Article

Conformally Compactified Minkowski Space: Myths and Facts

Arkadiusz Jadczyk
Center CAIROS, Institut de Mathématiques de Toulouse
Université Paul Sabatier, 31062 TOULOUSE CEDEX 9, France

Abstract

Maxwell's equations are invariant not only under the Lorentz group but also under the conformal group. Irving E. Segal has shown that while the Galilei group is a limiting case of the Poincaré group, and the Poincaré group comes from a contraction of the conformal group, the conformal group ends the road, it is *rigid*. There are thus compelling mathematical and physical reasons for promoting the conformal group to the role of the fundamental symmetry of space–time, more important than the Poincaré group that formed the group-theoretical basis of special and general theories of relativity. While the action of the conformal group on Minkowski space is singular, it naturally extends to a nonsingular action on the compactified Minkowski space, often referred to in the literature as "Minkowski space plus light-cone at infinity". Unfortunately in some textbooks the true structure of the compactified Minkowski space is sometimes misrepresented, including false proofs and statements that are simply wrong.

In this paper we present in, a simple way, two different constructions of the compactified Minkowski space, both stemming from the original idea of Roger Penrose, but putting stress on the mathematically subtle points and relating the constructions to the Clifford algebra tools. In particular the little-known antilinear Hodge star operator is introduced in order to connect real and complex structures of the algebra. A possible relation to Waldyr Rodrigues' idea of gravity as a plastic deformation of Minkowski's vacuum is also indicated.

1 Preliminaries

1.1 Notation

Let E be a real 6–dimensional vectors space endowed with a bilinear form (x,y) of signature $(4,2)$. Let Q be the diagonal 6×6 matrix

$$Q = \text{diag}\ (1,1,1,-1,1,-1). \tag{1.1}$$

We will call a basis e_i in E orthonormal if $(e_i,e_j)=Q_{i,j}$.[1] Any two orthonormal bases of E are related by a transformation from the group $O(4,2)$:

$$O(4,2)=\{R\in \text{Mat}(6,\mathbb{R}):\ R\,Q\,{}^tR=Q\}. \tag{1.2}$$

When a preferred orthonormal basis is selected in E, then E is denoted by $E_{2,2}$. For $x\in E_{4,2}$ we write

$$Q(x)={}^txQx=(x^1)^2+(x^2)^2+(x^3)^2-(x^4)^2+(x^5)^2-(x^6)^2. \tag{1.3}$$

Let G be the diagonal 4×4 matrix $G=\text{diag}(+1,+1,-1,-1)$. Let V be a four–dimensional complex vector space endowed with a pseudo–Hermitian form $(\cdot|\cdot)$ of signature (2.2). A basis e_i of $H_{2,2}$ is said to be orthonormal if $(e_i|e_j)=G_{ij}$. Any two orthonormal bases in $H_{2,2}$ are related by a transformation from the group $U(2,2)$:

$$U(2,2)=\{U\in \text{Mat}(4,\mathbb{C}):UGU^*=G\}. \tag{1.4}$$

When a preferred orthonormal basis is selected in V, then V is denoted by $H_{2,2}$.

[1]According to our conventions, in $E_{4,2}$, the first four coordinates x^1,x^2,x^3 and x^4 will correspond to Minkowski space coordinates x,y,z and t, while the coordinates x^5,x^6 will correspond to the added hyperbolic plane $E_{1,1}$.

1.2 The algebras $Cl_{4,2} \approx \mathbf{Mat}(4,\mathbb{R})$, $Cl^{+}_{4,2} \approx \mathbf{Mat}(4,\mathbb{C})$, and the groups $SO_{+}(4,2)$ and $U(2,2)$

Define the following six 4×4 antisymmetric matrices $\Sigma_\alpha = (\Sigma_\alpha^{AB})$:

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \Sigma_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Sigma_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\Sigma_4 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \Sigma_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \Sigma_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The following identities hold:

$$\overline{\Sigma_\alpha^{ij}} = \frac{1}{2}\epsilon^{ijkl} G_{km} G_{ln} \Sigma_\alpha^{mn}, \tag{1.5}$$

where $\alpha = 1, ..., 6$; $i, j, k, l, m, n = 1, ..., 4$.
Recall that for an antilinear operator A acting on a pseudo-Hermitian space V the adjoint A^* is defined by the formula: $(Av|w) = (A^*w, v)$. The following proposition holds:

Proposition 1.2.1. *Define the following six complex matrices*

$$\Gamma^i_{\alpha k} = \Sigma_\alpha{}^{ij} G_{jk},\ (\alpha = 1, ..., 6;\ i, j, k = 1, ..., 4). \tag{1.6}$$

Explicitly:

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \Gamma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma_4 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad \Gamma_5 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \Gamma_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let Γ_α be the antilinear operators on $H_{2,2}$ defined by the formula:

$$(\Gamma_\alpha f)^i = \Gamma^i_{\alpha j} \overline{f^j}, \quad f = (f^i) \in H_{2,2}.$$

Then the antilinear operators Γ_α are anti–self–adjoint: $\Gamma_\alpha = -\Gamma^*_\alpha$, and satisfy the following anti–commutation relations of the Clifford algebra of $E^{4,2}$:

$$\Gamma_\alpha \circ \Gamma_\beta + \Gamma_\beta \circ \Gamma_\alpha = 2\, Q_{\alpha_\beta}.$$

The space $H_{2,2}$ considered as an 8–dimensional **real** *vector space carries this way an irreducible representation of the Clifford algebra $Cl_{4,2}$. The Hermitian conjugation in $H_{2,2}$ coincides with the conjugation of $Cl_{4,2}$. The space $H_{2,2}$ considered as a 4-dimensional* **complex** *vector space carries a faithful irreducible representation of the even Clifford algebra $Cl^+_{4,2}$.*
For each $x = (x^1, ..., x^6) \in E_{4,2}$, let X be the matrix

$$X = \sum_{\alpha=1}^{6} x^\alpha \Gamma_\alpha, \tag{1.7}$$

then

$$X = \begin{pmatrix} 0 & ix_4 + x_6 & ix_1 + x_2 & -ix_3 - x_5 \\ -ix_4 - x_6 & 0 & -ix_3 + x_5 & -ix_1 + x_2 \\ ix_1 + x_2 & -ix_3 + x_5 & 0 & ix_4 - x_6 \\ -ix_3 - x_5 & -ix_1 + x_2 & -ix_4 + x_6 & 0 \end{pmatrix} \tag{1.8}$$

We have

$$\bar{X}^i{}_j = \frac{1}{2}\epsilon^{imnk}\, G_{mj} G_{nl}\ X^l{}_k, \tag{1.9}$$

and

$$\det(X) = \det(\bar{X}) = Q(x)^2, \tag{1.10}$$

where $\bar{X}$ is the complex conjugate matrix.

[2]

We recall [2, p. 387, Definition IX.4.C] that the group $Spin(E)$ consist of products of even numbers of vectors $x \in E$ with $Q(x) = 1$, and even numbers of vectors $y \in E$ with $Q(y) = -1$. The action of $Spin(E)$ on $E \subset Cl(4,2)$ is given by $x \mapsto \pi(g)xg^{-1}$, which is the same as gxg^{-1} when $g \in Spin(E) \subset Cl^+_{4,2}$.[3] If x, x' are two normalized in E, then their product xx' operates on $H_{2,2}$ via a **complex linear** transformation implemented by the matrix $X\bar{X}'$ of determinant one. It follows that, with our identification, the group $Spin(E)$ coincides with the group $SU(2,2)$. In fact, for $x \in E_{4,2}$ and $U \in SU(2,2,$ we have $UXU^* = X'$, where

$$X'^{\alpha} = L(U)^{\alpha}{}_{\beta}\, X^{\beta}, \tag{1.11}$$

where $U \mapsto L(U)$ is a homomorphism from $SU(2,2)) \approx Spin(4,2)$ onto $SO_+(E_{4,2})$ with kernel $\{1, -1, i, -i\}$.

2 The exterior algebra $\Lambda^2 H_{2,2}$

Let $\Lambda^2 H_{2,2}$ be the (complex) exterior algebra of $H_{2,2}$. It carries a natural pseudo-Hermitian form:

$$(x|y) = \frac{1}{p!}\, G_{i_1 j_1}..G_{i_p j_p} x^{i_1 \ldots i_p}\, \overline{y^{j_1 \ldots j_p}}. \tag{2.1}$$

If e_i is the orthonormal basis of $H_{2,2}$, we define the following six bivectors $E_\alpha \in \Lambda^2 H_{2,2}$:

$$E_\alpha = \frac{1}{2\sqrt{2}}\, \Sigma_\alpha^{ij}\, e_i \wedge e_j \tag{2.2}$$

They are normalized:

$$(E_\alpha | E_\beta) = Q_{\alpha\beta}. \tag{2.3}$$

In [3] an antilinear Hodge $\star$ operator has been defined in the case of the Euclidean signature. It can be readily extended to the pseudo–Euclidean case. Thus we define $\star :\ \Lambda^k H_{2,2} \to \Lambda^{4-k} H_{2,2}$ by the formula:

$$x \wedge \star y = (x|y)e, \quad x, y \in \bigwedge^k V, \tag{2.4}$$

where $e = e_1 \wedge \ldots \wedge e_4$, $x \in \Lambda^k H_{2,2}$, $y \in \Lambda^{4-k} H_{2,2}$. Notice that we have:

$$(x| \star y) = (-1)^{k(4-k)} (y| \star x), \tag{2.5}$$

We easily find that $\star\star\ x = (-1)^{k(n-k)}x,\ x \in \Lambda^k H_{2,2}$. It follows that on $\Lambda^2 H_{2,2}$ we have $\star^2 = 1$. Thus $\Lambda^2 H_{2,2}$, which is a vector space of complex dimension 6 splits into a direct sum of two **real** 6–dimensional subspaces

$$\Lambda^2 H_{2,2} = \Lambda^2_+ H_{2,2} \oplus \Lambda^2_- H_{2,2}, \tag{2.6}$$

where

$$\Lambda^2_\pm H_{2,2} = \{x \in \Lambda^2 H_{2,2} : \star x = \pm x\}. \tag{2.7}$$

[2] There are several errors in Ref. [1]: in the listing of the matrices Σ_α, the matrix Σ_2 is listed twice; in Lemma 5, Eq. (31) instead of Q_{jk} should be G_{jk}; in a formula below, Γ^α should be Γ_α; it is stated that the Hermitian conjugation coincides with the main antiautomorphism, which is incorrect, it should be 'conjugation'. Also the Lorentz Lie algebra block matrix at the end of section 8.3 should be in the upper left corner.

[3] We denote by π (resp. by τ) the main automorphism (resp. antiautomorphism) of the Clifford algebra.

The multiplication by the imaginary unit i gives a bijective correspondence between these subspaces. The bivectors E_α are self–dual: $\star E_\alpha = E_\alpha$, and form a basis in $\Lambda^2_+ H_{2,2}$.

Thus $E_{4,2}$ can be interpreted in two ways: either as a real linear subspace of anti–linear transformations of $H_{2,2}$ determined by the matrices of the form (1.7), or as the subspace $\Lambda^2_+ H_{2,2}$ of self–dual bivectors in $\Lambda^2 H_{2,2}$, as suggested originally by Kopczyński and Woronowicz [4], though these authors did not recognize the relation of their reality condition to the anti–linear Hodge operator (2.5).

2.1 Compactified Minkowski space

Early in the XX-th century Bateman and Cunningham [5, 6, 7] established invariance of the wave equation and of Maxwell's equations under conformal transformations. The central role in these transformations is being played by the conformal inversion, formally defined by

$$R: \, (\mathbf{x}, t) \mapsto \frac{(\mathbf{x}, t)}{\mathbf{x}^2 - c^2 t^2}. \tag{2.8}$$

It is singular on the light cone $x^2 = \mathbf{x}^2 - c^2t^2 = 0$. More general, special conformal transformations, of the form $RT(a)R$, where $T(a)$ is the translation by a vector a in $\mathbb{R}^4$, are also singular in the Minkowski space–time M. In order to avoid singularities the conformally compactified Minkowski space M^c - a homogeneous space for the conformal group $SO_+(4,2)$ is introduced. There are three different, though related to one another, ways of describing M^c : a) The group manifold of the unitary group $U(2)$, b) The projective quadric defined by the equation $Q(x) = 0$ in $E_{4,2}$, and c) The space of maximal isotropic subspaces of $H_{2,2}$. Let us discuss briefly the relation between b) and c). The relation to a) has been discussed in [1].
Let φ be the mapping $\varphi : \, E_{4,2} \to \Lambda_+ H_{2,2}$ defined by $\varphi(x) = x^\alpha E_\alpha$. The following lemma holds [1].

Lemma 2.1.1. *The mapping φ defines a bijective correspondence between generators of the null cone $Q(x) = 0$ in $E_{4,2}$ and maximal isotropic subspaces in $H_{2,2}$.*

In particular, $Q(x) = 0$, if and only if $\varphi(x)$ is a bivector of the form $v \wedge w$, where v and w are two linearly independent, mutually orthogonal, isotropic vectors in $H_{2,2}$. There is another method of viewing this correspondence, studied by René Deheuvels in [8, Théorème VI.6.D, p. 283], and generalized by Pierre Anglès [9]. We start with a brief recapitulation, with a slight change of notation, of the construction given in [9]. The construction is quite general. but we restrict ourselves to the case we are interested in. Let $E = E_{4,2}$, $V = H_{2,2}$. It is convenient to identify $x \in E$ with the antilinear operator $\varphi(x)$ acting on the spinor space V. Then the following theorem holds:

Theorem 2.1.2. [9, 1.5.5.1.1, p. 45] *The mapping*

> *{ isotropic line $\mathbb{R}x$ of $E_{4,2}$} $\mapsto$ maximal totally isotropic subspace $S(x)$ of V is injective and realizes a natural embedding of $\tilde{Q}(E)$, the projective quadric associated with E, into the Grassmannian $G(V, \frac{1}{2}\dim V)$ of subspaces of V of dimension $\frac{1}{2}\dim V$.*

The proof goes as follows: given a null vector $x \in E \subset L_\mathbb{R}(V) \approx Cl_{4,2}$, there exists another null vector $y \in E$ such that $(x, y) = 2$. Denoting by x, y antilinear operators on V representing x, y, in $L_\mathbb{C}(V)$, $(xy)^2 = xy$, $(yx)^2 = yx$, and $xy + yx = I$. Therefore xy and yx are two complementary idempotents in $L_\mathbb{C}(V)$. Evidently we have $\mathrm{Im}(xy) \subset \mathrm{Im}(x)$. Since $x^2 = 0$ we have that $\mathrm{Im}(x) \subset \mathrm{Ker}(x)$. On the other hand, if $v \in V$ is in $\mathrm{Ker}(x)$ then $v = (xy+yx)v = xyv$, therefore $\mathrm{Ker}(x) \subset \mathrm{Im}(xy) \subset \mathrm{Im}(x)$. It follows that $\mathrm{Im}(x) = \mathrm{Im}(xy) = \mathrm{Ker}(x)$. Now, owing to the fundamental property of the trace, we have $\mathrm{Tr}_\mathbb{R}(xy) = \mathrm{Tr}_\mathbb{R}(yx)$. But since xy and yx are complex linear, it follows that $\mathrm{Tr}_\mathbb{C}(xy) = \frac{1}{2}\mathrm{Tr}_\mathbb{R}(xy) = \frac{1}{2}\mathrm{Tr}_\mathbb{R}(yx) = \mathrm{Tr}_\mathbb{C}(yx)$. Since xy and yx are complementary idempotents, it follows that $\mathrm{Tr}_\mathbb{C}(xy) = \dim_\mathbb{C}(\mathrm{Im}(xy)) = \dim_\mathbb{C}(\mathrm{Im}(yx))$. Then, from $\mathrm{Im}(xy) \oplus \mathrm{Im}(yx) = V$, $\mathrm{Im}(xy) \cap \mathrm{Im}(yx) = \{0\}$, we deduce that $\dim(\mathrm{Im}(xy)) = \dim(\mathrm{Im}(yx)) = \frac{1}{2}\dim(V)$. In our particular case $\frac{1}{2}\dim(V) = 2$.

In order to complete the demonstration we notice that $(xy)^\tau = (xy)^* = yx$ therefore for all $s,t \in V$ we have $(xy\,s|xy) = (s,(xy)^*yy|t) = (s|yxxy\,t) = 0$. It follows that $\mathrm{Im}(xy)$ is a totally isotropic subspace, thus maximal totally isotropic. On the other hand from the equality $\mathrm{Im}(xy) = \mathrm{Im}(x)$ we find that this subspace is independent of the choice of y having the required properties.

In our case the correspondence $x \mapsto S(x)$ is not only an embedding, but also a **bijection**. This follows form the construction above, the covering homomorphism $U(2,2) \to O(4,2)$ and the fact that the pseudo–orthogonal group $O(4,2)$ (resp. unitary group $U(2,2)$) acts transitively on totally isotropic subspaces of E (resp. of V) of the same dimension - cf. [10, Corollaire 2, p. 74]

While isotropic lines $\mathbb{R}x$ in E (or maximal isotropic subspaces $S(x)$ in V) correspond to the points of the compactified Minkowski space M^c, there is an interesting duality: maximal isotropic subspaces (isotropic planes) in E are in one–to–one correspondence with isotropic (complex) lines in V, and they correspond to null geodesics in M^c.[4]

2.2 From maximal isotropic subspaces in E to isotropic lines in V.

We will use the method and the results of the previous subsection. Let N be a maximal isotropic subspace of E. Since, in our case, $E = E_{4,2}$, it follows that N is two–dimensional. It is then known [11, p. 77-78] that there exists another maximal isotropic subspace P such that $N \cap P = \{0\}$, vectors x_1, x_2 spanning N and vectors y_1, y_2 spanning P, with the property $(x_i, y_j) = 2\delta_{ij}, \quad i = 1,2$. We define $P_i = x_i y_i$, $Q_i = y_i x_i$, and it easily follows that $P_i^2 = P_i$, $Q_i^2 = Q_i$, $P_i + Q_i = I$, and, moreover, P_i and Q_j commute for $i \neq j$. It follows $R_1 = P_1P_2, R_2 = P_1Q_2, R_3 = Q_1P_2, R_4 = Q_1Q_2$ are four idempotents with $R_iR_j = 0, \quad i = 1,2,3,4$, and $R_1 + R_2 + R_3 + R_4 = I$. It is easy to see that $\mathrm{Tr}(R_1) = ... = \mathrm{Tr}(R_4)$. For instance

$$\begin{aligned}\mathrm{Tr}(R_1) &= \frac{1}{2}\mathrm{Tr}_{\mathbb{R}}(x_1y_1x_2y_2) = \frac{1}{2}\mathrm{Tr}_{\mathbb{R}}(y_2x_1y_1x_2) = \frac{1}{2}\mathrm{Tr}_{\mathbb{R}}(x_1y_1y_2x_2)\\ &= \mathrm{Tr}_{\mathbb{C}}(P_1Q_2) = \mathrm{Tr}(R_2),\end{aligned}$$

where we have use the fact y_2 anticommutes with x_1 and y_1, therefore commutes with x_1y_1. Therefore we have that $\dim \mathrm{Im}(R_i) = 1$. Since $\mathrm{Im}(R_1) \subset \mathrm{Im}(x_1)$, the subspace $\mathrm{Im}(R_1)$ is an isotropic line.
Let us now show that $\mathrm{Im}(R_1)$ depends only on the subspace N and not on the choice of the auxiliary subspace P or a particular choice of our vectors x_i and y_i. For this end we first notice that $\mathrm{Im}(R_1) = \mathrm{Im}(x_1x_2)$. Indeed, we evidently have $\mathrm{Im}(x_1y_1x_2y_2) \subset \mathrm{Im}(x_1x_2)$. On the other hand assume that $s \in \mathrm{Im}(x_1x_2)$, then, for some $t \in V$, we have

$$s = x_1x_2\,t = x_1x_2(x_1y_1x_2y_2 + x_1y_1y_2x_2 + y_1x_1x_2y_2 + y_1x_1y_2x_2).$$

Multiplying, using commutation properties and nilpotency of x_i, we find that only the last term survives and it can be written as

$$x_1x_2x_2y_1x_1y_2x_2\,t = x_1y_1x_2y_2(x_1x_2\,t).$$

Therefore $s \in \mathrm{Im}(R_1)$, and so we have shown that $S(x_1,x_2) \overset{\mathrm{df}}{=} \mathrm{Im}(R_1) = \mathrm{Im}(x_1x_2)$ does not depend on the choice of y_i with the properties as above. Finally, if $x'_i = \sum_{j=1}^2 a_{ij}x_j$ is a nonsingular transformation of the basis x_i of N, then a simple calculation gives that $x'_1x'_2 = \det(a)\,x_1x_2$, and therefore $\mathrm{Im}(x_1x_2) = \mathrm{Im}(x'_1x'_2)$.

There is an alternative way of looking at this correspondence: let v be a non–zero null vector in V, and let $N(v)$ be defined as:

$$N(v) = \{x \in E :\, xv = 0\}. \tag{2.9}$$

Then $N(v)$ is a maximal isotropic subspace of E and $\mathbb{C}v$ corresponds to $N(v)$ according to the construction above. Indeed, it follows immediately from the construction that $S(x_1,x_2) \subset N(v)$. To show that, in fact, equality holds, it is enough to show that $N(v)$ is an isotropic subspace of E. If $x^2 \neq 0$, and $x \in N(v)$, then for all $w \in V$ we have $0 = (xv|xw) = -(w|x^2v)$, and therefore we must have $x^2 = 0$.

[4]It is important to notice that naturally M^c carries only a conformal structure rather than a Riemannian metric. But null geodesics are the same for each Riemannian metric in the conformal class.

2.3 $E_{4,2}$ as the arena for Lie spheres of $\mathbb{R}^3$

The space $E_{4,2}$ carrying a natural linear representation of the conformal group $O(4,2)$ has been introduced by Sophus Lie in 1872 [12], and developed further by W. Blaschke in 1929 [13]. We will follow the modern presentation of Ref. [14], though we will change the coordinate labels so as to adapt them to our notation introduced in section 1.1. It is instructive to see how the Lorentz, Poincaré and the conformal group naturally enter the scene without any philosophical load of Einstein's relativity.
We can assume absolute time. We can also assume absolute space, its points represented by coordinates of $\mathbb{R}^3$, and compactified by adding one point: ∞, and absolute time, represented by points of $\mathbb{R}$. An oriented sphere with center at $\mathbf{x}$ and radius r can be also interpreted as the coordinate of an 'event' in space–time, taking place at $\mathbf{x}$ at time r/c.[5] We allow for r to be negative, with the interpretation that negative radius corresponds to the negative orientation of the sphere. The radius r can be interpreted as the radius of a spherical wave at time t, if the wave, propagating through space with the speed light, was emitted at $\mathbf{x}$ at time $t=0$. The image being that when the spherical wave reduces to a point, it turns itself inside–out, thus reversing its orientation. As a limit case there will also be spheres of infinite radius - represented by oriented 2–planes - these will correspond to events that took place in infinitely distant past or future. Points (spheres of zero radius), spheres, and planes (spheres of infinite radius) are bijectively represented by generator lines of the null cone in $E_{4,2}$ as follows (cf. [14, p. 16]):

Euclidean	**Lie**
points: $\mathbf{x}\in\mathbb{R}^3$	$[(\mathbf{x},0,\frac{1+\mathbf{x}^2}{2},-\frac{1-\mathbf{x}^2}{2},0,0)]$
∞	$[(\mathbf{0},0,1,1)]$
spheres: center $\mathbf{x}$, signed radius t	$[(\mathbf{x},t,\frac{1+\mathbf{x}^2-t^2}{2},-\frac{1-\mathbf{x}^2+t^2}{2})]$
planes: $\mathbf{x}\cdot\mathbf{n}=h$, unit normal $\mathbf{n}$	$(\mathbf{n},1,h,h)]$

Table 1: Correspondence between Lie spheres and points of the compactified Minkowski space. $[x]$ denotes the equivalence class modulo $\mathbb{R}^*$.

3 Myths and facts

One of the 'myths' we have already encountered above. Minkowski space and its compactification arise naturally through the studies of geometry of spheres in $\mathbb{R}^3$ - it not necessary to invoke Einstein's relativity principle, or restrict the range of available velocities, as it is usually being done. Second myth, more serious one, can be summarized in one sentence: 'conformal infinity' is the result of the conformal inversion of the light cone at the origin of M.' Such a statement can be found, for instance, in [15, p. 127], where the authors write:

> *"... after compactification the tangent space $T_x(M)$ is enlarged by the point at infinity y with coordinates $(0,0,...,0,1)$ and by the isotropic cone C_x, with vertex at this point y whose equation is the same as the equation of the cone C_x, namely $g_{ij}x^ix^j=0$.'(...) To include these points in the domain of the mapping defined by the inversion in $\mathbb{R}^n_q$, we enlarge the space $\mathbb{R}^n_q$* **not only by the point at infinity, ∞, corresponding to the point a but also by the isotropic cone C_∞ with the vertex at this point.**

Conformal inversion is implemented by $O(4,2)$ transformation $(\mathbf{x},t,v,w)\mapsto(\mathbf{x},t,-v,w)$. Conformal infinity of M is represented by generator lines of the quadric $Q(x)=0$ of the form $(\mathbf{x},t,v,v)$. According to Table 1, the light cone $\mathbf{x}^2=t^2$ is represented in E by generator lines of the form $[(\mathbf{x},t,1/2,-1/2)]$. Applying conformal inversion we get $[(\mathbf{x},t,-1/2,-1/2)]$. **Clearly the whole 2-sphere S^2 of** $[(\mathbf{x},1,0,0)]$, $\mathbf{x}^2=1$ **is missing. This two–sphere is located at conformal infinity and is pointwise invariant**

[5] We will assume the system of units in which c, the speed of light, is numerically equal to 1.

under the conformal inversion. The third myth is closely related, and it is usually summarized by giving to conformal infinity the name 'light cone at infinity' [16] or, sometimes, 'double light cone at infinity'. For instance in [**?**,] we can find the following paragraph:

> *From the point of view of the conformal structure of space-time, "points at infinity" can be treated on the same basis as finite points. Minkowski space can be completed to a highly symmetrical conformal manifold by the addition of a null cone at infinity- the "absolute cone".(...) Let x^μ be the position vector of a general event in Minkowski space-time relative to a given origin O. Then the transformation to new Minkowskian coordinates $\hat{x}^\mu$ given by*
>
> $$\hat{x}^\mu = \frac{x^\mu}{x_\alpha x^\alpha}, \quad x^\mu = \frac{\hat{x}^\mu}{\hat{x}_\alpha \hat{x}^\alpha}, \tag{3.1}$$
>
> *is conformal ("inversion with respect to O"). Observe that the whole null cone of O is transformed to infinity in the $\hat{x}^\mu$ system and that infinity in the x^μ system becomes the null cone of the origin $\hat{O}$ of the $\hat{x}^\mu$ system. ("Space–like" or "time-like" infinity become $\hat{O}$ itself but "null" infinity becomes spread out over the null cone of O.) Thus, from the conformal point of view* **"infinity" must be a null cone.**

For instance Huggett and Tod write [17, p. 36]:

> *"This is the intersection of N with a null hyperplane through the origin. All such hyperplanes are equivalent under $O(4, 2)$ so to see what these extra points represent, we consider the null hyperplane $v+w = 0$. ... we see that the points of M corresponding to generators of $\mathcal{N}$ which lie in this hyperplane are just the null cone of the origin. Thus $P\mathcal{N}$ consists of $\tau(M)$ with* **an extra cone at infinity***."*

Not only they write so in words, but also miss this S^2 in their formal analysis. A pictorial representation of the conformal infinity (suppressing just one dimension) is that of one of the degenerate cases of Dupin cyclides[6] - so called *needle (horn) cyclide* [18, Fig. 6, p. 80], [14, Fig. 5.7, p. 156], or, in French, *croissant simple* [19]: In fact we have the following theorem:

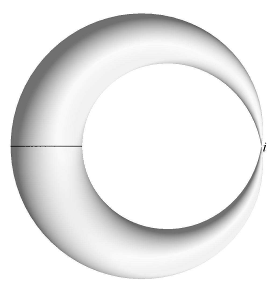


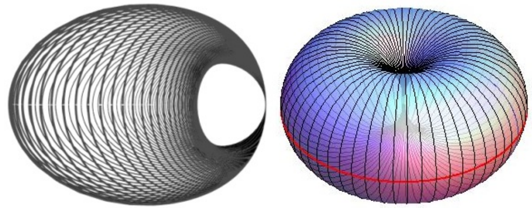
Figure 1: Pictorial representation of the conformal infinity with one dimension skipped. *'Double light cone at infinity'*, with endpoints identified. While topologically correct such a name is misleading as it suggests non differentiability at the base, where the two half-cones meet - cf. [1]. The ill-fated sphere S^2 is marked.

[6]The exact connection of these two concepts is not known to the present author at the time of this writing.

Theorem 3.0.1. *The conformal infinity of M, isomorphic do the set $\{U \in U(2) : \det(I - U) = 0\}$ can be described as the three-dimensional surface in $\mathbb{R}^5$, coordinates (t, x, y, z, v, w), defined by a pair of quadratic equations*

$$x^2 + y^2 + z^2 + v^2 = 1, \quad t^2 + w^2 = 1. \tag{3.2}$$

With one space dimension (say, z) skipped, it can be faithfully represented in $\mathbb{R}^3$ as a generalized, horned, Dupin's cyclide.



4 Concluding comments

We discussed, briefly, several interesting properties of the compactified Minkowski space - an important homogeneous space for the conformal group, the group of symmetries of Maxwell equations discovered long ago. Conformal group appears to be important in several areas of physics, mathematics, also in computer graphics and pattern recognition. It is quite possible that its full potential has not yet been exhausted. One of the interesting properties of the conformal group is that it is the first element in a sequence of group contractions [20]: conformal group, Poincaré group, Galilei group. The Lie algebra of the conformal group, as demonstrated by I. Segal in the final part of his 1951 paper [21][7], does not result as a limit of some other Lie algebra. Segal has constructed the foundations of his cosmological model [23] starting from the compactified Minkowski space and its universal covering space, a model with some difficulties, but also with some promises if connected with elementary particle physics (cf. [24], (see also R. I, Ingraham, via a somewhat different, but related path [25], and a recent paper extending the idea of Segal's chronogeometry in new directions, by A. V. Levichev [26]). Segal himself did not really touch the problem of placing gravity within his framework. But recent papers (cf. [26] and references therein) seem to point at the possibility of realizing gravity along the lines suggested also by Waldyr Rodrigues in his monograph written with V. V. Fernández [27]: gravitation is somehow related to 'quantum fluctuations of the vacuum'. The geometrical arena for such a description can be either Minkowski space or, what seems to be more attractive to the present author, one of the homogeneous spaces of the conformal group (or its covering).

The final remark concerns the question whether the choice of signature for Minkowski space, $(1, 1, 1, -1)$ vs. $(-1, -1, -1, 1)$ may have any physical significance. It seems that this question is unsettled yet. We refer the Reader to [28, 29, 30], where different aspects of this problem are being discussed.

5 Acknowledgements

Some of the content discussed in this paper was presented during one of the Author's talks at the 9th International Conference on Clifford Algebras and their Applications (ICCA9), 15-20 July 2011, Bauhaus-University Weimar, Germany. The author is grateful to A. v. Humboldt-Foundation for supporting the ICCA9 conference participation. QFG support during the preparation of this paper is acknowledged aith thanks. Thanks are also due to Pierre Anglès for reading the paper, his kind encouragement and advice, and for constructive and critical comments.

[7] Cf. also [22]

References

[1] Arkadiusz Jadczyk, *On Conformal Infinity and Compactifications of the Minkowski Space*, Advances in Applied Clifford Algebras, DOI: 10.1007/s00006-011-0285-5, 2011

[2] René Deheuvels, *Formes quadratiques et groupes classiques*, Presse Universitaires de France, Paris, 1981

[3] Leszek Z. Stolarczyk, *The Hodge Operator in Fermionic Fock Space*, Collect. Czech. Chem. Commun. **70**, 979–1016, 2005

[4] W. Kopczyński and L. S. Woronowicz, *A geometrical approach to the twistor formalism*, Rep. Math. Phys. Vol **2**, 35–51, 1971

[5] H. Bateman, *The Conformal Transformations of a Space of Four Dimensions and their Applications to Geometrical Optics*, Proc. London. Math. Soc. (ser. 2), **7**, 70–98, (1909)

[6] E. Cunningham, *The Principle of Relativity in Electrodynamics and Extension Thereof*, Proc. London. Math. Soc. (ser. 2), **8**, 77–98, (1910)

[7] H. Bateman, *The Transformation of the Electrodynamical Equations*, Proc. London. Math. Soc. (ser. 2), **8**, 223–264, (1910)

[8] René Deheuvels, *Tenseurs et Spineurs*, Presse Universitaires de France, Paris, 1993

[9] Pierre Anglès, *Conformal Groups in Geometry and Spin Structures*, Birkhauser, Progress in Mathematical Physics, Vol. 50, 2008

[10] N. Bourbaki, *Éléments de Mathématique, Algèbre, Chapitre 9*, Springer, Berlin, 2007

[11] Chevalley C., *The algebraic theory of Spinors*, Columbia University Press, New-York, 1954

[12] Sophus Lie, *Über Komplexe, inbesondere Linien- und Kugelkomplexe, mit Anwendung auf der Theorie der partieller Differentialgleichungen*, Math. Ann., **5**, 145?-208, 209?-256 (Ges. Abh. 2, 1?121), 1872

[13] Wilhelm Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie*, Vol. **3**, Springer-Verlag, Berlin, 1929

[14] Thomas E. Cecil, *Lie Sphere Geometry*, Springer–Verlag, New York, 2000

[15] Maks A. Akivis, Vladislav V. Goldberg, *Conformal Differential Geometry and its Generalizations*, A Wiley Interscience Publications, New York (1996)

[16] Roger Penrose, *The Light Cone at Infinity*, in “Relativistic Theories of Gravitation”, ed. L. Infeld, Pergamon Press, Oxford, 369–373, 1964

[17] S. A. Huggett, K. P. Tod, *An Introduction to Twistor Theory*, Cambridge Univ. Press, 1994

[18] Michael Scrott, Boris Odehnal, *Ortho-Circles or Dupin Cyclides*, Journal of Geometry and Graphics, **1**, 73–98, 2006

[19] Robert Ferréol, *CYCLIDE DE DUPIN, Dupin’s Cyclide, dupinsche Zyklide*, http://www.mathcurve.com/surfaces/cycliddedupin/cyclidededupin.shtml

[20] E. Inonu and E. P. Wigner, *On the contraction of groups and their representations*, Proc. NatAcad. Sci **39**, 510–524, 1953; cf also Erdal Inonu, *A Historical Note on Group Contractions*, http://www.ysfine.com/wigner/inonu.pdf

[21] I. E. Segal, *A class of operator algebras which are determined by groups*, Duke Math, J. **18**, 221–265, 1951

[22] I. E. Segal, *Maxwell's Equations in the Einstein Universe and Chronometric Cosmology*, Astrophysical Journal, Suppl. Series, **100**, 307–324, 1995

[23] I. E. Segal, *Mathematical Cosmology and Extragalactic Astronomy*, Academic Press, New York, 1976

[24] I. E. Segal, H. P. Jacobsen, B. Ørsted, S. M. Paneitz and B. Speh, *Covariant chronogeometry and extreme distances: Elementary particles*, Proc. Natl. Acad. Sci. USA, **9**, 5261–5265, 1981

[25] R. I. Ingraham, *Particle masses and the fifth dimension*, Ann. Found. Louis de Broglie, **29** Hors série, 989–1004

[26] A. V. Levichev, *Pseudo–Hermitian realization of the Minkowski world through DLF theory*, Phys. Scr. **83**, doi:10.1088/0031-8949/83/01/015101, 2011

[27] V. V. Fernández and W. A. Rodrigues Jr., *Gravitation as a Plastic Distortion of the Lorentz Vacuum*, Fundamental Theories of Physics 168, Springer, Heidelberg, 2010.

[28] William M. Pezzaglia Jr., John J. Adams, *Should Metric Signature Matter in Clifford Algebra Formulations of Physical Theories?*, `http://arxiv.org/abs/gr-qc/9704048`

[29] M. Berg, C. DeWitt-Morette, S. Gwo, E. Kramer, *The Pin Groups in Physics: C, P, and T*, `http://arxiv.org/abs/math-ph/0012006`

[30] David Miralles, Josep Manel Parra, and Jayme Vaz, Jr., *Signature Change and Clifford Algebras*, International Journal of Theoretical Physics, Vol. **40**,