Essay

The Modern Analysis of the Problem of Multisecting an Angle

Temur Z. Kalanov*

ABSTRACT

The work is devoted theoretical and practical analysis of an actual problem – the problem of multisecting (in particular, trisecting) an angle, i.e. the problem of division of a given arbitrary angle into the given set of equal parts using only a compasses and an unmarked straightedge. General statement of a problem is formulated. The mathematical analysis of the problem (within the framework of the theories of similarity of triangles and of similarity of concentric circles) and the logical analysis of the problem are proposed. It is proved that practical solution of the problem of multisecting an arbitrary angle with only a compasses and an unmarked straightedge is impossible because an arc cannot be transformed to the straight line segment with a compasses and a straightedge.

Key Words: philosophy of geometry, trisecting an angle, multisecting an angle.

1. Introduction

The problem of multisecting an angle (i.e. the problem of division of a given arbitrary angle into the given set of equal parts using only a compasses and an unmarked straightedge) is a modern geometrical problem. It arises as generalization of the classical problem in ancient Greek mathematics – the problem of trisecting an angle (i.e. the problem of division of a given arbitrary angle into three equal parts using only a compasses and an unmarked straightedge). As is known, the result of efforts over the centuries on the solution of the famous problem of trisecting an angle has been summed up in 19th century. In 1837 Pierre Wantzel has reduced the problem of trisecting an angle to the solution of the equation $x^3 - 3x - 2\cos\alpha = 0$ and has proved that the problem of trisecting an angle has no solution except for some special cases. However, the question why this problem has no general solution was not analyzed from logical and practical points of view. Therefore, the search of solution is continued till now.

The modern analysis of the works on the problem of trisecting an angle shows that unsuccessfulness of attempts to solve the problem completely, undertaken by mathematicians, is explained by three circumstances. Firstly, the idea of the general approach to the problem has not been guessed right. Secondly, the statement of the problem of trisecting an angle as a particular case of the problem of multisecting an angle has not been proposed. Thirdly, logical analysis of the problem of multisecting an angle has not been proposed. The purpose of the present work is to propose: (a) the idea of the general approach to the problem; (b) the theoretical (mathematical) analysis of the problem within the framework of the theory of similarity of triangles and of the theory of similarity of concentric circles; (c) the practical analysis of the problem within the framework of formal logic.

* Correspondence: Temur Z. Kalanov, Home of Physical Problems, Pisatelskaya 6a, 100200 Tashkent, Uzbekistan. E-mail: tzk_uz@yahoo.com, t.z.kalanov@mail.ru, t.z.kalanov@rambler.ru Note: This paper went through two rounds of peer reviews and was recommended to be published as an essay.

2. The theoretical analysis of the problem of multisecting an angle

As is known, operation of multiplication of the given angle by an integral number $n = 1, 2, 3, \ldots$ – direct operation – is always carried out, and operation of division of a given arbitrary angle into the given number of equal parts – inverse operation – represents known difficulty. In this connection, following idea of the approach to a geometrical problem of division of an angle arises. Operation of multiplication of one quantity should be connected with operation of division of other quantity by such relation that operation of multiplication of one quantity would lead to operation of division of other quantity. This idea is embodied, for example, into the theory and design of the proportional compasses used for practical division of a straight line segment into some equal parts. Thus, the idea of the general approach to the geometrical problem of multisecting an angle is that the general solution of this problem must be based on the theory of proportions and on the theory of similarity of geometrical objects.

Analysis of the theory of similarity of geometrical objects leads to the following general statement of the problem of multisecting an angle: taking into consideration the theory of similarity of concentric circles, one should find the relation between quantity of radius of a circle, length of an arc of a circle, and quantity of the central angle, which describes multisecting the central angle with multiplication of quantity of radius by integral number $n = 1, 2, 3, \ldots$.

The theoretical analysis of the problem of multisecting an angle is based on following known geometrical propositions.

1) **Definition**. The angle formed by two radiuses of the same circle is called central angle.

2) **Theorem**. If two central angles in the same circle are equal, then the arcs corresponding to them are equal.

3) **Theorem**. If two arcs in the same circle are equal, then the central angles corresponding to them are equal.

4) **Definition**. The straight line segment connecting two points of a circle is called chord.

5) **Theorem**. Equal arcs are tied by equal chords.

6) **Theorem**. Equal chords tie equal arcs.

7) **Theorem**. If diameter is perpendicular to a chord, then it divides this chord and arc tied by this chord into half.

8) **Theorem**. Diameter passed through the middle of the chord is perpendicular to the chord and divides the arcs tied by this chord into half.

9) **Theorem**. Diameter passed through the middle of the arc divides the chord which ties this arc into half and is perpendicular to this chord.

10) **Definition**. Triangles are called similar if their angles are equal in pairs and the like sides are proportional.

11) The theorems of similarity of triangles and the theorems of similarity of concentric circles.

12) Arcs of the given circle are measured by the arc on which central angle is leaned.

13) **Definition**. The ratio of length of an arc of a circle, for which given angle is central, to length of radius of this arc is called radian measure of angle.

14) If radius R of circle and radian measure α of arc are known, then one can calculate length l of this arc. Really, in accordance with definition of radian measure,

$$\alpha = \frac{l}{R}, \quad \text{or} \quad l = \alpha R .$$

15) Ratio of lengths of arcs of two concentric circles is ratio of lengths of their radiuses for the same central angle α:

$$\frac{l_i}{l_j} = \frac{R_i}{R_j}, \text{ or } \quad \frac{l_i}{R_i} = \frac{l_j}{R_j}, \qquad i,\ j = 1, 2, 3, ...$$

where $i,\ j$ are order numbers of concentric circles. In other words, the ratio of length of arc of circle to its radius does not depend on radius for the same central angle α. Quantity of this ratio is changed if the central angle is changed.

The proof of existence of the solution of the problem of multisecting an arbitrary angle $\alpha = const$, based on the theory of similarity of concentric circles, follow from these propositions. Really, if:

(a) two circles designated by the order numbers 1 and n are given,

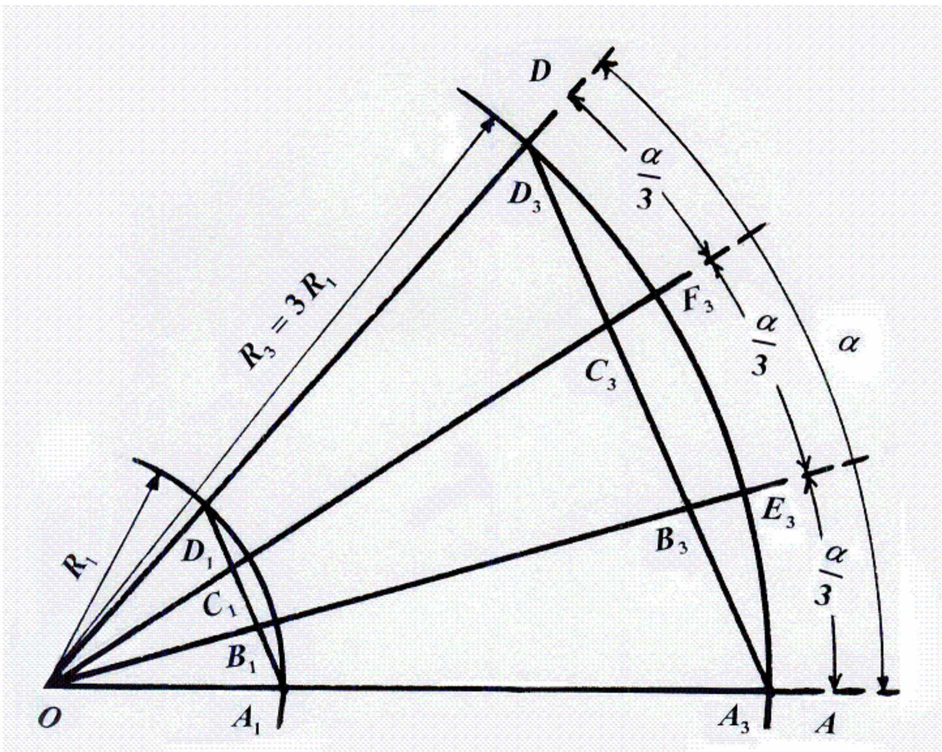
(b) the operation of multisection of an angle α is expressed by the ratios α/n, $l_n/n = R_n\alpha/n$, and $R_1/R_n = 1/n$ where $n = 1, 2, 3, ...$,

then the relations $l_1/l_n = 1/n$, α/n take place for arbitrary R_1. In other words, if $l_n/n = R_n\alpha/n$, $R_n = nR_1$, and $l_n = nl_1$, then length l_n of arc and angle α are divided into n equal parts: $l_n/l_1 = n$ and α/n under arbitrary R_1. Since the multiplication operation $R_n = nR_1$ is always carried out, the multiplication operation $l_n = nl_1$ and, consequently, the division operation α/n is always carried out as well .

Thus, the existence of the theoretical solution of the problem of multisecting an arbitrary angle α is proven.

3. The practical analysis of the problem of multisecting an angle

The practical analysis of the problem of multisecting an arbitrary angle α is based on the above-stated mathematical analysis and on formal logic. The purpose of the analysis is to prove that the practical solution of the problem of trisecting (multisecting) an arbitrary angle α exists. But this solution is reached with only special operation: namely, transformation of form of line. Therefore, transforming form of line with a compasses and an unmarked straightedge is impossible. The example of the practical solution is shown on the figure below.



The practical solution of the problem of trisecting
an arbitrary angle $\angle AOD = \alpha$

The practical solution of the problem of trisecting an arbitrary angle α consists of following operational steps:

1) Draw an arbitrary angle $\angle AOD = \alpha$, where the point O is the vertex of the angle.

2) Draw the first (basic) circle of an arbitrary radius R_1, where $n = 1$ is order number of the circle, O is the circle centre.

3) Draw the second (concentric) circle of the radius $R_3 = 3R_1$, centered in the point O, where $n = 3$ is order number of the circle. Radiuses R_1 and R_3 are connected by the relation $R_n / R_1 = n$, $n = 3$.

4) Designate the points of intersections of circles and of sides of angle $\angle AOD$ by the letters A_1, D_1 and A_3, D_3 .

5) Draw the chords $\overline{A_1 D_1}$ and $\overline{A_3 D_3}$. These chords are parallel each other.

6) Find the following proportions from similarity of triangles $\Delta\, A_1 O D_1$ and $\Delta\, A_3 O D_3$:

$$\frac{\overline{A_3 O}}{\overline{A_1 O}} = \frac{\overline{OD_3}}{\overline{OD_1}} = \frac{\overline{A_3 D_3}}{\overline{A_1 D_1}} = \frac{R_3}{R_1} .$$

7) In consecutive order, set down three segments $\overline{A_1 D_1}$ on chord $\overline{A_3 D_3}$ using compasses. Designate the formed points by letters B_3 and C_3.

8) Transform the arc $\cup A_3 D_3$ to the straight line segment $\overline{\cup A_3 D_3}$. This operation is called straightening the arc $\cup A_3 D_3$. The segment $\overline{\cup A_3 D_3}$ and the chord $\overline{A_3 D_3}$ are parallel each other.

9) Pass the segments $\overline{OE_3}$ and $\overline{OF_3}$ through the points B_3 and C_3 on the chord $\overline{A_3 D_3}$. The obtained segments $\overline{\cup A_3 E_3}$, $\overline{\cup E_3 F_3}$, and $\overline{\cup F_3 D_3}$ are equal each other.

10) Transform the segment $\overline{\cup A_3 D_3}$ to the arc $\cup A_3 D_3$. This operation is called bending the straight line to the arc.

11) Pass the segments $\overline{OE_3}$ and $\overline{OF_3}$ through the points E_3 and F_3 on arc $\cup A_3 D_3$.

12) Take into consideration that

$$\cup\, A_3 E_3 = \cup\, E_3 F_3 = \cup\, F_3 D_3 = \frac{\cup A_3 D_3}{3} .$$

Then

$$\angle\, A_3 O E_3 = \angle\, E_3 O F_3 = \angle\, F_3 O D_3 = \frac{\angle AOD}{3} = \frac{\alpha}{3} .$$

Thus, the problem of trisecting an arbitrary angle α has been practically solved. The obtained practical solution of the problem of trisecting an angle α signifies that operation of multisection an arbitrary angle α exists and is a trivial procedure now. The practical solution proves that a solution of the problem of trisecting (multisecting) an arbitrary angle α with only a compasses and an unmarked straightedge is impossible because an arc cannot be transformed to the straight line segment with a compasses and a straightedge.

4. Discussion

1. The problem of multisecting (in particular, trisecting) an angle, i.e. the problem of division of a given arbitrary angle into the given set of equal parts using only a compasses and an unmarked straightedge, differs essentially from the problem of multisecting an straight line. A straight line (as a simplest form of line) can be divided into a set of identical and of equal parts with only a compasses and an unmarked straightedge. However, an angle (as a system of two intersecting straight lines) and a curve line (for example, an arc of circle) cannot be divided into a set of identical and of equal parts with only a compasses and an unmarked straightedge in all cases because: firstly, a curve line is not a sum and consequence of set of straight line segments; secondly, a compasses and an straightedge determine distance between points along straight line. (Russian mathematician N. Lobachevsky was probably the first who realized this fact). Unsuccessfulness of attempts to solve the problem completely, undertaken by mathematicians, is explained by that a curve line cannot be transform to a straight line with a compasses and an straightedge.

2. Geometry studies the form of a material object. Form is a system of parts (elements). System (i.e. whole, made of parts) is a set of the elements which are in relations and connections with each other and form certain integrity, unity. Part (i.e. element, line) is characterized by a form (i.e. qualitative determinacy) and size (i.e. quantitative determinacy). Therefore, the problem of dividing the line has two aspects: qualitative and quantitative aspects. The qualitative aspect is decomposition of a given form in the set of non-identical forms (for example, Fourier expansion) or transformation of a given form to another form (for example, transformation of an arc to a straight line segment). Quantitative aspect is division of a given form into a set of identical forms. For example, decomposition of an arc in the set of identical arcs is a quantitative aspect (the quantitative operation), and decomposition of an arc in the set of chords or straightening of an arc is a qualitative aspect (qualitative operation). In carrying out the operation of division, one should take into consideration the following requirement of formal logic: the arc should be measured by a unit arc, the angle should be measured by a unit angle, and the straight line should be measured by a unit straight line. But if one takes into account only the lengths of the lines (for example, if one neglects the difference between the form of the arc and the form of the chord), the discrepancy between theoretical and practical solutions arise.

3. Discrepancy between theoretical and practical solutions does not arise if one can carry out transformation of forms of line.

5. Conclusion

Thus, it has been proved that the problem of multisecting (in particular, trisecting) an arbitrary angle has the theoretical and practical solutions in all cases. The proposed solution of the problem is based on known geometrical propositions, formal logic, and practice. Consequently, it represents scientific truth.

Scientific and cognitive significance of my work is that my work promotes a critical analysis of classical geometry within the framework of the correct methodological basis – unity of formal logic and of rational dialectics. In order to understand why theoretical geometry is differs from practical geometry, or, in general, why theory differs from practice, one should analyze the geometry on the base of the philosophical categories “theory and practice”, "content and form", "quality and quantity", "part and whole". Then this will lead to the realization of Einstein's opinion that theoretical geometry (for example, axiomatic approach) should contain definitions of concepts and should represent a field of natural sciences.

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