**Article**

# Experimental Observations on the Uncomputability of The Riemann Hypothesis: Part I

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## ABSTRACT

This paper seeks to explore whether the Riemann hypothesis falls into a class of putatively unprovable mathematical conjectures, which arise as a result of unpredictable irregularity. It also seeks to provide an experimental basis to discover some of the mathematical enigmas surrounding these conjectures, by providing Matlab and C programs which the reader can use to explore and better understand these systems (see appendix 6 in Part II). Part I of this article includes: Introduction; The Riemann hypothesis and the Zeta Function; and The Quantum Chaos Connection.

**Key Words:** Riemann Hypothesis, uncomputability, enigma, experimental observation.

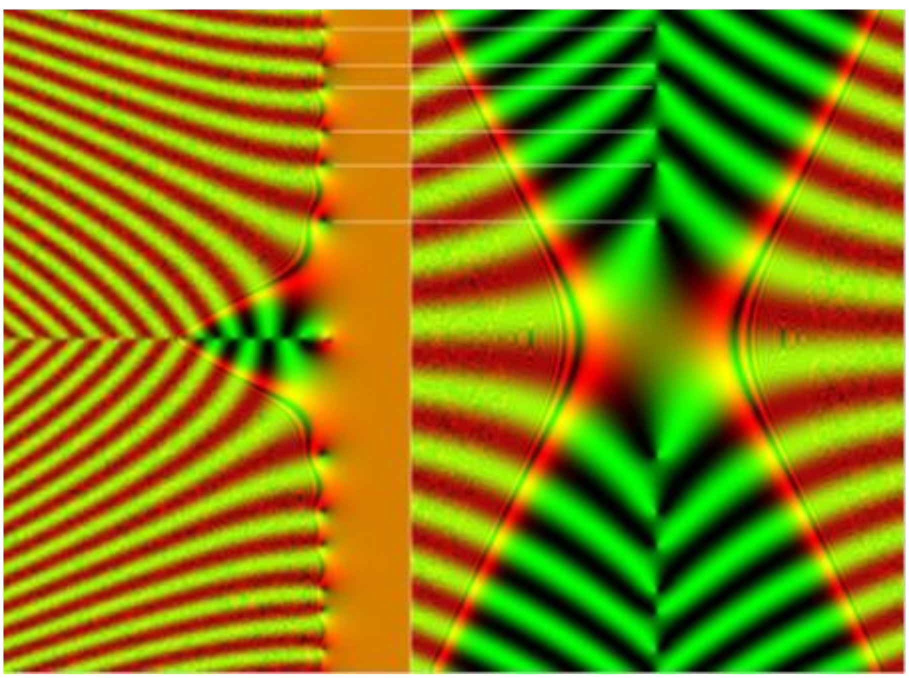

Fig 1: The Riemann functions $\zeta(z)$ and $\xi(z)$: absolute value in red, angle in green. The pole at $z = 1$ and the non-trivial zeros on $x = ½$ showing in $\zeta(z)$ as a peak and dimples. The trivial zeros are at the angle shifts at even integers on the negative real axis. The corresponding zeros of $\xi(z)$ show in the central foci of angle shift with the absolute value and angle reflecting the function's symmetry between $z$ and $1 - z$. If there is an analytic reason why the zeros are on $x = ½$ one would expect it to be a manifest property of the reflective symmetry of $\xi(z)$.

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## Introduction:

The Riemann hypothesis[i,ii] remains the most challenging unsolved problem in mathematics at the beginning of the third millennium. The other two problems of similar status, Fermat's last theorem[iii] and the Poincare conjecture[iv] have both succumbed to solutions by Andrew Wiles and Grigori Perelman in *tours de force* using a swathe of advanced techniques from diverse mathematical areas. Fermat's last theorem states that no three integers *a*, *b*, *c* can satisfy $a^n + b^n = c^n$ for *n* > 2. The Poincare conjecture states that any 3-manifold (a space locally like n-dimensional space, such as a sphere, torus or Klein bottle) on which any loop can be continuously shrunk to a point is a 3-sphere. Both of these, despite being difficult problems, have a justifiable case that a solution ought to exist, and that they are not undecidable propositions.

The anticipation in the mathematical community thus remains largely focused on the notion that the Riemann hypothesis is in-principle a provable proposition, which is also eminently plausible, and indeed confirmed by all numerical instances so far discovered. Nevertheless the problem has resisted all attempts to close in on it from areas at least as diverse as the other two, so one might not be entirely foolish to suggest that there may be mathematical barriers to proving the Riemann hypothesis that are fundamentally different.

This paper comes with a Mac XCode application and source code and a Matlab toolbox downloadable from: http://www.dhushara.com/DarkHeart/

## The Riemann hypothesis and the Zeta Function

"*Mathematicians have tried in vain to discover some order in the sequence of prime numbers but we have every reason to believe that there are some mysteries which the human mind will never penetrate.*" Euler 1751

The Riemann hypothesis states that the Zeta function[v] $\zeta(z) = \sum_{n=1}^{\infty} n^{-z} = \prod_{p\ prime} \left(1 - p^{-z}\right)^{-1}$ [1] has all its non-trivial zeros on the vertical line x = ½ in the complex plane. This like the previous two problems is an intuitively obvious result, which has great plausibility, since billions of its zeros do lie on the line, but is it, if it is logically equivalent to potentially unprovable, or even undecidable propositions about large primes?

The relationship between the zeta sum formula and the product of primes was discovered by Leonhard Euler and is equivalent to prime sieving.

$$\frac{1}{2^s}\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right) = \frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\cdots \text{ so } \left(1-2^{-s}\right)\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right) = \left(1+\frac{1}{3^s}+\frac{1}{5^s}+\cdots\right),$$

$$\frac{1}{3^s}\left(1+\frac{1}{3^s}+\frac{1}{5^s}+\cdots\right) = \frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\cdots$$

$$\text{so } \left(1-3^{-s}\right)\left(1-2^{-s}\right)\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right) = \left(1-3^{-s}\right)\left(1+\frac{1}{3^s}+\frac{1}{5^s}+\cdots\right) = 1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\cdots$$

$$\prod_{p\ \text{prime}} \left(1-p^{-s}\right)\sum_{n=1}^{\infty}\frac{1}{n^s} = 1, \text{ so } \sum_{n=1}^{\infty} n^{-s} = \prod_{p\ \text{prime}} \left(1-p^{-s}\right)^{-1}, \text{ real}(s) > 1$$

Zeta also has relationships with the primes at real integer values of *s*, where the Euler product formula can be used to show that the probability that *s* integers are relatively prime is $1/\zeta(s)$.

While both sides of [1] are convergent only for real(*s*) > 1, using the above construction, we can see that we can extend convergence to real(*s*) > 0 using Dirichlet's Eta[vi]:

$$\eta(s)=\left(1-2^{1-s}\right)\zeta(s)=\sum_{n=1}^{\infty}(-1)^{n+1}n^{-s},\text{ so }\zeta(s)=\left(1-2^{1-s}\right)^{-1}\sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}$$

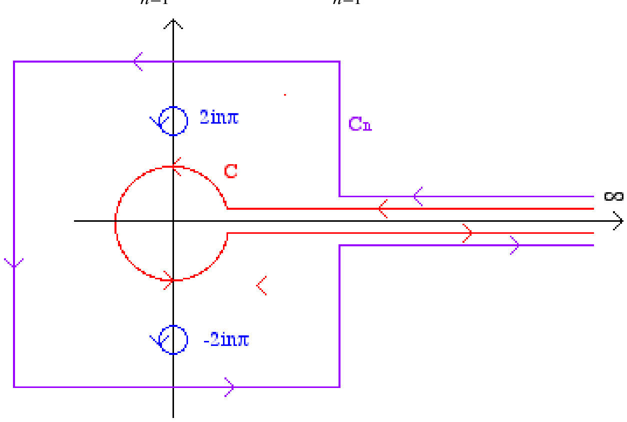
In 1739 Euler gave a bizarre proof which would be frowned on by teachers of modern mathematics, by mischievously playing with seemingly impossible values of the real zeta function: $1+x+x^2+x^3+\cdots=\frac{1}{1-x}$, so differentiating $1+2x+3x^2+4x^3+\cdots=\frac{1}{(1-x)^2}$.

Substituting $x = -1$, he then produced $1-2+3-4+\cdots=\frac{1}{4}$. But this is by definition $\eta(-1)$, so expressing zeta in terms of eta, as above, he got $\zeta(-1)=1+2+3+4+\cdots=\frac{-1}{12}$, which caused Abel to declare "The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever." However over a hundred years later in 1859 Riemann would show that these are precisely the values of eta and zeta realized by analytic continuation. Riemann[vii] analytically extended the zeta function to the entire complex plane, except the simple pole at *z* = 1, by considering the integral definition of the gamma function

We will now give a complete derivation of the analytic continuation of $\zeta(z)$[viii] to make the process as clear as possible. We start with the following integral expression using the gamma function:

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt=n^s\int_0^\infty x^{s-1}e^{-nx}dx,\text{ so }\frac{\Gamma(s)}{n^s}=\int_0^\infty x^{s-1}e^{-nx}dx,$$

$$\text{and }\Gamma(s)\zeta(s)=\Gamma(s)\sum_{n=1}^{\infty}\frac{1}{n^s}=\int_0^\infty x^{s-1}\sum_{n=1}^{\infty}e^{-nx}dx=\int_0^\infty\frac{x^{s-1}}{e^x-1}dx,\ \mathrm{Re}(s)>1$$




Now let $I(s)=\int_C\frac{z^{s-1}}{e^z-1}dz=-\int_0^\infty\frac{x^{s-1}}{e^x-1}dx+\int_0^\infty\frac{(xe^{2\pi i})^{s-1}}{e^x-1}dx=\left(e^{2\pi is}-1\right)\Gamma(s)\zeta(s),\ \mathrm{Re}(s)>1,$

where C is a path from ∞ round a small circle not enclosing any of the poles at $\pm 2\pi ni$ and back to ∞, as the integral round the circular region tends to 0 for Real($s$)>1 as the radius tends to 0.

So $\zeta(s) = \dfrac{e^{-\pi is}\Gamma(1-s)}{2\pi i} I(s)$, and this gives a uniformly convergent integral function of $s$ providing an analytic continuation of $\zeta(s)$ over the entire plane.

If we now take another path enclosing a larger square with corners $(2n+1)\pi(\pm 1 \pm i)$ as shown above we have:

$$I(s) = \int_{C_n} \frac{z^{s-1}}{e^z - 1} dz + 4\pi i e^{\pi is} \sin\left(\frac{\pi s}{2}\right) \sum_{m=1}^{n} (2m\pi)^{s-1}$$

$$\to 0 + 4\pi i e^{\pi is} \sin\left(\frac{\pi s}{2}\right) \sum_{m=1}^{n} (2m\pi)^{s-1} = 4\pi i e^{\pi is} \sin\left(\frac{\pi s}{2}\right) (2\pi)^{s-1} \zeta(1-s)$$

since the outer integral tends to 0 and the residue pairs separating the two contour paths above are

$$(2\text{m}\pi \text{e}^{\pi i/2})^{s-1} + (2\text{m}\pi \text{e}^{3\pi i/2})^{s-1} = (2m\pi)^{s-1} e^{\pi i(s-1)} 2\cos\left(\frac{\pi(s-1)}{2}\right) = -2(2m\pi)^{s-1} e^{\pi is} \sin\left(\frac{\pi s}{2}\right)$$

So $\zeta(s) = \dfrac{e^{-\pi is}\Gamma(1-s)}{2\pi i} 4\pi i e^{\pi is} \sin\left(\dfrac{\pi s}{2}\right)(2\pi)^{s-1}\zeta(1-s) = 2^s \pi^{-1+s} \sin\left(\dfrac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$, with

inverse transformation $\zeta(1-s) = 2^{1-s}\pi^{-s} \sin\left(\dfrac{\pi(1-s)}{2}\right)\Gamma(s)\zeta(s) = 2(2\pi)^{-s} \cos\left(\dfrac{\pi s}{2}\right)\Gamma(s)\zeta(s)$.

We can now derive the other symmetrical forms of the functional equation:

$$\zeta(s) = 2^s \pi^{-1+s} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) = \pi^{s-1/2} \frac{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s),$$

$$\text{since } \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \text{ and } \Gamma(s)\Gamma\left(\frac{1}{2} + s\right) = 2^{1/2-2s} 2^{1/2} \pi^{1/2} \Gamma(2s) = 2^{1-2s} \pi^{1/2} \Gamma(2s)$$

$$\text{and so } \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \Gamma\left(1 - \frac{s}{2}\right) \pi^{-\frac{1-s}{2}} \zeta(1-s)$$

This reflectivity relation then became the basis for Riemann's $\xi$ function, which is symmetrical about x = ½ and whose zeros are identical to those of $\zeta(s)$.

$$\text{Now if } \xi(s) = \Gamma\left(\frac{s}{2} + 1\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{s}{2}\right)\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s), \text{ we get}$$

$$\xi(1-s) = \Gamma\left(\frac{1-s}{2}\right)\frac{1-s}{2}(-s)\pi^{-\frac{1-s}{2}}\zeta(1-s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}(1-s)\frac{s}{2}(s-1)\zeta(s) = \xi(s)$$

thus enabling $\zeta(s)$ to be extended to $\square \setminus \{1\}$ using reflection about the line x = ½ .

The trivial zeros then arise from the zeros of the sine on the negative real axis and the non-trivial from $\zeta(s)$ itself in the critical strip.

By careful contour integration Riemann then established that the irregularly-spaced zeros between 0 and *t* grow according to $\frac{t}{2\pi}\left(\log\frac{t}{2\pi}-1\right)$. More precisely, the number of zeros between 0 and t has been found based on contour integration round the zeros and poles using functional equation and the Cauchy argument principle $\oint_C \frac{f'(z)}{f(z)}dz = 2\pi i(Z-P)$ to be:

$$N(t)=\frac{1}{\pi}\mathrm{Arg}\big(\xi(s)\big)=\frac{1}{\pi}\mathrm{Arg}\left(\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\,\frac{s}{2}(s-1)\zeta(s)\right)\sim\frac{t}{2\pi}\left(\log\frac{t}{2\pi}-1\right),\ s=\frac{1}{2}+it$$

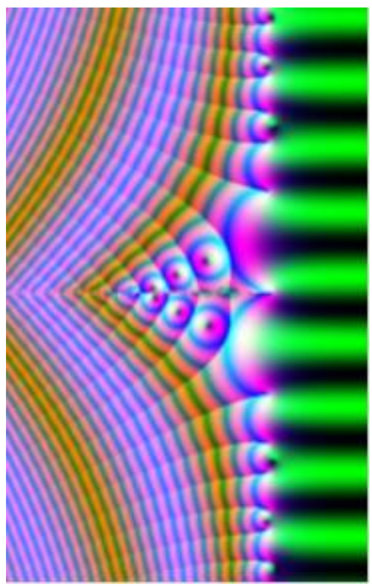


Fig 1b: log $\zeta(s)$ shows all zeta zeros at the end of branch curves.

The argument of the product is the sum of the arguments, but the last term $S(t)=\frac{1}{2\pi}\mathrm{Arg}(\zeta)$, measuring fluctuations about the average, which jumps by 1 at each zero, and declines with derivative $\sim\log(t)$, grows extremely slowly in the average as $\sim(\log(\log(t))^{1/2}$. Numerical calculations confirm that *S* grows very slowly: $|S(T)| < 1$ for $T < 280$, $|S(T)| < 2$ for $T < 6800000$, and the largest value of $|S(T)|$ found so far [ix] is around 3.2. It is thus hard to predict the eventual behavior of the zeros, for extremely large numbers are required before this factor becomes significant and hence the fact that very large computed zeros are on the critical line doesn't necessarily establish the likelihood that all non-trivial zeros are on the line.

By turning to the logs of $\xi(t)$ and $\zeta(s)$ Riemann establishes a formula for the primes:

$$\log\big(\zeta(s)\big) = -\sum \log(1-p^{-s})$$
$$= \sum p^{-s} + \frac{1}{2}\sum p^{-2s} + \frac{1}{3}\sum p^{-3s} + \ldots \; = \sum s\int_p^\infty x^{-s-1}dx + \sum s\int_{p^2}^\infty x^{-s-1}dx + \ldots$$
$$\text{so } \frac{\log\big(\zeta(s)\big)}{s} = \int_1^\infty \big(f(x)\big)x^{-s-1}dx = \int_1^\infty \left(F(x) + \frac{1}{2}F\big(x^{1/2}\big) + \frac{1}{2}F\big(x^{1/3}\big) + \ldots\right)x^{-s-1}dx,$$

where $F(x)$ is the number of primes less than $x$ growing by ½ at primes.

Using a Fourier transform, he establishes that the primes can be counted by forming an infinite series of integrals over the zeros of $\zeta(s)$, remarking in the process his hypothesis that the zeros of $\xi(t)$ were real, or equivalently, those of $\zeta(s)$ were on $x = ½$:

$$\Pi_0(x) = f(x) = \mathrm{Li}(x) - \sum_\rho \mathrm{Li}(x^\rho) + \int_x^\infty \frac{1}{x^2+1}\frac{dx}{x\log x} - \log(2), \text{ where } \mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$$

The first term corresponds to the pole, the second to the non-trivial zeros, the third to the trivial zeros and the last is $\log(\xi(0)) = \log(1/2)$. By the Möbius inversion formula we arrive at:

$$\pi_0(x) = R(x) - \sum_\rho R(x^\rho) + \frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{\ln(x)}\right) - \frac{1}{\ln(x)},\; R(x) = \sum_{n=1}^\infty \frac{\mu(n)}{n}\mathrm{li}(x^{1/n}) = 1 + \sum_{k=1}^\infty \frac{(\ln(x))^k}{k!k\zeta(k+1)},$$

where $\pi_0(x)$ is the prime counting function[x], $\mu(x)$ the Möbius function $(-1)^k$, $n = p_1 \cdot \ldots \cdot p_k$ distinct primes and 0 otherwise (appendix 4). The simplest version of this formula and the one we shall use[xi] is: $\psi(x) = \sum_{n\le x}\Lambda(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{1}{2}\log\big(1-x^{-2}\big) - \log(2\pi)$, where the von Mangoldt function $\Lambda(x) = \log p$ if $x = p^k$ and 0 otherwise (appendix 4), $\rho = 1/2 + it$ are the zeros of $\zeta(s)$, and the summation is performed over zeros of increasing $|t|$.

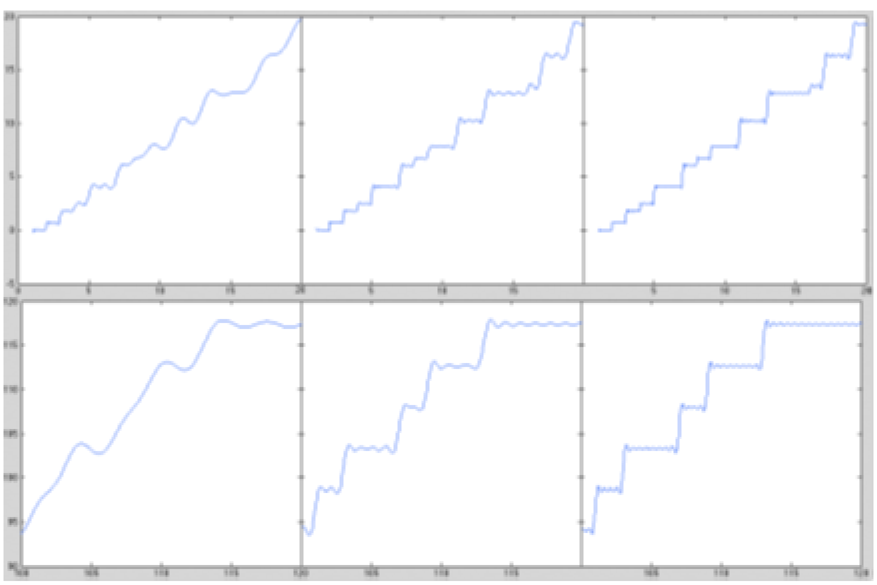


Fig 2: The prime counting function $\psi(x)$ iterated for 10, 40 and 100 zero pairs of $\zeta(z)$ in the range o 1 to 20 and for 100, 400 and 1000 zero pairs in the range 100 to 120 shows evidence of wave superposition of contributions

from each of the successive zeros, and of the increasing number of zeros required to resolve higher primes corresponding to the logarithmically closer spacing of higher zeros.

To derive this[xii], we start with Perron's formula, again based on contour integration:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{su}\frac{ds}{s} = \begin{cases} 0, & u<0 \\ 1/2, & u=0, \\ 1, & u>0 \end{cases} \text{ with } u=\log(x/n), \text{ giving } \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{n}\right)^{s}\frac{ds}{s} = \begin{cases} 0, & u<0 \\ 1/2, & u=0 \\ 1, & u>0 \end{cases}$$

$$\psi(x) = \sum_{\substack{p \text{ prime} \\ m\ge 1,\ p^m \le x}} \log(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{\substack{p \text{ prime} \\ m\ge 1}}\frac{\log(p)}{p^{ms}}\frac{x^s}{s}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{\substack{p \text{ prime} \\ m\ge 1}} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds, \ \mathrm{Re}(s)>1$$

Now if we move the contour away to the left to Re($s$)=-$N$ for a large odd integer $N$ for each zero and pole of $\zeta(z)$ we have order 1 so the values are $-\frac{\zeta'(0)}{\zeta(0)}$ at $s$=0, x at $s$=1, and $-\frac{mx^{\rho}}{\rho}$ for a zero of order $m$. Hence we have $\psi(x) = x - \sum_{\substack{\zeta(\rho)=0 \\ \mathrm{Re}(\rho)>-N}}\frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2\pi i}\int_{-N-i\infty}^{-N+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}ds$

Now the integrand tends to 0 as $N$ tends to infinity, and the zero $\rho=-2m$ contributes $1/\left(2mx^{2m}\right)$, giving a summed weighting of $-\frac{1}{2}\log\left(1-\frac{1}{x^2}\right)$, so we get

$$\psi(x) = x - \sum_{\substack{\zeta(\rho)=0 \\ 0<\mathrm{Re}(\rho)<1}}\frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right)$$

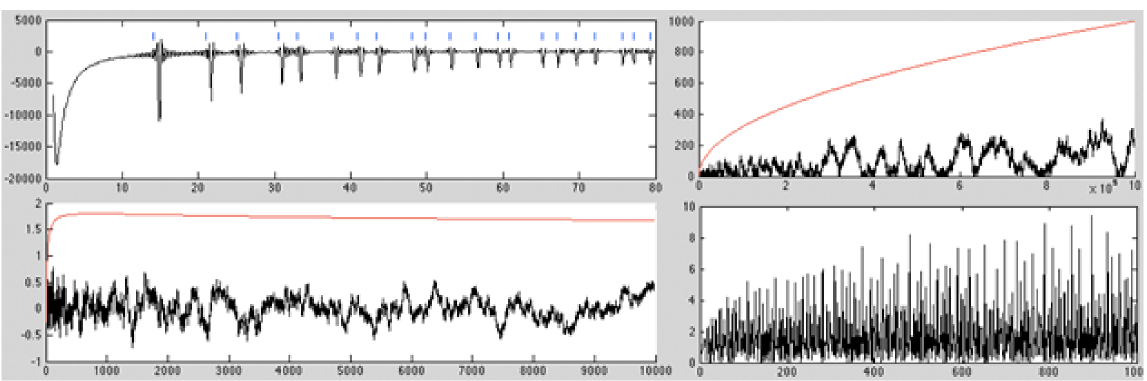


Fig 2b (Top left): Fourier transform of the prime counting function, showing coincidence with zeta zeros (blue above) confirms the zeta zeros are in a sense 'holograms' of the primes . (Bottom left) Function $\Delta(x) = \left(\pi_0(x) - R(x) + \frac{1}{\ln x} - \frac{1}{\pi}\tan^{-1}\frac{\pi}{\ln x}\right)\frac{\ln x}{\sqrt{x}}$ showing the fluctuations of the Riemann estimate about $\sqrt{x}\,/\ln(x)$ compared with the same function substituting Li($x$) for $\pi_0(x)$ (red). RH has been shown to be equivalent to the statement $|\pi(x) - li(x)| < x^{1/2}\log(x)/8\pi$ because the explicit formula shows the magnitude of the oscillations of primes around their expected position is controlled by the real parts of the zeros of the zeta function. Littlewood proved that there exists a value where $\pi(x) > \mathrm{Li}(x)$ without determining what these values would be. Skewes then produced $10^{10^{10^{34}}}$ an unimaginably huge upper bound, of which Hardy said "the truth has defeated not only all the evidence of the facts and of common sense, but even a mathematical imagination as

powerful as that of Gauss" [xiii]. The bound came down more recently thanks to te Riele and later Hudson and Bayes to $10^{310}$. (Top right) A similar conjecture by Mertens that $\mathrm{M}(n)=\sum_{k=1}^{n}\mu(k)<n^{1/2}$ (see Möbius function) which would have proved the Riemann hypothesis was also found false at a value of around $10^{30}$ by Odyzko's colleague Herman te Riele. (Bottom right) Lindelof hypothesis that $|\zeta(1/2+it)|$ grows more slowly that any fixed power of *t* remains unproven despite being 'easier' than RH.

Applying this formula is effectively assuming the Riemann hypothesis, because in performing a calculation, we are working on the basis that all the zeros we are using are on *x* = ½, and since it is the real part of $x^{\rho}=x^{1/2+it}=e^{(1/2+it)\log x}=x^{1/2}(cos(t\log x)+i\sin(t\log x))$ which determines the magnitude of the contribution of each of these terms to the fluctuations in the prime distribution, the prime distribution that emerges is a practical application of the Riemann hypothesis.

The effect of the process is to create an integral transform of the distribution of primes, with the terms involving the non-trivial zeros forming a series of superimposed wave functions very similar to a Fourier series as can be seen from fig 2. This integral transform converges rapidly for the first few primes but more slowly as the primes grow larger.

Looking from the other end of the transform, we can examine how the product formula converges to the zeta zeros as the number of primes in the product increases. There are two ways of doing this. In fig 2b is shown the Fourier sin transform $F(t)=\frac{1}{\ln X}\int_{0}^{\ln X}\sin(ty)\frac{\psi(e^{y})-e^{y}}{e^{y/2}}dy$ of the prime counting function $\psi(x)$ of fig 2, showing coincidence between the major fluctuations of the transform and the zeta zeros.

Let us examine how zeros with real part larger or smaller than ½ would affect the explicit formula for the prime distribution.

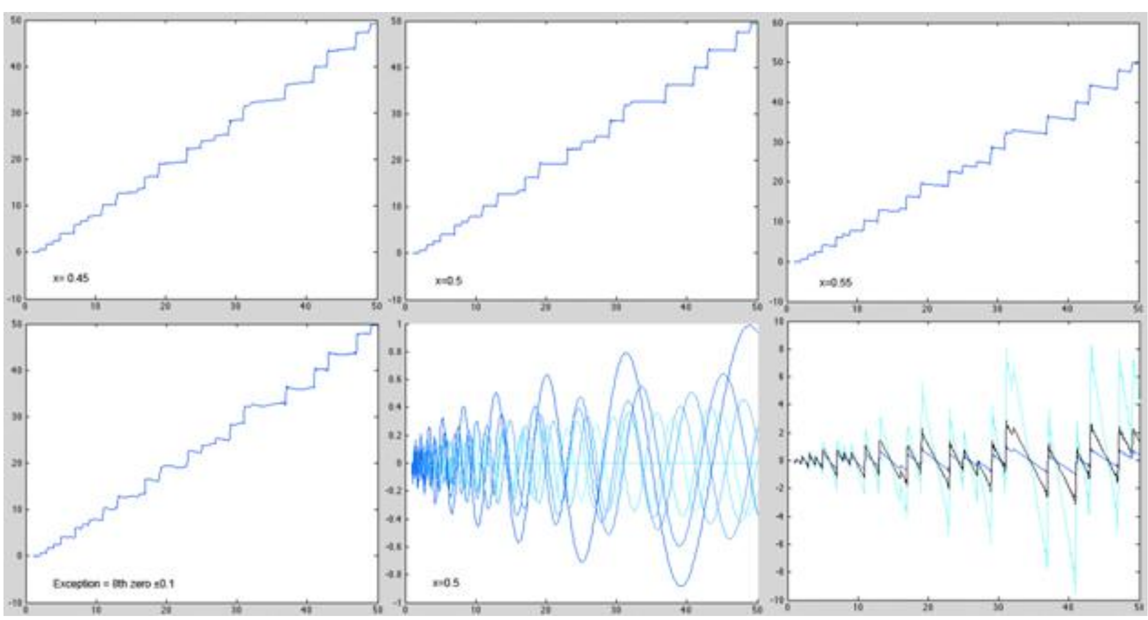


Fig 2c: The explicit formula gives a strong indication of why all the zeros may have to be on the critical line. When they are, (centre), the individual contributions of pairs $x^\rho / \rho$ result in quadratically increasing amplitudes of the oscillating wave function (below, summed lower right black). When these are off the line (above left and right) the prime distribution is no longer constant between primes because the summed $x^\rho / \rho$ pairs have the wrong trends in amplitude. This effect remains pronounced even if only one zero is moved off the critical line to form a pair ±0.1 off consistent with the symmetry about the critical line of $\xi(z)$. (Lower right) Shifting the real parts alters the amplitude trend of successive prime fluctuations in the summed terms (0.2 blue, 0.5 black, 0.8 cyan).

When we examine the formula for the summed wave functions:

$$x^\rho = e^{(p+iq)\ln(x)} = x^p(\cos(\ln(x)q) + i\sin(\ln(x)q)), \text{ so } x^\rho + x^{\bar\rho} = 2x^p\cos(\ln(x)q)$$

we can see that , the increased amplitude of the summed $x^\rho$ terms at primes as shown in fig 3 depends only in the imaginary values of the zeros, as is confirmed from the formula below for the summed conjugate pairs $x^\rho + x^{\bar\rho}$.

By contrast, in the terms in the explicit formula:

$$\frac{x^\rho}{\rho} = \frac{e^{(p+iq)\ln(x)}}{p+iq} = \frac{x^p(\cos(q\ln x) + i\sin(q\ln x))}{p+iq} = \frac{x^p}{p^2+q^2}(\cos(q\ln x) + i\sin(q\ln x))(p-iq)$$

$$\text{so } \frac{x^\rho}{\rho} + \frac{x^{\bar\rho}}{\bar\rho} = 2\,\text{real}\left(\frac{x^\rho}{\rho}\right) = 2\frac{x^p}{p^2+q^2}(p\cos(q\ln x)) + q\sin(q\ln x)) \sim 2\frac{x^p}{p^2+q^2}q\sin(\ln(x)q)$$

the leading term $x^p$, which is inverse quadratic for $x$=1/2 is pivotal in the sum of the wave functions remaining constant on intervals in the zeta prime distribution. Explaining qualitatively why this is so, would pertain, not just to the first few zeros, but to **all** zeros, solving RH.

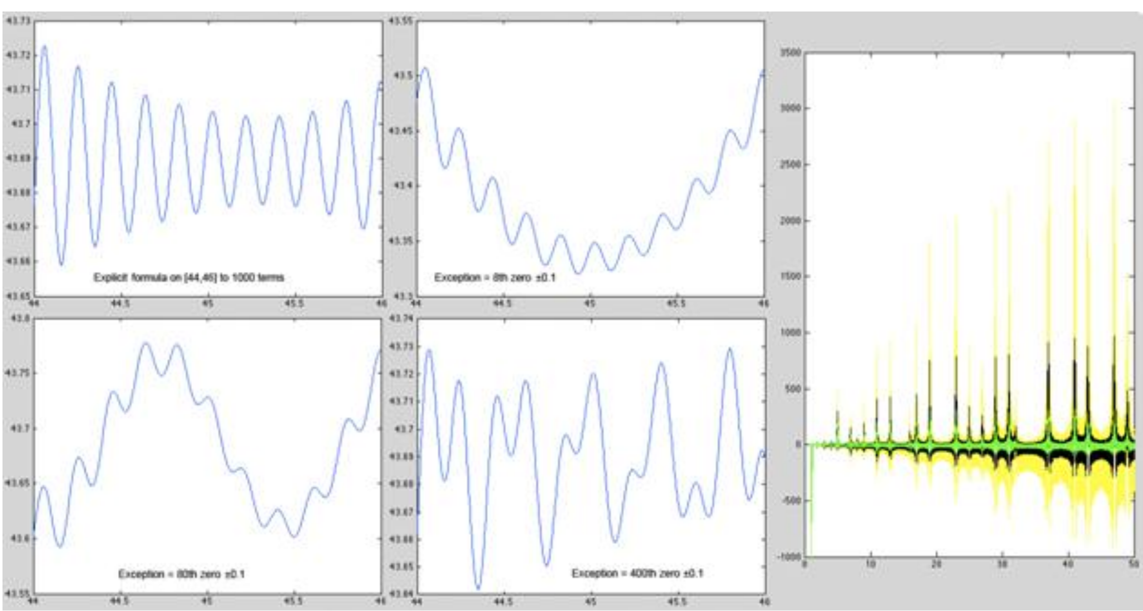


Fig 2d: (Right) When the terms $x^{\rho}$ are summed, they have bursts of amplitude at the primes even when their real parts are varied significantly. This affects their relative amplitudes at each x but not their positions at the primes and prime powers. This process for $x$ = ½ forma a partial duality to the Fourier transform in fig 2b. (Left) However when even a single zero is moved off the critical line to form a symmetric pair consistent with the symmetry of $\xi(z)$, the convergence to a constant function on non-primes is disrupted by a rogue amplitude. The effect remains pronounced even when the zero is the 400$^{th}$ one.

While the Fourier sin transform of fig2(b) differs from the process forming the zeta zeros, and the above product formula equals the zeta function only for real($z$)>1, the explicit formula has been proven even by Riemann himself to be a step square wave function on primes and prime powers. Therefore showing the constraints necessary to produce a square wave superposition requires the real parts to be on the critical line would establish RH.

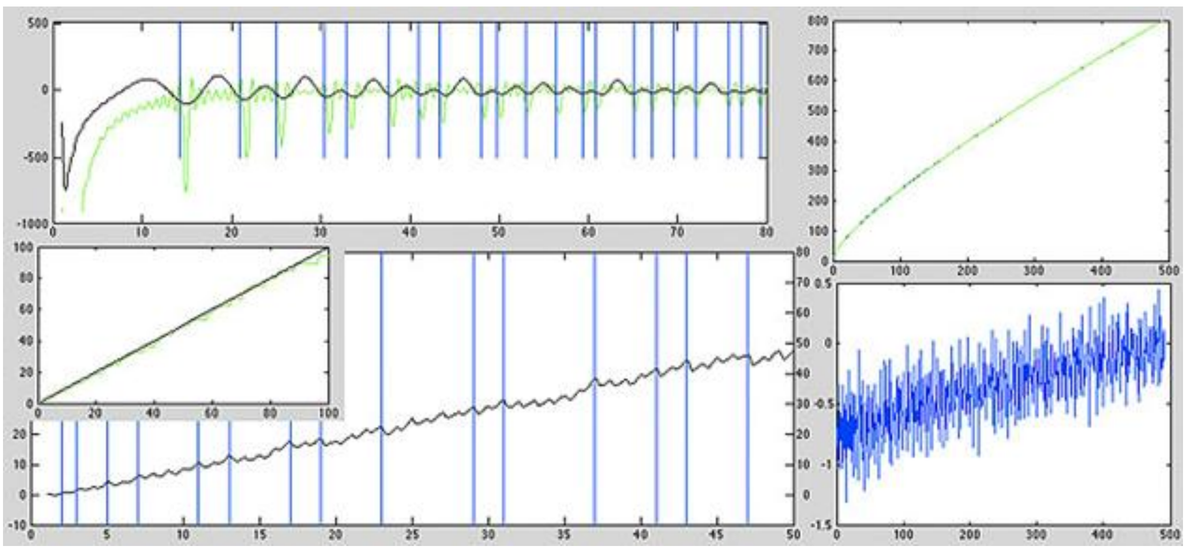


Fig 2e: (Top left) Fourier sin transform of integer step function (black) overlaid on that of the prime counting function (green) and actual zeta zeros (blue). (Top right) Step function minima distribution blue closely coincides with zeta zero distribution (green). (Lower right) Difference between the two. (Lower left) Applying Riemann's explicit formula to the step minima at real part ½ results in a function with integer local maxima with neighbourhood peaks at primes. (Inset) The step function and summed Mangoldt prime distribution compared.

Part of the intrinsic difficulty of RH is that we are trying to compare an irregular transform consisting of zeta zeros with the irregular prime distribution. Thus to a certain extent the zeros and primes are mutually encoded, so that it is difficult to establish RH without knowing the prime distribution and vice versa. To look for a comparative test function with more regular properties, we now examine the Fourier sin transform of the integer step function as shown in fig 2e, which I discovered accidentally has intriguing properties.

Attempting to construct an eta-like associated Dirichlet function for the zeta analogue of this distribution, which would be convergent on the critical strip, and thus reveal its exact zeros, would be very difficult, because the $k$-th coefficient, instead of being 1, would be the total number of possible factorizations of $k$, which may grow faster than the fixed powers of $k$ in the denominator.

The Fourier sin transform gives a smoother transformed function than the Mangoldt psi prime counting function, and its minima can be easily found. Bizarrely, the minima are irregular and correspond closely with zeta zeros, with an overall distribution closely following the actual zeros up into the thousands. When the explicit formula generating the prime distribution from the zeta zeros is applied to the transformed step minima, we gain an ascending function with integer local maxima and neighbourhood peaks at the primes, subtly combining both features. The fact that we gain a meaningful distribution at all suggests there should be an analytic continuation like the zeta function underlying the duality.

A second view of the other end of the zeta transform can be seen from examining the Euler product itself. In fig 3 are the truncated products for primes less up to 2, 3, and powers of 10 up to a million terms. Each of the terms $\left(1-p^{-s}\right)^{-1}$ in the product formula is finite in the critical strip except for the pole at 1. Since $p^{-s+\frac{2\pi}{\log p}i} = p^{-s}e^{\log p\frac{2\pi}{\log p}i} = p^{-s}e^{2\pi i} = p^{-s}$, each term is also periodic with period $\frac{2\pi i}{\log p}$ with minima on the critical line *x* = ½ at $\frac{(2n+1)\pi}{\log p}$.

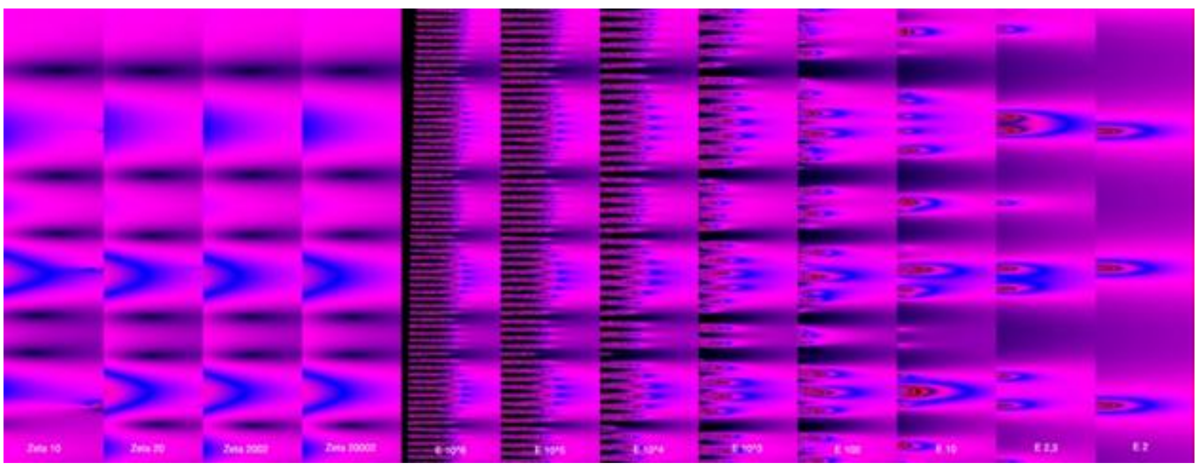


Fig 3: Convergence of the sum formula in the critical strip (from left) is compared with that of the product formula (from right) in the range *t* =10-40 for the absolute value of $\zeta(z)$. The pattern of the $\zeta(z)$ zeros is clearly manifest in the multiplicative superposition of the terms in the product formula in which the zeta zeros form the combined minima, although generally not coinciding with the minima of any one of the prime terms.

The narrowing of gaps between the zeros for increasing *t* thus results entirely from the phase relationships between successive prime periods centered on the *x* axis. For any finite product of $\left(1-p^{-s}\right)^{-1}$ the positions of the zeta zeros on the critical line neither represent minima nor zeros of the finite product, which declines to zero along the line $x+i\rho,\ x\to 0$ where $\rho$ is a zero of $\zeta(s)$, but grows to a peak for intermediate values before declining to infinitesimal values for negative *x*.

While the sum and product formulas for large finite values closely approximate one another for x > ½ the behavior of the finite product formula even for *p* in the millions becomes increasingly different from $\zeta(s)$, for *x* < ½, with a set of escalating peaks and troughs increasing slowly in number with $\log p$, so that even for primes up to a million there are still only a small number of these peaks and troughs between any two zeros of, $\zeta(s)$ causing a significant deviation from $\zeta(s)$, even for exponentially large *p*.

The lack of a proof of the Riemann hypothesis doesn't just mean we don't know **all** the zeros are on the line $x = ½$, it means that, despite all the zeros we know of lying neatly and precisely smack bang on the line $x = ½$, no one knows why **any** of them do, for if we had a definitive reason why the **first** zero ½ + 14.13472514 $i$ has real value precisely ½, we would have a definitive reason to know why they **all** do. Neither do we know why the imaginary parts have the values they do.

So let's have a look at the dynamics firstly of the product formula in the critical strip, noting that the equivalence between this and zeta strictly holds only for $x > 1$. We can see immediately that on the critical line, the product formula has become unstable and doesn't converge consistently to 0. If we look at the first zero, we can see some very odd things are happening.

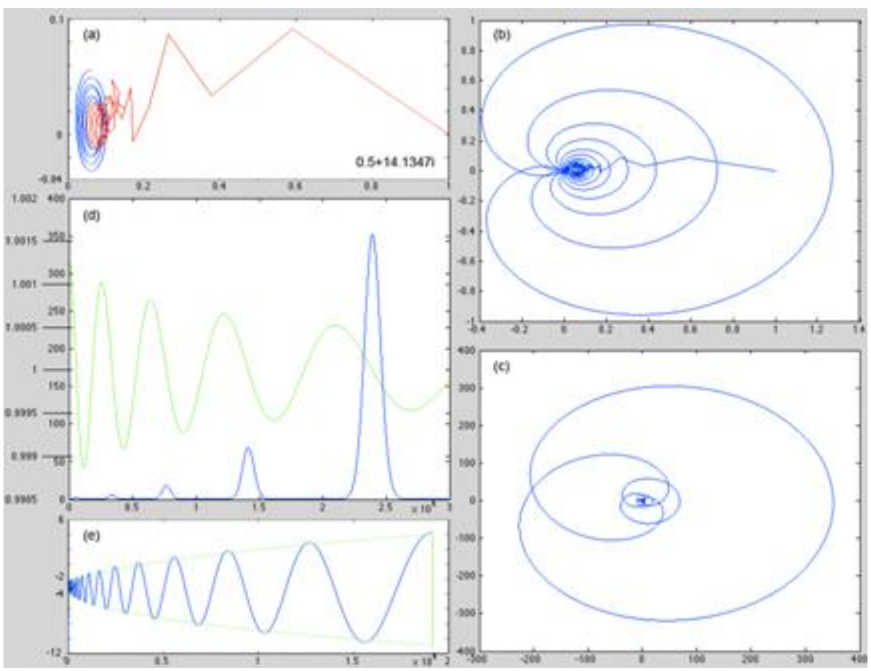



Fig 3b: The orbit of the first zero for increasing terms in the product formula initially has erratic values for the first few primes (a) driven by prime fluctuations, but as the primes increase enters into an unstable orbit which grows to a peak (b), with successive higher peaks which return to values close to 0 (c). Looking at the absolute value (d) of individual terms (green) and the cumulative product (blue), we can see that oscillating trends in the prime wave functions are now causing exponential bursts in the cumulative product. Viewing this in terms of the log of the product shows an approximate quadratic growth in the amplitude as the number of terms increases.

As shown in fig 3b, the initial erratic prime-driven values of the product, begin to settle into a more regular trend, which leads eventually to regular oscillations in the terms, resulting in exponentiating peaks tending to unbounded values, while intervening values tend to 0. We will thus get differing answers for the limit depending on how the product terms are grouped.
As we move up the zeros, this process becomes more complicated, with a vastly longer sequence of erratic steps forming a fractal orbit resembling a stochastic process.

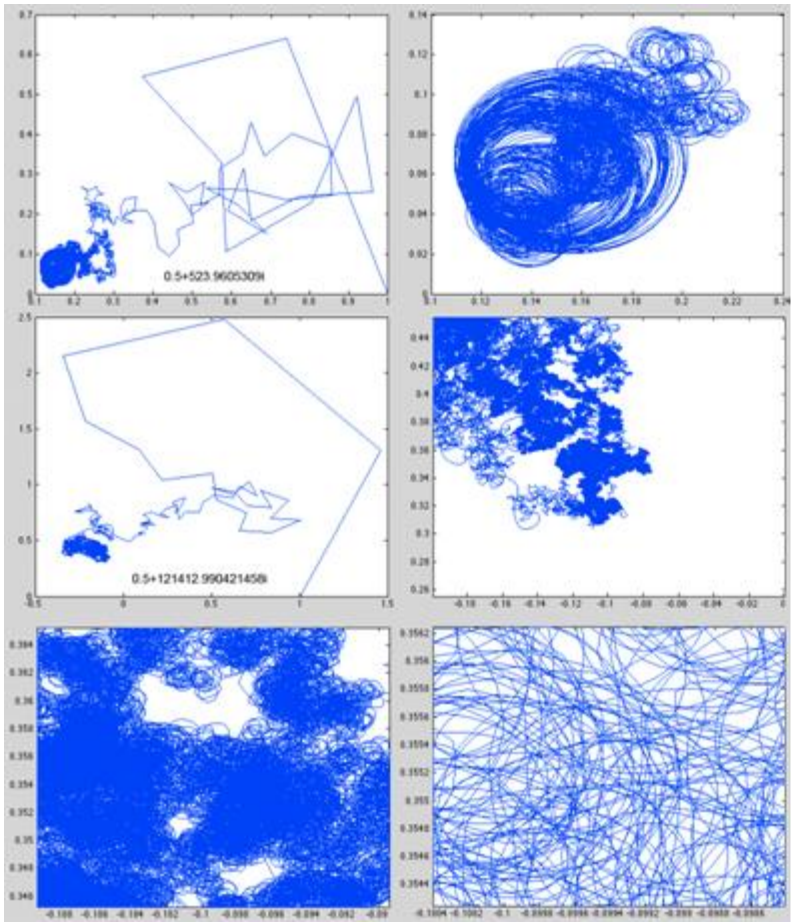



Fig 3c: Two higher zeros showing extended erratic transients in the cumulative product.
Fractal expansions of the orbit of the zero with *y*=121412.

By the time we reach values as high as 121412, the erratic transients have become so long that it is unclear that for realizable values of primes, the orbit pattern will have settled into quasi-regular oscillations.

An indication of how high the values of primes would have to be to see any resolution of the orbits of larger zeros can be seen from fig 3e, where even a zero as small as 523 takes until primes of the order of 3 million to begin to enter the oscillatory phase.

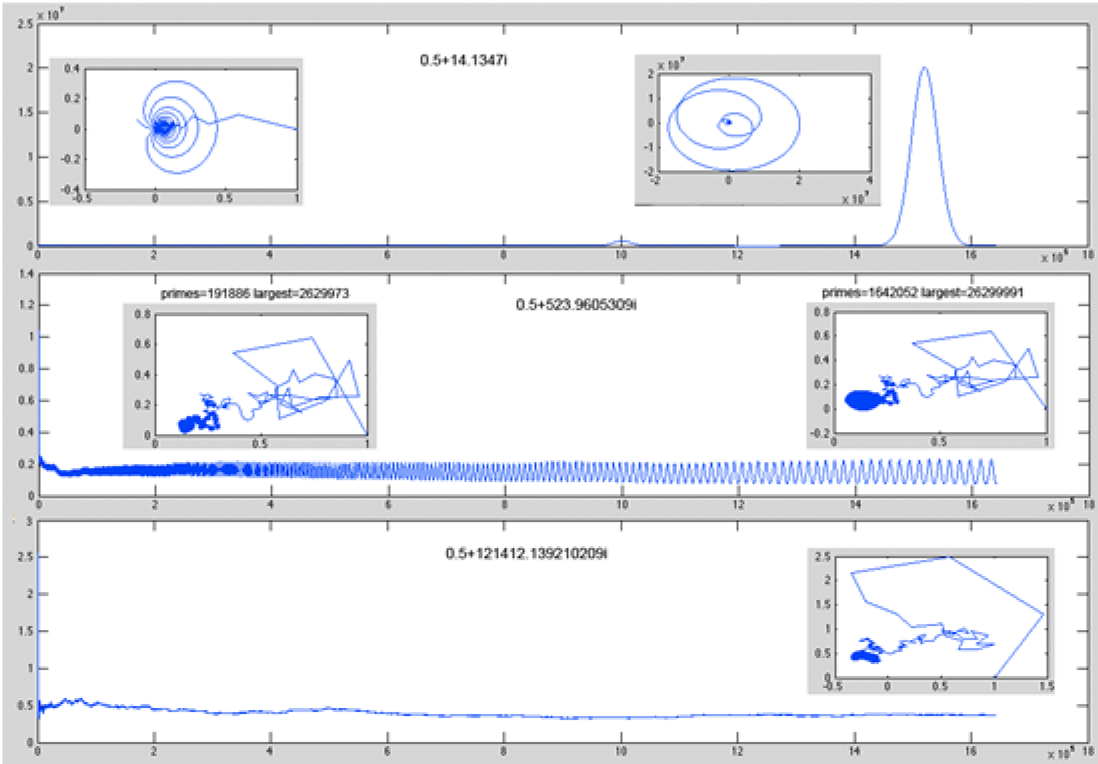



Fig 3d: When we evaluate the cumulative product up to the 1,642,052th prime 26299991, we find the first zero *y*~14 (top) has grown to a peak of around 10 million, while the zero *y*~523 (middle) has only begun to enter its first oscillatory burst around the 200,000th prime of around 3 million and *y*~121412 is as yet showing no signs of having fully explored its fractal dynamics.

The trend to exponential bursting is portrayed in fig 3e, where a complex plot of the values of the product and zeta are compared for differing numbers of cumulative terms.

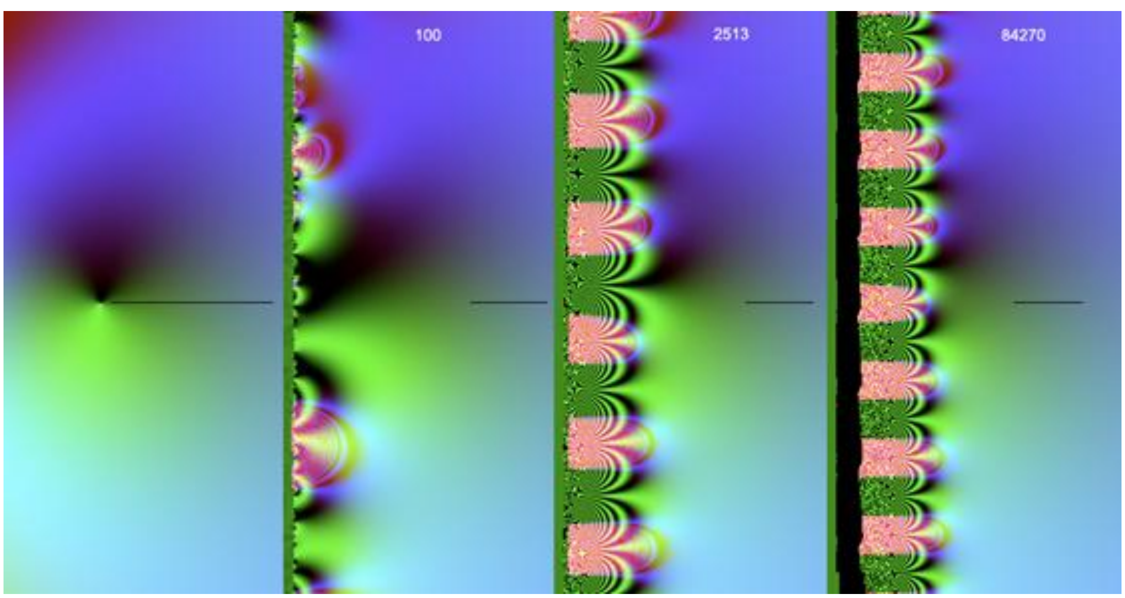



Fig 3e: Cumulative product and zeta compared at the first zero for 100, 2513 and 84270 prime terms. The value of the product at 84270 is not 0 but 6.25. Exponentially larger values occur at successive peaks as show in fig 3b.

Now we turn to looking at the additive representation, where each of the added functions $f(z)=1/(k^z)$ is a complex exponential varying exponentially in modulus with x and sinusoidally in angle with y. The effect of additive superposition is akin to a wave superposition of each of these functions. Since the eta representation of zeta is valid in the critical strip, we can easily investigate how rapidly the zeta function converges to its zeros, as the number of iterations is increased. When we do this we find that for a given zero, there are a series of erratic transients followed by a steady winding toward zero past a critical iteration number. However as we go up to higher zeros, these transients become longer and longer and involve winding in and out of a series of non-zero values resulting from the combined wave interference of the successive iterates. In fig 3f we can see just how complicated this process has become for the 2000th zero.

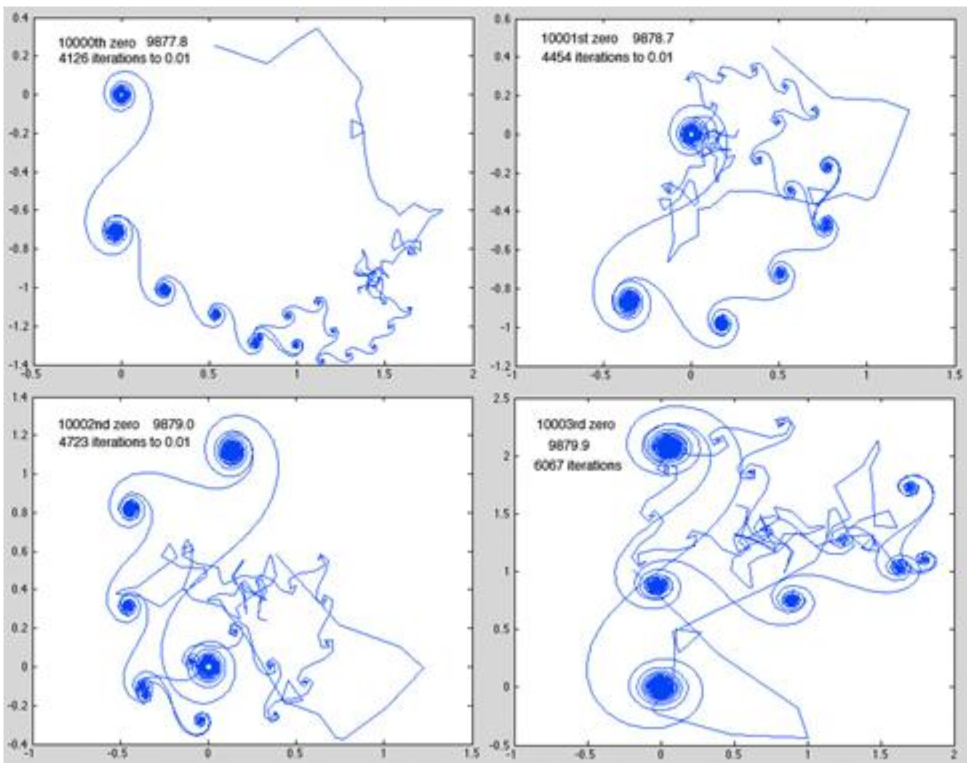



Fig 3f: Convergence of zeta to 0 at the four zeros starting at the 10000th zero shows erratic transients involving winding in and out of around 20 separate false zeros before eventual convergence. There is no consistent trend at all between successive zeros, each of which has its own idiosyncratic way of wandering towards 0 taking completely different numbers of iterations to get within 0.01 for 3 steps.

Although it is tantalizingly obvious that all the zeros lie enticingly on the line $x=1/2$, it would appear extremely unlikely that a general argument can explain why ALL of them do because the spiraling behavior involves the real and imaginary parts equally and the imaginary parts of the zeros form an uncomputable sequence of values. Similarly although we know every Collatz sequence for positive integers ends in a 3 cycle, proving all do is unsolved. If we accept we can't find a formula for the roots of a degree 5 polynomial, thanks to Galois, expecting to find those of zeta, despite its apparent symmetry may be a delusion.

We can make an estimate of how rapidly zeta converges to zero for successive zeros, we do this we arrive at a highly erratic relationship for individual zeros. This proves to be a result of the way zeta is expressed in terms of eta, as revealed in the smooth trend for eta with a power law of $\sim 13x^{0.68}$.

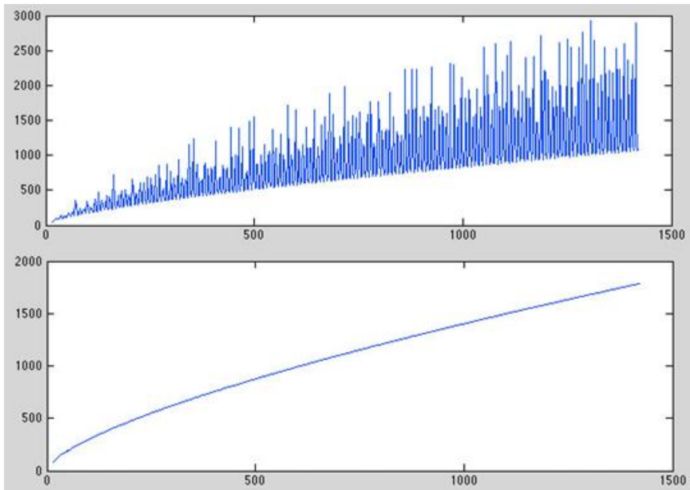



Fig 3g: (Above) The number of iteration steps in the eta-derived zeta series required to get 5 steps with 0.005 of 0 varies erratically from one zero to the next, but this is a disguised effect of the presence of the $1/(1-2^{1-z})$ term so becomes a smooth curve for eta (below).

We can also explore the source of the false convergences to non-zero values forming the spirals in the above sequences. As can be seen from the next image plotting the real and imaginary components and the angle and modulus of the individual terms, major shifts of convergence coincide with interference effects, when a number of terms in sequence have a similar angle due to constructive interference in the waves of angle in the imaginary direction for each term in the series, thus contributing a systematic shift in values, while steady trends in angle tend to cancel. Close to the values a change from $n^{-z}$ to $(n+1)^{-z}$ causes a near perfect frame shift of the angle to a multiple of $\pi$ later permitting a cumulative position shift. This also causes a problem for RH because of the lack of an obvious relationship between a sum of square roots $n^{1/2}$ defined by x=1/2 and the logarithmic variation of the imaginary waves defined by Cis($y$ln($n$)). The mode-locking phases in fig 3b can be calculated directly by finding where the waves match phase:

$$y\ln(n+1) = y\ln(n) + k\pi,\ \ln\left(\frac{n+1}{n}\right) = \frac{k\pi}{y},\ 1+\frac{1}{n} = e^{k\pi/y},\ n = \frac{1}{e^{k\pi/y}-1}.$$

This corresponds closely to the mode shifts, as shown in fig 3b in yellow.

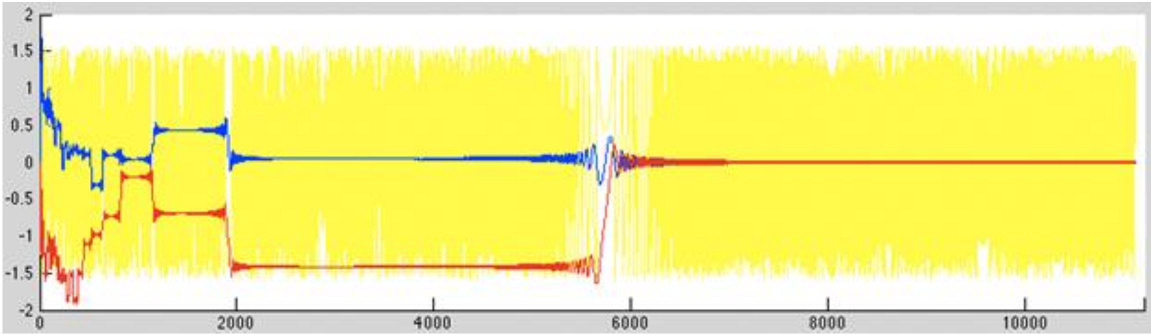



Fig 3h: Trends in real and imaginary parts of the sequence of terms converging to the 20000th zero are compared with trends in the angle and modulus of the individual terms. Showing convergence shifts correspond to angular phase-locking.

Although the Riemann hypothesis (RH) has not been proved, all of the $10^{13}$ of zeros so far found in the range up to $10^{20}$ lie on the critical line. It has been established that an infinite number of zeros lie on the critical line, that over 40% of the zeros do, and that all but an infinitely small proportion lie within $\varepsilon$ of $x = ½$.

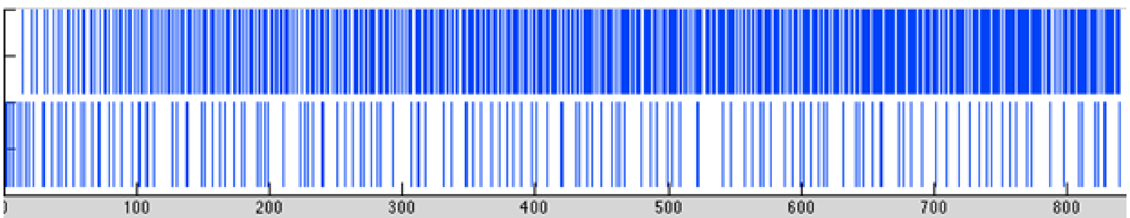



Fig 4: Above: *t* values for the zeta zeros. Below: the distribution of primes. A complementary relationship results from the superposition of the product formula terms, whose periodicities are $2\pi / \log p$. On the average, the zeros are distributed as $t / 2\pi\left(\log(t / 2\pi) - 1\right)$ and the primes, from the prime number theorem as $\pi(x) \sim x / \log x$.

However these results do not guarantee RH is true. Littlewoood who had been given RH as a PhD thesis topic by Hardy and proved an infinite number of zeros are on the critical line said[xiv] "There is no evidence whatever for it (unless one counts that it is always nice when any function has only real roots). One should not believe things for which there is no evidence." We have seen that $S(t)$ grows extremely slowly with $t$ so that major fluctuations in the zeros might not emerge with the large numbers so far computed. Other properties of the zeta function, such as changes in the topology of real($\zeta(s)$)=0 between 121414 and 121416, shown in fig 5, emerge only with moderately large numbers. RH is equivalent to the conjecture that the prime counting function $\pi(x) = \mathrm{Li}(x) + O(x^{1/2+\varepsilon}),\ \forall \varepsilon > 0$, where $\mathrm{Li}(x) = \int_2^\infty \frac{dt}{\log t}$.

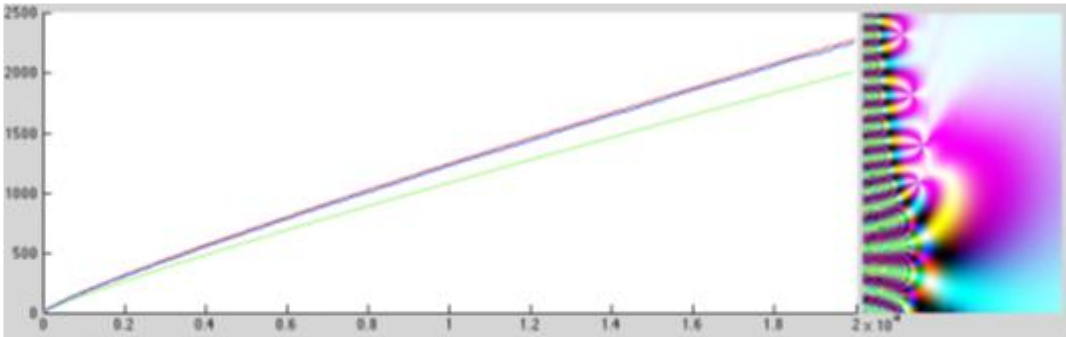



Fig 5: Left: Curves of $\pi(x)$ (blue) Li($x$) (red) and x/log(x) (green) showing the fit of the estimates.
Right: Region near $t$=121415 may be the first place where the curves real(($s$)) = 0 wrap around each other.
The zeros have slight errors of position, off $x$ = ½, due to the limit on the number of terms in the sum formula used.

In another classic result closely related to the zeta zeros, it has been proven that $\pi(x)-\text{li}(x),\ \text{li}(x)=\int_0^x \frac{dt}{\log t}$ changes sign infinitely often, although the difference is negative for all calculated primes. Skewes' bounds[xv] for a change of sign of $10^{10^{10^{34}}}$ assuming RH (see fig 2b), and $10^{10^{10^{963}}}$ not assuming it, show such changes can occur far beyond numbers so far computed. Although lower computer bounds of 1.39822 × $10^{316}$ where there are at least $10^{153}$ consecutive such integers near this value without assuming RH have been established, these are still astronomical by comparison with the known zeta zeros, so further anomalies in zeta zeros could appear.

On the other hand the Riemann hypothesis might be unprovable, yet conditionally true, like theories of physics, such as relativity and quantum theory, which, if all acid tests to refute the theory's predictions fail, remains valid until a counterexample is found under new physical conditions. If RH were found to be formally undecidable, demonstrating the inability to prove it false would be grounds to declare it true, as it would have been proven that no counterexample, i.e. offline zero, could exist.

Finally it might turn out to be both unprovable and uncertain, because it can only be resolved by a computation that suffers Turing machine halting undecidability. Whether many cellular automata will terminate, and the Collatz conjecture, have similar unprovability problems associated with unpredictable intermittencies associated with growth and decay. If RH proved unprovable on analytical arguments alone, such as the obvious internal symmetry of the functional equation manifest so obviously in $\xi$, RH might simply prove to be logically equivalent to all its complementary formulations in terms of primes and number theory, so that it could only be proven if and only if these arithmetic results could be proved independently. For example, the prime number theorem, which was first proved based on the zeta function now has elementary proofs, showing the belief that such number theory results could be proved only by analytic techniques was unfounded. Conrey[9] admits as much inclosing his review in Notices of the AMS:

A major difficulty in trying to construct a proof of RH through analysis is that the zeros of L-functions behave so much differently from zeros of many of the special functions we are used to seeing in mathematics and mathematical physics. For example, it is known that the zeta-function does not satisfy any differential equation. ... It is my belief that RH is a genuinely arithmetic question that likely will not succumb to methods of analysis. There is a growing body of evidence indicating that one needs to consider families of *L*-functions in order to make progress on this difficult question.

Nevertheless, although Conrey has placed his faith in the various *L*-functions which generalize zeta, associated with structures such as the elliptic curves pivotal in solving Fermat's last theorem, and the families $L(s,\chi_d)$ associated with finite fields, for which Andre Weil had proved RH, which can be associated with orthogonal, unitary and symplectic symmetry types, he notes an ultimate impasse:

"There is a growing body of evidence that there is a conspiracy among *L*-functions – a conspiracy that is preventing us from solving RH".

This arises from the inability to eliminate a spurious zero near 1, the Landau-Siegel zero, which stymies predictions. Two quotes from Peter Lax[xvi,xvii] give one of the clearest indications of the status of RH as the inscrutable *mysterium tremendum*:

"The Riemann hypothesis is a very elusive thing. You may remember in Peer Gynt there is a mystical character, the Boyg, which bars Peer Gynt's way wherever he goes. The Riemann hypothesis resembles the Boyg!"

"Did you know John Nash, the protagonist of the film 'A Beautiful Mind'?" "I did, and I had enormous respect for him. He solved three very difficult mathematical problems and then he turned to the Riemann hypothesis, which is deep mystery. By comparison, Fermat's is nothing. With Fermat's - once they found a connection to another problem - they could do it. But the Riemann hypothesis, there are many connections, and still it cannot be done. Nash tried to tackle it and that's when he broke down."

Another interesting indication of this impasse, which also highlights irregular features in zeta, akin to randomness, arises if we examine the L-function $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)$, where $\mu(n) = \begin{cases} (-1)^k, & n = p_1 \cdot \ldots \cdot p_k \\ 0, & n = c \cdot p^2 \end{cases}$, is the Möbius L-function (appendix 4). An equivalent of RH is that $M(x) = \sum_{n \le x} \mu(n) = O(x^{1/2+\varepsilon})$, which would guarantee the Möbius function above would converge for x > ½, and show there were no poles (and hence no zeta zeros), however Mertens' conjecture that $M(x) \le x^{1/2}$ was proved false and $M(x) = O(x^{1/2})$ is in serious doubt. The above equivalence of RH to $M(x)$ means $\mu(n)$ is as likely to have a 1 as a -1, thus behaving essentially like a random function, which Chaitin[xviii] noted in considering whether RH might be unprovable. Denjoy's probabilistic argument for RH is precisely this - comparing the spacing of the zeros with a random walk - that if $\mu'(n)$ is a random sequence of 1's and -1's then the simple random walk, $M'(x) = \sum_{n \le x} \mu'(n) = O(x^{1/2+\varepsilon})$ with probability 1. Hence this says the parity of the number of prime factors of a number varies randomly. The problem as Terrence Tao has pointed out[xix] is that the primes show both pseudo random and ordered behavior, making a proof difficult, because the process is then not able to be captured in a finite symbolic description.

Finally one fundamentally important universality property: somewhere in the crtical strip the zeta function fits arbitrarily closely any smooth non-zero function in a small neighbourhood [xx]. This is done by first approximating the function by a finite product of primes in the product formula and then showing the total product, i.e. zeta, comes arbitrarily close to this for a suitable neighbourhood of 3/4+$iT$.

## The Quantum Chaos Connection

A major breakthrough was thought to have happened when Montgomery made contact with the physicists studying nuclear energy levels at Princeton[xxi] and found that the pair correlations in the gaps between the zeta zeros followed the same Gaussian unitary ensemble statistics as chaotic quantum systems and energy levels of large nuclei:

$$\sum_{\frac{2\pi\alpha}{\log T}\le\gamma-\gamma'\le\frac{2\pi\beta}{\log T}} 1 : \; N(T)\int_{\alpha}^{\beta}\left(1-\left(\frac{\sin\pi u}{\pi u}\right)^{2}\right)du$$

Montgomery was also taken to discuss the twin prime conjecture[xxii] with Gödel, who likewise worried that this might be undecidable. The GUE statistic and its time-reversible real variant, the GOE, or Gaussian orthogonal ensemble, appear in many forms of quantum system whose classical analogue is chaotic. The corresponding form for fermions, rather than bosons is the symplectic GSE. These include both the many body problem of nuclear energetics, highly excited atoms in a magnetic field and the quantum stadium problem. One of the greatest moments in this interaction of fields[9,16] was the Keating's development of a formula for the zeta moments, using characteristic polynomials of unitary matrices (see appendix 1).

Initially this correspondence between fields caused great excitation in both the mathematics and physics communities, and a number of eminent researchers tried, so far in vain, to prove RH by discovering a system of random Hermitian matrices whose eigenvalues would be real and might correspond to the zeros of $\xi(s)$ thus showing they had to be real and hence those of $\zeta(s)$ would be on $x$ = ½. However this program has so far not borne fruit, despite many concerted attempts[8].

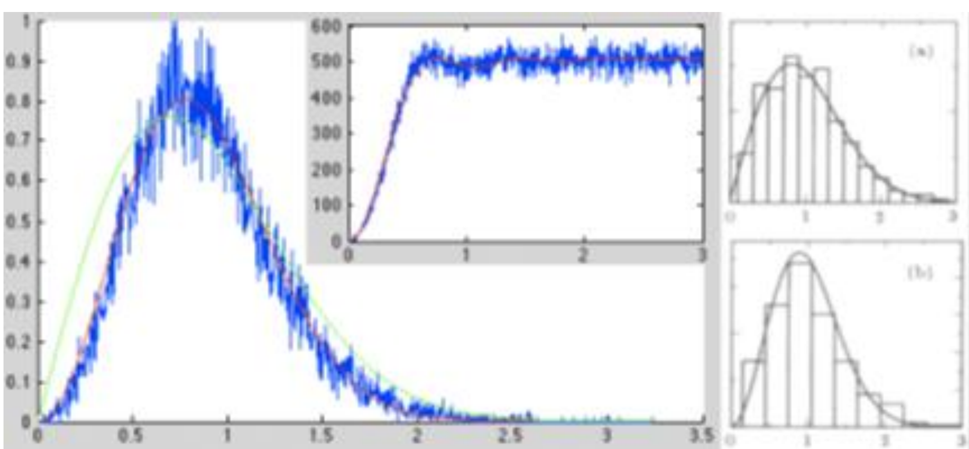


Fig 6: Left: The spacing between zeta zeros 100-10,000 is compared with a Gaussian Unitary Ensemble distribution $32\pi^{-2}s^{2}e^{(-4s^{2}/\pi)}$(red), showing coincidence of the two statistics, and a Gaussian Orthogonal Ensemble $\pi s e^{(-\pi s^{2})}/2$ (green) characteristic of the Wigner distribution of atomic nuclear energies. Inset: Pair correlations for the first 10^5 zeros compared with the theoretical GUE distribution. Right: Quantum dot stadium eigenvalues[xxiii] display both (a) GOE and (b) GUE forms of random matrix distribution under increasing magnetization of the electron's orbit.

In attempting to create a convergence between Hermitian operators and the zeta function, researchers have constructed a variety of candidates, some very complex and others deceptively simple. Berry[xxiv] has presented one of the most straightforward of these, the semi-classical operator $H=x\mathbf{p}$, and attempting to modify it to establish an operator having correspondence with zeta, demonstrating several putative connections between this and the zeta zeros. However the space on which this acts is not elucidated and the complex plane would need to be 'sewn up' in Berry's own words into a region which makes the dynamics quantally bound. Secondly there is no elucidated relationship between the primes and the periodic orbits of the Riemann dynamics.

Connes[xxv] has constructed a Hermitian operator whose eigenvalues are the non-trivial Riemann zeros. His operator is the transfer (Perron-Frobenius) operator of a classical transformation. Berry comments that such operators formally resemble quantum hamiltonians, but these usually have very complex non-discrete spectra with singular eigenfunctions. Connes gets a discrete spectrum by making the operator act on an abstract space here the primes acting on the Euler product are built in using a space of *p*-adic numbers and their units. The proof of the Riemann hypothesis is then transformed into establishing the proof of a certain classical trace formula.

Selberg has constructed a zeta function related to hyperbolic motion on constant curvature surfaces generated by discrete groups[xxvi]. The product formula is not over primes, but over all primitive periodic orbits for the motion of the surface considered.

$$Z(s)=\prod_{ppo}\prod_{m=0}^{\infty}\left(1-e^{-l_p(s+m)}\right)$$

where $l_p$ are the lengths of the orbits, and *s* is complex. This function like the Euler product is defined only for real($Z(s)$) > 1, but can be analytically continued to the entire complex plane:

$$Z(s)=e^{\left(\mu\int_0^{s-1/2}u\tan\pi u du\right)}Z(1-s)$$

As a result, Z(s) has both trivial zeros at 1, 0, -1, -2 etc. and a set of non-trivial zeros putatively on the critical line $x = ½$. $Z(s)$ has a similar trace formula to the Weil explicit formula for sums over the zeros of zeta. The correspondence between primes and periodic orbits, provides a correspondence between zeros and eigen-momenta in which ln($p$) corresponds to the orbital period $T_p$, resulting in an equivalent expression of the prime/periodic orbit number theorem:

$$N(\ln(p)<T)\overset{T\to\infty}{\to}\frac{e^T}{T}\leftrightarrow N(T_p<T)\overset{T\to\infty}{\to}\frac{e^T}{T}$$

Fundamental to the problem are two issues. The first is that the duality already seen in the relationship between zeta and the prime products is already the duality transform one is seeking. That is the system that decodes the zeta zeros is the distribution of numerical primes itself, so seeking an analogue from other mathematical areas cannot necessarily simplify the problem.

Secondly these GUE systems may show similarities in their statistics to zeta's zeros, because they share overall features combining structured constraints and pseudo-randomness in common with the primes and zeta zeros, without necessarily being isomorphic to them. In a sense there is a regress occurring, in which attempting to model GUE systems to zeta's zeros results in more elaborate mathematical constructions which share zeta's characteristics but neither provide a breakthrough in proving RH, nor result in a real valued quantum operator.

The quantum stadium is a direct analogue of the classical chaotic stadium billiard which displays the classical butterfly effect of chaos - sensitive dependence on initial conditions - and for almost all orbits produces a dense trajectory filling the stadium as shown in fig 7 (a). Within this classical system is a dense set of repelling periodicities, any arbitrarily small deviation from which results in a dense orbit, or a differing periodicity.

The quantum versions of this system behave in a fundamentally different manner. While the initial stages of a trajectory follow the classical picture, after a limited period of time, called the quantum break time, they have a cumulatively increasing probability of entering one of the eigenvalues of the system. These eigenvalues turn out to correspond to the closed orbits of the classical system, which have now become probability maxima of the quantum system because wave spreading has effectively compensated for sensitive instability of the orbit, resulting in wave-periodicity and so-called scarring of the quantum wave function by probability maxima along these closed orbits, which also extend to fractal eigenstates of open chaotic systems[xxvii].

Moreover, unlike the eigenvalues of ordered quantum systems such as the Lyman, Balmer and Paschen series of orbital electrons, whose energy separation converges to zero at infinity, the chaotic eigenvalues display energy separation statistically distributed as a GOE or GUE suppressing small energy transitions between eigenvalues. Semi-classical simulations of such systems, using a small classical wave packet, generally give similar results, showing the suppression of chaos and the separation of eigenvalues is directly caused by wave spreading.

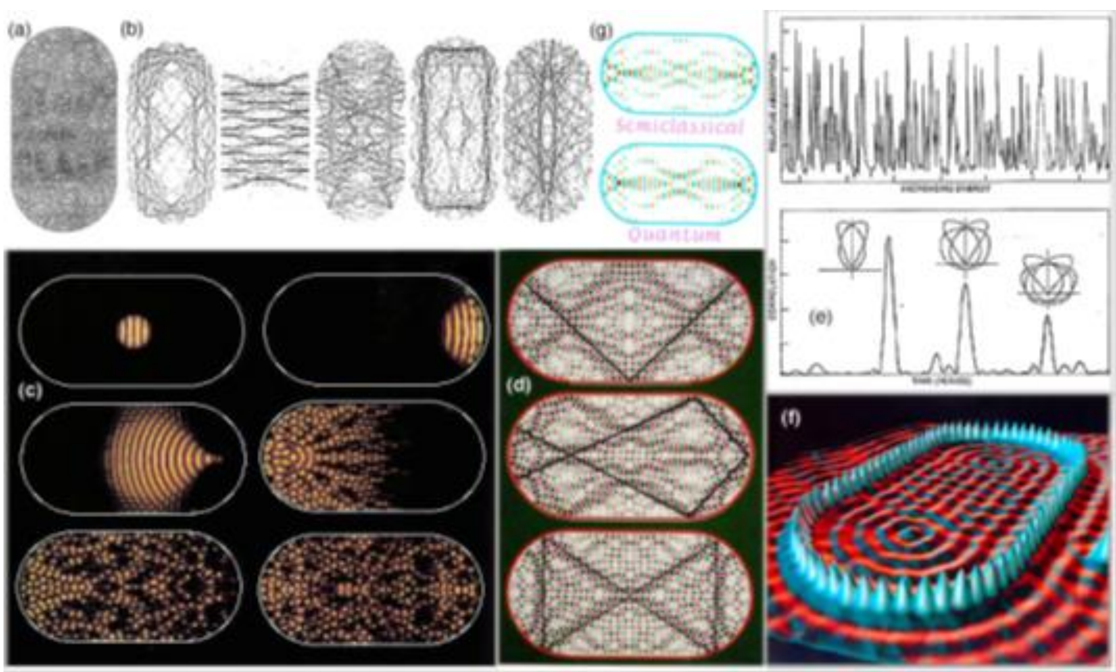



Fig 7: Quantum chaos: The classical stadium billiards is chaotic. A given trajectory has sensitive dependence on initial conditions. As well as space-filling chaotic orbits (a) [xxviii], the stadium is densely filled with repelling periodic orbits, three of which are shown in black in (d). Because they are repelling, neighbouring orbits are thrown further away, rather than being attracted into a stable periodic orbit, so arbitrary small deviations lead to a chaotic orbit, causing almost all orbits to be chaotic. The quantum solution of the stadium potential well (b) [xxix] and (d) [xxx] shows 'scarring' of the wave function along these repelling orbits, thus repressing the classical chaos, through probabilities clumping on the repelling orbits. A semi-classical simulation (c) shows why this is so. A small wavelet bounces back and forth, forming a periodic wave pattern, because even when slightly off the repelling orbit the wave still overlaps itself and can form standing wave constructive interference when its energy and frequency corresponds to one of the eigenvalues of a periodic orbit, even though the orbit is classically repelling. The quantum solution is

scarred on precisely these orbits (d). This causes resonances such as absorption peaks of a highly magnetically excited atom (e) to coincide with the eigenfunctions of the repelling periodic orbits, just as the orbital waves of an atom constructively interfere with themselves, in completing an orbit to form a standing wave, like that of a plucked string. The result is that, over time, in the quantum system, although the behaviour may be transiently chaotic, it eventually settles into a periodic solution. Experimental realizations such as the scanning tunneling view of an electron on a copper sheet bounded by a stadium of carefully-placed iron atoms (f) [xxxi], confirm the general picture, although, in this experiment, tunneling leaked the wave function outside too much to demonstrate proper scarring. The semi-classical approach matches closely to the full quantum calculation (g).

In systems like the quantum stadium, the closed orbits and their eigenvalues are playing a role similar to the primes in that they are orthogonal or uncoupled to one another, are determined by constraints which result in a discrete spectrum and form an irrationally related subset of the phase space. Primes among the numbers behave similarly in that they have no common factors, form a discrete spectrum having no consistent rational formulation and act as a set of discrete generators of all the other integers. Thus the correspondence may be analogical but not homologous.

The end result is that for a variety of closed quantum systems, wave spreading eventually represses classical chaos by scarring, causing the periodic eigenfunctions to become eventual solutions of any time-dependent problem, although the initial trajectory behaves erratically, just as does an orbit in the classical situation. For example, a periodically kicked quantum rotator [xxxii, xxxiii] will stochastically gain energy, just as in the classical situation, until a quantum break time [xxxiv], after which it will become trapped in one of the quantum solutions. A highly excited atom in a magnetic field will have its absorbance peaks at the periodic solutions, and quantum tunneling will likewise use scarred eigenvalues as its principle modes of tunneling [xxxv, xxxvi].

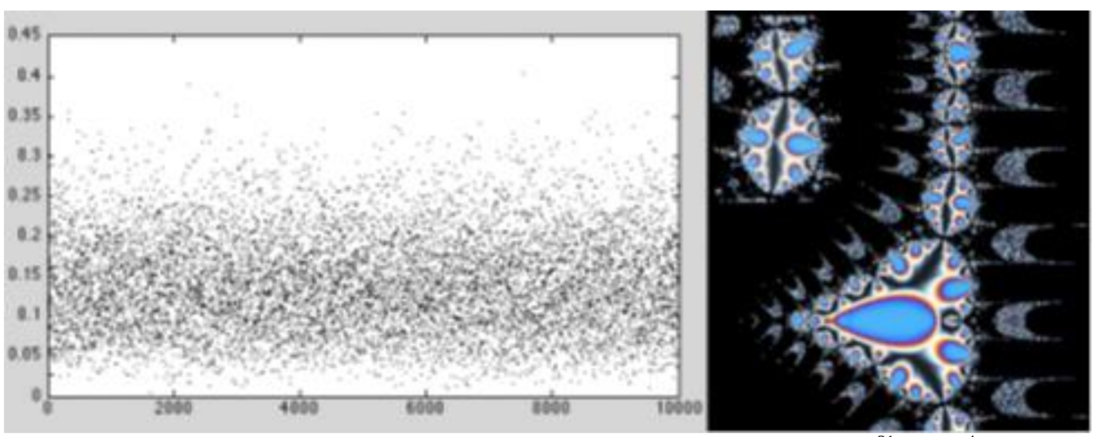



Fig 8: Left: Pattern of differences between successive zeros of zeta in the range from $10^{21}$ for $10^4$ successive zeros determined by Odlyzko[xxxvii] shows a Hurst exponent corresponding to a fractal dimension of 1.9 with pronounced negative persistence, differing significantly from corresponding statistics of GUE random matrices. Right: Julia set of zeta by 'inverse iteration' of the zeros[xxxviii] highlighting their superimposed first 20 successive inverse images, shows all zeros are in the basin of attraction of the fixed attractor $\alpha =\sim -0.2959$. The positions of the 1st and 2nd non-trivial zeros can be seen as small annuli just to the right of the centre dark cleft in the first two Julia bulbs shown in inset.

Evidence supporting differences between these two types of system comes from studies of the fractal dimension of the graph of zeta zero gaps for large zeros, which shows a Hurst exponent of 0.095 corresponding to a fractal dimension of ~1.9, with anti-persistence, indicating large gaps are followed by smaller ones, self-similarity over a wide range of values and significant differences from corresponding GUE systems. When corresponding block sizes of zeros and random matrices are used, Hurst exponents for the zeros and matrices are 0.34 and 0.65, suggesting fundamental differences in fractal structure.

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