimensional static plane symmetric vacuum solutions of the field equations of modified theory of gravity i.e., *f(R)* theory of gravity using the metric approach and solved the field equations with the assumption of constant scalar curvature which may be zero or non zero on the lines of Sharif and Farasat (2009). In the present paper, we have extended this work to higher six dimensional plane symmetric space-time. Thus here we have obtained three static plane symmetric solutions in V_6 which are corresponding to the well known solutions in Einstein's general theory of relativity. The work of Sharif and Farasat (2009) in V_4 and our earlier five dimensional work explained in [2] can be reproduced from our work carried out in the present paper by reducing the dimensions.

Keywords: static plane symmetric vacuum solutions, modified theory of gravity, gravity, six dimensional plane symmetric space-time.

1. Introduction

The $f(R)$ theory of gravity is the modification of general theory of relativity proposed by Einstein. The study of the solutions in $f(R)$ theory of gravity is an important source of inspiration for research in general theory of relativity regarding the problem of singularity. In the $f(R)$ theory of gravity there are two approaches to find out the solutions of modified Einstein's field equations. The first approach is called metric approach and second one is known as Palatini formalism. The $f(R)$ gravity theory is modified by replacing R with $f(R)$ in the standard Einstein's Hilbert action and $f(R)$ is a general function of the Ricci scalar. If we consider R in place of $f(R)$ then the action of standard Einstein's Hilbert can be obtained.

Many authors have shown keen interest in exploring different issues in $f(R)$ theories of gravity [Weyl, and Eddington in (1919 and 1922), Buchdahl (1970), Sotiriou (2006), S. N. Pandey (2008), etc.]. Recently in the paper [1], Sharif and Farasat Shamir (2009) have solved the field

[*] Correspondence Author: J. K. Jumale, Department of Physics, R. S. Bidkar College, Hinganghat - 442 301 Wardha, India.
E-mail: .jyotsnajumale@yahoo.com

equations of modified $f(R)$ theory of gravity and obtained vacuum solutions of four dimensional static plane symmetric space-time using metric approach with the assumption of constant scalar curvature.

In the recent years there has been considerable interest in theories with higher dimensional space-time. In this direction many researchers have solved higher dimensional Einstein's field equations in general theory of relativity and investigated their wave solutions using different higher dimensional space-times [Wesson (1983,84) and Reddy D.R.K. (1999), Adhao (1994), Thengane (2000), Patil (2003), Khadekar (2002), Ambatkar (2002), Jumale (2006)]. Inspired by work in string theory and other field theories, we have extended the work of M. Sharif and M. Farasat Shamir (2009), to higher five dimensional space-time in the paper [2] and obtained similar exact plane symmetric vacuum solutions. It has been observed that the five dimensional work regarding vacuum solutions of Einstein's modified theory of gravity can further be extended to higher six dimensional space-time to obtain a total of three static plane symmetric vacuum solutions with the assumption of constant scalar curvature which may be zero or non-zero in $f(R)$ gravity and therefore an attempt has been made in the present paper. Thus, in this paper, we propose to solve the Einstein's modified field equations of $f(R)$ theory of gravity using the metric approach with constant scalar curvature in six dimensional space-time. The corresponding field equations of $f(R)$ gravity theory in V_6 are given by

$$F(R)R_{ij} - \frac{1}{2} f(R)g_{ij} - \nabla_i \nabla_j F(R) + g_{ij} \Box F(R) = kT_{ij}, \quad (i, j = 1,2,3,4,5,6) \tag{1}$$

where $F(R) \equiv \dfrac{df(R)}{dR}$, $\quad \Box \equiv \nabla^i \nabla_i$ (2)

with ∇_i the covariant derivative and T_{ij} is the standard matter energy momentum tensor. These are the fourth order partial differential equations in the metric tensor. The fourth order is due to the last two terms on the left hand side of the equation. If we consider $f(R) = R$, these equations of $f(R)$ theory of gravity reduce to the field equations of Einstein's general theory of relativity in V_6.

After contraction of the field equation (1), we get

$$F(R)R - 3f(R) + 5\Box F(R) = kT. \tag{3}$$

In vacuum this field equation (3) reduces to

$$F(R)R - 3f(R) + 5\Box F(R) = 0. \tag{4}$$

This gives a relationship between $f(R)$ and $F(R)$ which can be used to simplify the field equations and to evaluate $f(R)$.

From the equation (4), we observe that any metric with constant scalar curvature is a solution of the contracted equation (4), say $R = R_0$. Therefore, we have

$$F(R_0)R_0 - 3f(R_0) = 0. \quad (5)$$

is called constant curvature condition. Differentiating equation (4) with respect to R , we obtain

$$-2F(R)R' + F'(R)R + 5[\Box F(R)]' = 0 \quad (6)$$

which gives a consistency relation for $F(R)$.
The paper is organized as follows: In the section-2, we have obtained two non-linear differential equations with three unknowns. Section-3 is dealt with the plane symmetric static solutions of the vacuum field equations in $f(R)$ gravity using constant scalar curvature and in the section-4, we summarize and conclude the result.

2. Six dimensional plane symmetric space-time

We consider the six dimensional general static plane symmetric space-time given by

$$ds^2 = A(x)dt^2 - C(x)dx^2 - B(x)(dy^2 + dz^2 + du^2 + dv^2) \quad (7)$$

For the sake of simplicity, we take $C(x) = 1$ therefore, the space-time (7) becomes

$$ds^2 = A(x)dt^2 - dx^2 - B(x)(dy^2 + dz^2 + du^2 + dv^2) . \quad (8)$$

The corresponding Ricci scalar becomes

$$R = \frac{1}{2}\{\frac{2A''}{A} - (\frac{A'}{A})^2 + \frac{4A'B'}{AB} + \frac{8B''}{B} + 2(\frac{B'}{B})^2\} , \quad (9)$$

where prime represents derivative with respect to x . Using equation (4), it follows that

$$f(R) = \frac{1}{3}\{5\Box F(R) + F(R)R\} . \quad (10)$$

Inserting this value of $f(R)$ in the vacuum field equations, we obtain

$$\frac{F(R)R_{ij} - \nabla_i \nabla_j F(R)}{g_{ij}} = \frac{1}{6}[F(R)R - \Box F(R)] . \quad (11)$$

Since the metric (8) depends only on x, we have equation (11) as the set of differential equations for $F(x)$, A and B. Therefore, from equation (11) we consider

$$A_i = \frac{F(R)R_{ii} - \nabla_i \nabla_i F(R)}{g_{ii}}, \tag{12}$$

is independent of the index i and hence $A_i - A_j = 0$ for all i and j. Thus $A_6 - A_1 = 0$ gives

$$F\,[\frac{2A'B'}{AB} + 2(\frac{B'}{B})^2 - \frac{4B''}{B}] + \frac{A'F'}{A} - 2F'' = 0, \tag{13}$$

$A_6 - A_2 = 0,\ \ A_6 - A_3 = 0,\ \ A_6 - A_4 = 0$ and $A_6 - A_5 = 0$ yield

$$F\,[\frac{2A''}{A} - (\frac{A'}{A})^2 + \frac{3A'B'}{AB} - \frac{2B''}{B} - 2(\frac{B'}{B})^2] + \frac{2A'F'}{A} - \frac{2B'F'}{B} = 0\,. \tag{14}$$

In this way we have obtained two non-linear differential equations with three unknowns namely A, B and F. The solution of these equations could not be found straight forwardly. However, we can find a solution using the assumption of constant curvature.

3. Constant Curvature solutions in V_6

For constant curvature solution, we consider $R = R_0$ therefore, we have

$$F'(R_0) = 0 = F''(R_0)\,. \tag{15}$$

With this condition equations (13) and (14) reduce to

$$[\frac{2A'B'}{AB} + 2(\frac{B'}{B})^2 - \frac{4B''}{B}] = 0\,. \tag{16}$$

$$[\frac{2A''}{A} - (\frac{A'}{A})^2 + \frac{3A'B'}{AB} - \frac{2B''}{B} - 2(\frac{B'}{B})^2] = 0\,. \tag{17}$$

Equation (16) and (17) can be solved by the power law assumption, i.e., $A \,\alpha\, x^a$ and $B \,\alpha\, x^b$, where a and b are any real numbers. Thus we use $A = k_1\, x^a$ and $B = k_2\, x^b$, where k_1 and k_2 are constants of proportionality. Therefore, we have

$$a = -\frac{6}{5}, \quad b = \frac{4}{5} \tag{18}$$

and hence the solution becomes

$$ds^2 = k_1 x^{\frac{-6}{5}} dt^2 - dx^2 - k_2 x^{\frac{4}{5}}[dy^2 + dz^2 + du^2 + dv^2]. \tag{19}$$

It can be shown that these values of a and b lead to $R = 0$.

We re-define the parameters, i.e., $\sqrt{k_1}\, t \to T$, $\sqrt{k_2}\, y \to Y$, $\sqrt{k_2}\, z \to Z$ and $\sqrt{k_2}\, u \to U$, $\sqrt{k_2}\, v \to V$ the above metric takes the form

$$ds^2 = x^{\frac{-6}{5}} dT^2 - dX^2 - x^{\frac{4}{5}}[dY^2 + dZ^2 + dU^2 + dV^2] \tag{20}$$

which is exactly the same as the well-known *Taub's metric*.

Now we assume $A(x) = e^{2\mu(x)}$ and $B(x) = e^{2\lambda(x)}$ so that the space-time (8) takes the form

$$ds^2 = e^{2\mu(x)} dt^2 - dx^2 - e^{2\lambda(x)}[dy^2 + dz^2 + du^2 + dv^2]. \tag{21}$$

The corresponding Ricci scalar is given by

$$R = 2\mu'' + 8\lambda'' + 2\mu'^2 + 8\mu'\lambda' + 20\lambda'^2. \tag{22}$$

Using equation (12), $A_6 - A_1 = 0$ gives

$$4[-\lambda'' - \lambda'^2 + 2\mu'\lambda']F + F'\mu' - F'' = 0 \tag{23}$$

and $A_6 - A_2 = 0$, $A_6 - A_3 = 0$, $A_6 - A_4 = 0$ and $A_6 - A_5 = 0$ yield

$$[\mu'' - \lambda'' + \mu'^2 - 4\lambda'^2 + 3\mu'\lambda']F + F'\mu' - F'\lambda' = 0. \tag{24}$$

For constant curvature solutions, the above equations reduce to

$$\lambda'' + \lambda'^2 - \mu'\lambda' = 0. \tag{25}$$

$$[\mu'' - \lambda'' + \mu'^2 - 4\lambda'^2 + 3\mu'\lambda'] = 0. \tag{26}$$

Equation (25) can be written as

$$\lambda'(\frac{\lambda''}{\lambda'}+\lambda'-\mu')=0 \tag{27}$$

which leads to the following two cases :

I: $\lambda'=0$ II : $(\frac{\lambda''}{\lambda'}+\lambda'-\mu')=0$.

Now we solve the field equations for these two cases.

Case I : $\lambda'=0$ which implies after integration with respect to x that

$$\lambda=c\,, \tag{28}$$

where c is an integration constant.

Inserting this value in equation (26) and integrating the resulting equation, we obtain

$$\mu=\ln(hx+he)\,, \tag{29}$$

where h and e are constants of integration. Thus the metric (21) becomes

$$ds^2=(hx+he)^2dt^2-dx^2-e^{2c}[dy^2+dz^2+du^2+dv^2]\,. \tag{30}$$

The corresponding scalar curvature is

$$R=0\,. \tag{31}$$

It is to be noted that the metric (30) is a solution corresponds to the *self-similar* solution of the infinite kind for the parallel dust case.

Case II: $(\frac{\lambda''}{\lambda'}+\lambda'-\mu')=0$ which implies that

$$\mu'=\frac{\lambda''}{\lambda'}+\lambda'\,. \tag{32}$$

After integration with respect to x, we get

$$\mu=\lambda+\ln\lambda'+d. \tag{33}$$

Using the assumption of constant scalar curvature, it follows that

$$R = \frac{2\lambda'''}{\lambda'} + 22\lambda'' + 30\lambda'^2 = \text{constant} \tag{34}$$

which is a third order non-linear differential equation. The general solution of this equation seems to be difficult. However, a special choice, out of a large set of possible solutions is that $\lambda(x) = fx + g$, where f and g are arbitrary constants. Consequently, the metric (21) takes the form

$$ds^2 = e^{2(fx+\bar{g})}dt^2 - dx^2 - e^{2(fx+g)}[dy^2 + dz^2 + du^2 + dv^2] \tag{35}$$

where $\bar{g} = g + \ln f + d$. The corresponding Ricci scalar reduces to

$$R = 30 f^2. \tag{36}$$

Now re-defining $e^{\bar{g}}t \to T$, $e^g y \to Y$, $e^g z \to Z$, $e^g u \to U$ and $e^g v \to V$ it follows that

$$ds^2 = e^{2fx}[dT^2 - dY^2 - dZ^2 - dU^2 - dV^2] - dx^2. \tag{37}$$

This corresponds to the well-known *anti-de Sitter* space-time in general relativity.

4. Concluding Remark

In the paper [1], M. Sharif and M. Farasat Shamir (2009) have solved the field equations of modified $f(R)$ theory of gravity and obtained vacuum solutions of static plane symmetric space-time using metric approach with the assumption of constant scalar curvature in four dimensional space-time. This work regarding vacuum solutions of Einstein's modified theory of gravity has further been extended to five dimensional space-time in our paper refer it to [2].

In this paper, we have extended the study regarding vacuum solutions in $f(R)$ theory of gravity to six dimensional space-time and obtained a total of three static plane symmetric vacuum solutions with the assumption of constant scalar curvature which may be zero or non-zero in $f(R)$ gravity using metric approach.

We observed that out of these three solutions two are exactly similar to *Taub's* solution and *anti deSitter* space time while the third solution corresponds to the *self-similar* solution already available in the literature.

The solution obtained here is more general to that of earlier static plane symmetric vacuum solutions obtained in V_4 and V_5.

It is shown that results of static plane symmetric vacuum solutions obtained in six dimensional plane symmetric space time are similar to those of V_4 and V_5, essentially retaining their mathematical format.

It is pointed out that the work of Sharif and Farasat (2009) in V_4 and our work in V_5 regarding static plane symmetric vacuum solutions emerge as special cases of our work carried out in the present paper.

It has been observed that, plane symmetric solutions in $f(R)$ theory of gravity concerning the non-static plane symmetric space-time and non-vacuum Einstein's modified field equations in $f(R)$ theory of gravity in six dimension can further be investigated.

Acknowledgement: We are thankful to Professor S N Pandey from India for his constant inspiration.

References

[1] Sharif, M. and Shamir, M.F., Plane Symmetric Solutions in $f(R)$ Gravity,(2009).

[2] G P Urkude et. al, Five dimensional static plane symmetric vacuum solutions in $f(R)$ gravity, communicated to EJTP,(2012).

[3] Adhao K S., Some aspects of plane gravitational waves in general relativity and bi-metric relativity. *Ph.D.Thesis Nagpur University, Nagpur*, (1994).

[4] Ambatkar B. G., Some geometrical and physical aspects of H and F_{ij} . *Ph.D. Thesis*,(02)

[5] Buchdahl, H. A., Mon. Not. Roy. Astr. Soc. **150** (1970)1.

[6] Eddington, A. S.. The Mathematical Theory of Relativity (Cambridge University Press, Cambridge, 1923)

[7] Jumale J. K.. Some plane wave solutions of field equations in general relativity" *Ph.D. Thesis, Nagpur University, Nagpur*, (2006).

[8] Khadekar G.S. (2002). arXIV: gr-qc/0211058 v1.

[9] Pandey S.N., Journal & Proceedings of the Royal Society of New South Wales. Vol.141, p. 45-50, (2008).

[10] Patil K D (2003), *Indian Academy of Sciences*, Vol. 60, No. 3, pp. 423-431.

[11] Reddy D.R.K. (1999), *Astrophy.Space.Sci*., 1-5.

[12] Sotiriou, T.P., Class. Quantum Grav. **23** (2006)5117.

[13] Thengane K.D, Plane gravitional waves in general theory of relativity. *Ph.D.Thesis Nagpur University*, *Nagpur*, (2000)

[14] Weyl, H., Ann. Phys. **59** (1919)101.

[15] Wesson P S (1983), *Astro. Astrophys.,* **119**, 145.

[16] Wesson P S (1984), *Gen. Rel. Grav.* **16**, 193