Article

Energy momentum pseudo-tensors in n-dimensional space-time V_n

Sanjay R. Bhoyar[#] & A. G. Deshmukh[&]

[#]Department of Mathematics, College of Agriculture, Darwha 445 202, India
[&]Dept. of Mathematics, Govt. Vidarbha Inst. of Science and Humanities, Amravati,Joint Director (Higher education),Nagpur Division ,Nagpur, India

Abstract

This paper, which asserts the existence of *Z*= (*z* - *t*) type plane gravitational waves carrying some energy and momentum in the direction of their propogation in n-dimensional space-time V_n.

Keywords: metric tensor, plane gravitational waves, energy momentum pseudo tensor of Einstein, energy momentum tensor of Landau and Lipschitz. .

Introduction

The plane gravitational waves g_{ij} are mathematically exposed by H.Takeno [1] in general relativity. He has studied $(z-t)$and (t/z)- type plane gravitational waves and obtained the line element for both waves .The work of Takeno has been carried out by Adhao and Karde [2] to higher dimension V_5 and V_6 by deducing the line elements for both $Z=(z-t)$and $Z=(t/z)$- type purely plane gravitational waves . Futhermore the work of Adhav and Karade [2] extended to n-dimensions by Thengne and Karade [3], Zade and Karde [4] by reformulating the plane gravitational waves in V_n. Bhoyar and Deshmukh [5] deduced the metric in V_n for $Z=(z-t)$type plane gravitational waves.

H.Takeno [1] investigated that both $Z=(z-t)$and $Z=(t/z)$- type purely plane gravitational waves carries some energy and momentum in the direction of their propogation by calculating non-vanishing components of energy momentum pseudo tensors of Einstein and Landau Lipshitz and in four dimensional space-time V_4.Extension of this work has been carried out by Gawande and Kandalkar[6], Bhoyar and Deshmukh [7] in V_5 and V_6 respectively in case of $Z=(z-t)$-type purely plane gravitational waves. In this paper we have shown that $Z=(z-t)$-type plane gravitational waves can also carry some energy and momentum in the direction of their propagation in n-dimensional space-time V_n introduced by Bhoyar and Deshmukh [5]. Surprisingly all the results are retain in format of Takeno [1].

Definition

We annex the definition of plane gravitational waves as detailed in Thengane and Karade [3] for n- dimensional space-time as follows:

A plane wave g_{ij} is a non-flat solution of the field equations

$$R_{ij} = 0, \qquad i, j = 1,2,..n. \tag{1}$$

in any empty region of the space- time

$$ds^2 = g_{ij} dx^i dx^j \tag{2}$$

with $g_{ij} = g_{ij}(Z),\ Z = Z(x_1, x_2, .. x_{n-1}, t)$ where $t = x_n,\ z = x_{n-1}$.

in some suitable co-ordinate system such that

$$g^{ij} Z_{,i} Z_{,j} = 0, \quad Z_{,i} = \partial Z / \partial x^i \tag{3}$$

and

$$Z = Z(x_{n-1}, t) \text{ such that } Z_{,(n-1)} \neq 0, \quad Z_{,n} \neq 0. \tag{4}$$

The signature convention adopted is,

$$g_{\mu\mu} < 0, \begin{vmatrix} g_{\mu\mu} & g_{\mu\nu} \\ g_{\nu\mu} & g_{ww} \end{vmatrix} > 0, \quad \begin{vmatrix} g_{\mu\mu} & g_{\mu\nu} & g_{\mu w} \\ g_{\nu\mu} & g_{\nu\nu} & g_{\nu w} \\ g_{\mu w} & g_{w\nu} & g_{ww} \end{vmatrix} < 0, \ldots,$$

(not summed for μ, ν, w = 1,2,…,(*n*-1)).

Any determinant of order $(n-2) = \begin{cases} > 0, \text{ when } n \text{ is even} \\ < 0, \text{ when } n \text{ is odd.} \end{cases}$

And $$\begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ g_{(n-1)1} & g_{(n-1)2} & \cdots & g_{(n-1)(n-1)} \end{vmatrix} < 0, \text{ when } n \text{ is even,} \tag{5}$$

$$\begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ g_{(n-1)1} & g_{(n-1)2} & \cdots & g_{(n-1)(n-1)} \end{vmatrix} > 0, \quad \text{when } n \text{ is odd,}$$

$$g_{nn} > 0.$$

Denote $$g = \det g_{ij}. \tag{6}$$

$$g = \begin{cases} > 0, \text{ when } n \text{ is odd} \\ < 0, \text{ when } n \text{ is even.} \end{cases} \tag{7}$$

The n-Dimensional Plane Wave Metric

Adopting the space-time deduced by us [5] for $Z = (z - t)$-type plane gravitational waves in n-dimensional space-time

$$\begin{aligned} ds^2 = & -J_{11}(dx^1)^2 - 2J_{12}(dx^1)(dx^2) - 2J_{13}(dx^1)(dx^3) - 2J_{14}(dx^1)(dx^4) \ldots - 2J_{1(n-2)}(dx^1)(dx^{n-2}) \\ & -2J_{22}(dx^2)^2 - 2J_{23}(dx^2)(dx^3) - 2J_{24}(dx^2)(dx^4) - 2J_{25}(dx^2)(dx^5) - 2J_{2(n-2)}(dx^2)(dx^{n-2}) \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & - J_{(n-2)(n-2)}(dx^{n-2})^2 - (C-D)(dx^{n-1})^2 - 2D(dx^{n-1})(dx^n) + (C+D)(dx^n)^2. \end{aligned} \tag{8}$$

Where $J_{11}, J_{12}, \ldots, J_{(n-2)(n-2)}$ C *and* D are function of Z with $C > |D|$ satisfying (2.5), (2.6) and (5.3.8) of Zade [4] i.e

$$\overline{L_2} - \overline{\rho}_n + \frac{\rho_n^{\ 2}}{2} - L_2 \rho_n + \frac{L_1}{4} = 0.$$

On the line of Takeno (1961) by putting $C = 1$ and $D = 0$, the metric (8) reduces to (9) as

$$\begin{aligned} ds^2 = & -J_{11}(dx^1)^2 - 2J_{12}(dx^1)(dx^2) - 2J_{13}(dx^1)(dx^3) - 2J_{14}(dx^1)(dx^4) \ldots - 2J_{1(n-2)}(dx^1)(dx^{n-2}) \\ & -2J_{22}(dx^2)^2 - 2J_{23}(dx^2)(dx^3) - 2J_{24}(dx^2)(dx^4) - 2J_{25}(dx^2)(dx^5) - 2J_{2(n-2)}(dx^2)(dx^{n-2}) \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & - J_{(n-2)(n-2)}(dx^{n-2})^2 - (dx^{n-1})^2 + (dx^n)^2. \end{aligned} \tag{9}$$

The components of metric tensor g_{ij} for (9) are as follows

$$[g_{ij}] = \begin{bmatrix} -J_{11} & -J_{12} & \cdots & -J_{1(n-2)} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ -J_{(n-2)1} & -J_{(n-2)2} & \cdots & -J_{(n-2)(n-2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

Where $g = \det.(g_{ij})$

$$g = \begin{vmatrix} -J_{11} & -J_{12} & \dots & -J_{1(n-2)} & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 & 0 \\ -J_{(n-2)1} & -J_{(n-2)2} & \dots & -J_{(n-2)(n-2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}. \tag{10}$$

Simplifying, we get $g = mn = \begin{cases} > 0, \text{ when } n \text{ is even} \\ < 0, \text{ when } n \text{ is odd.} \end{cases}$

Where $m = \begin{vmatrix} -J_{11} & -J_{12} & \dots & -J_{1(n-2)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -J_{(n-2)1} & -J_{(n-2)2} & \dots & -J_{(n-2)(n-2)} \end{vmatrix}$ and $n = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1.$ (11)

Some Useful Formulae

For metric (9), we made following formulae which are useful to calculate the components of pseudo tensors.

$\sqrt{-g} = \sqrt{m}, \left(\sqrt{-g}\right)_{,i} = \left(0,0,0,0,\dots,\Psi\sqrt{m}, -\Psi\sqrt{m}\right)$, where $\Psi = \left(\overline{m}/2m\right)$ *and* $\overline{m}$ means derivative of m with respect to Z .

$$\begin{aligned} &\begin{Bmatrix} i \\ (n-2)i \end{Bmatrix} = 0, \quad \begin{Bmatrix} i \\ (n-1)i \end{Bmatrix} = -\begin{Bmatrix} i \\ ni \end{Bmatrix} = \Psi, \\ &\begin{Bmatrix} b \\ a(n-1) \end{Bmatrix} = -\begin{Bmatrix} a \\ b(n-1) \end{Bmatrix}, \quad g^{pq}\begin{Bmatrix} r \\ pq \end{Bmatrix} = 0, \quad g^{pq}\begin{Bmatrix} r \\ ps \end{Bmatrix}\begin{Bmatrix} s \\ qr \end{Bmatrix} = 0, \\ &\begin{Bmatrix} p \\ ac \end{Bmatrix}\begin{Bmatrix} d \\ bp \end{Bmatrix} = 0, \quad \begin{Bmatrix} b \\ (n-1)a \end{Bmatrix}\begin{Bmatrix} a \\ (n-1)b \end{Bmatrix} = -\begin{Bmatrix} b \\ (n-1)a \end{Bmatrix}\begin{Bmatrix} q \\ nb \end{Bmatrix} = \begin{Bmatrix} b \\ na \end{Bmatrix}\begin{Bmatrix} a \\ nb \end{Bmatrix}, \\ &\begin{Bmatrix} q \\ (n-1)p \end{Bmatrix}\begin{Bmatrix} p \\ (n-1)q \end{Bmatrix} = -\begin{Bmatrix} q \\ (n-1)p \end{Bmatrix}\begin{Bmatrix} p \\ nq \end{Bmatrix} = \begin{Bmatrix} q \\ np \end{Bmatrix}\begin{Bmatrix} p \\ nq \end{Bmatrix}, \end{aligned} \tag{12}$$

Where

i = 1,2,3,……. ,n ; a, b, c, d = 1, 2, 3, … (n-2) ; p, q, r, s = (n-1),n and summation convention is used with respect to these indices.

Pseudo-tensor of Einstein

Using expressions (9)-(12), we calculate the components of energy-momentum Pseudo-tensor $t_i^{\cdot j}$ introduced by Einstein

$$16\pi\sqrt{-g}\ t_i^{\cdot j} = \begin{Bmatrix} j \\ mn \end{Bmatrix}\left(\sqrt{-g}\,g^{mn}\right)_{,i} - \left(\log\sqrt{-g}\right)_{,m} + \delta_i^j\left[\begin{Bmatrix} h \\ mk \end{Bmatrix}\begin{Bmatrix} k \\ nh \end{Bmatrix} g^{mn}\sqrt{-g} - g^{mn}\begin{Bmatrix} h \\ mn \end{Bmatrix}\left(\sqrt{-g}\right)_{,h}\right] \quad . \tag{13}$$

With the components of Christoffel's symbol made from (9) (calculations are omitted for brevity sake), (11) and (12), expression (13) gives,

$$t_{n-1}^{\cdot n-1} = -t_{n-1}^{\cdot n} = -t_{n-1}^{\cdot n} = t_n^{\cdot n} = \Omega,\ \text{Other}\ \ t_i^{\cdot j} = 0, \tag{14}$$

Where

$$\Omega = \frac{\tau}{16\pi m} \quad \text{and}$$

$$\tau = \begin{vmatrix} -\bar{J}_{11} & -\bar{J}_{12} & \cdots & -\bar{J}_{1(n-2)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -\bar{J}_{(n-2)1} & -\bar{J}_{(n-2)2} & \cdots & -\bar{J}_{(n-2)(n-2)} \end{vmatrix}.$$

Here Ω is a function of z and does not vanish in general.

Pseudo-Tensor of Landau and Lifshitz

Now we calculate the components of the symmetric energy momentum pseudo-tensor t_{ij} proposed by Landau and Lipshitz given by expression

$$\begin{aligned} 16\pi t^{*ij} = &\left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\left[2\begin{Bmatrix} h \\ kl \end{Bmatrix}\begin{Bmatrix} m \\ hm \end{Bmatrix} - \begin{Bmatrix} m \\ kh \end{Bmatrix}\begin{Bmatrix} h \\ lm \end{Bmatrix} - \begin{Bmatrix} h \\ kh \end{Bmatrix}\begin{Bmatrix} m \\ lm \end{Bmatrix}\right] \\ &+ g^{ik}g^{mn}\left[\begin{Bmatrix} i \\ kh \end{Bmatrix}\begin{Bmatrix} h \\ mn \end{Bmatrix} + \begin{Bmatrix} i \\ mn \end{Bmatrix}\begin{Bmatrix} h \\ kh \end{Bmatrix} - \begin{Bmatrix} j \\ nh \end{Bmatrix}\begin{Bmatrix} h \\ km \end{Bmatrix} - \begin{Bmatrix} j \\ km \end{Bmatrix}\begin{Bmatrix} h \\ nh \end{Bmatrix}\right] \\ &+ g^{jk}g^{mn}\left[\begin{Bmatrix} i \\ kh \end{Bmatrix}\begin{Bmatrix} h \\ mn \end{Bmatrix} + \begin{Bmatrix} i \\ mn \end{Bmatrix}\begin{Bmatrix} h \\ kh \end{Bmatrix} - \begin{Bmatrix} i \\ nh \end{Bmatrix}\begin{Bmatrix} h \\ km \end{Bmatrix} - \begin{Bmatrix} i \\ km \end{Bmatrix}\begin{Bmatrix} h \\ nh \end{Bmatrix}\right] \\ &- g^{hk}g^{mn}\left[\begin{Bmatrix} i \\ hm \end{Bmatrix}\begin{Bmatrix} j \\ kn \end{Bmatrix} - \begin{Bmatrix} i \\ hk \end{Bmatrix}\begin{Bmatrix} j \\ mn \end{Bmatrix}\right], \qquad \left(t^{*ij} = t^{*ji}\right) \quad . \end{aligned} \tag{15}$$

With the components of Christoffel's symbol made from (9), (11) and (12), expression (15) we obtained the following result:

$$t^{*(n-1)(n-1)} = t^{*(n-1)n} = t^{*n(n-1)} = t^{*nn} = \Omega^*,\ \text{other}\, t^{*ij} = 0\,. \tag{16}$$

Where

$$\Omega^* = -\frac{\left\{\tau + \dfrac{\overline{m}^2}{2m}\right\}}{16\pi\, m}, \tag{17}$$

$$\tau = \begin{vmatrix} -\overline{J}_{11} & -\overline{J}_{12} & \cdots & -\overline{J}_{1(n-2)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -\overline{J}_{(n-2)1} & -\overline{J}_{(n-2)2} & \cdots & -\overline{J}_{(n-2)(n-2)} \end{vmatrix}$$

Again $\Omega*$ is not zero in general.

$$-16\pi\,\Omega = \frac{\left(-\overline{\overline{m}} + \dfrac{\overline{m}^2}{2m}\right)}{m} \text{ and } 16\pi\,\Omega^* = -\frac{\overline{\overline{m}}}{2m}. \tag{18}$$

Again these values are functions of *Z* and do not vanish in general.

Conclusions

We conclude that:

i] If the assertion of the energy momentum pseudo tensors (13) or (15) that $t_i^{;j}$ or t^{*ij} expresses the energy- momentum due to the gravitational field is correct, *hence the gravitational waves given by* (9) *carry some energy and momentum in the direction of their propogation in* V_n.

ii] From our investigations the results deduced by Bhoyar et.al [7], Gawande et.al [6] and H.Takeno [1] were easily obtained by taking n=6, n=5 and n=4 respectively.

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