Article

Energy momentum pseudo-tensors in n-dimensional space-time $V_n$

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Abstract

This paper, which asserts the existence of $Z = (z - t)$ type plane gravitational waves carrying some energy and momentum in the direction of their propagation in n-dimensional space-time $V_n$.

Keywords: metric tensor, plane gravitational waves, energy momentum pseudo tensor of Einstein, energy momentum tensor of Landau and Lipschitz.

Introduction

The plane gravitational waves $g_{ij}$ are mathematically exposed by H.Takeno [1] in general relativity. He has studied $(z - t)$ and $(t/z)$- type plane gravitational waves and obtained the line element for both waves. The work of Takeno has been carried out by Adhao and Karde [2] to higher dimension $V_5$ and $V_6$ by deducing the line elements for both $Z = (z - t)$ and $Z = (t/z)$ - type purely plane gravitational waves. Furthermore the work of Adhav and Karade [2] extended to n-dimensions by Thengane and Karade [3], Zade and Karde [4] by reformulating the plane gravitational waves in $V_n$. Bhoyar and Deshmukh [5] deduced the metric in $V_n$ for $Z = (z - t)$ type plane gravitational waves.

H.Takeno [1] investigated that both $Z = (z - t)$ and $Z = (t/z)$- type purely plane gravitational waves carries some energy and momentum in the direction of their propagation by calculating non-vanishing components of energy momentum pseudo tensors of Einstein and Landau Lipshitz and in four dimensional space-time $V_4$. Extension of this work has been carried out by Gawande and Kandalkar[6], Bhoyar and Deshmukh [7] in $V_5$ and $V_6$ respectively in case of $Z = (z - t)$-type purely plane gravitational waves. In this paper we have shown that $Z = (z - t)$-type plane gravitational waves can also carry some energy and momentum in the direction of their propagation in n-dimensional space-time $V_n$ introduced by Bhoyar and Deshmukh [5]. Surprisingly all the results are retain in format of Takeno [1].

Definition

We annex the definition of plane gravitational waves as detailed in Thengane and Karade [3] for n-dimensional space-time as follows:
A plane wave $g_{ij}$ is a non-flat solution of the field equations

$$R_{ij} = 0, \quad i, j = 1, 2, \ldots n. \quad (1)$$

in any empty region of the space-time

$$ds^2 = g_{ij}dx^i dx^j \quad (2)$$

with $g_{ij} = g_{ij}(Z)$. $Z = Z(x_1, x_2, \ldots x_{n-1}, t)$ where $t = x_n$, $z = x_{n-1}$.

in some suitable co-ordinate system such that

$$g^{ij}Z_{,i}Z_{,j} = 0, \quad Z_{,i} = \partial Z / \partial x^i \quad (3)$$

and

$$Z = Z(x_{n-1}, t) \text{ such that } Z_{,(n-1)} \neq 0, \quad Z_{,n} \neq 0. \quad (4)$$

The signature convention adopted is,

$$g_{\mu\mu} < 0, \begin{vmatrix} g_{\mu\mu} & g_{\mu\nu} \\ g_{\nu\mu} & g_{\nu\nu} \end{vmatrix} > 0, \begin{vmatrix} g_{\mu\mu} & g_{\mu\nu} & g_{\mu\nu} \\ g_{\nu\mu} & g_{\nu\nu} & g_{\nu\nu} \\ g_{\nu\mu} & g_{\nu\nu} & g_{\nu\nu} \end{vmatrix} < 0, \ldots ,$$

(not summed for $\mu, \nu, w = 1, 2, \ldots (n-1)$).

Any determinant of order $(n - 2) = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ g_{(n-1)1} & g_{(n-1)2} & \cdots & g_{(n-1)(n-1)} \end{vmatrix}$

And

$$\begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ g_{(n-1)1} & g_{(n-1)2} & \cdots & g_{(n-1)(n-1)} \end{vmatrix} < 0, \text{ when } n \text{ is even,} \quad (5)$$

$$\begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ g_{(n-1)1} & g_{(n-1)2} & \cdots & g_{(n-1)(n-1)} \end{vmatrix} > 0, \text{ when } n \text{ is odd,}$$

$$g_{nn} > 0.$$

Denote

$$g = \det g_{ij}. \quad (6)$$
\[ g = \begin{cases} >0, \text{ when } n \text{ is odd} \\ <0, \text{ when } n \text{ is even.} \end{cases} \tag{7} \]

### The n-Dimensional Plane Wave Metric

Adopting the space-time deduced by us \([5]\) for \(Z=(z-t)\)-type plane gravitational waves in \(n\)-dimensional space-time

\[
d s^2 = -J_{11}(dx^1)^2 - 2J_{12}(dx^1)(dx^2) - 2J_{13}(dx^1)(dx^3) - 2J_{14}(dx^1)(dx^4) \ldots - 2J_{1(n-2)}(dx^1)(dx^{n-2}) \\
- 2J_{22}(dx^2)^2 - 2J_{23}(dx^2)(dx^3) - 2J_{24}(dx^2)(dx^4) \ldots - 2J_{2(n-2)}(dx^2)(dx^{n-2}) \\
\ldots \\
- J_{(n-2)(n-2)}(dx^{n-2})^2 - (C-D)(dx^{n-1})^2 - 2D(dx^{n-1})(dx^n) + (C+D)(dx^n)^2. \tag{8} \]

Where \(J_{11}, J_{12}, \ldots, J_{(n-2)(n-2)}\) and \(C, D\) are function of \(Z\) with \(C > |D|\) satisfying (2.5), (2.6) and (5.3.8) of Zade [4] i.e

\[
\bar{L}_z = \bar{\rho}_n + \frac{\rho_n^2}{2} - L_z \bar{\rho}_n + \frac{L_z}{4} = 0. \]

On the line of Takeno (1961) by putting \(C=1\) and \(D=0\), the metric (8) reduces to (9) as

\[
d s^2 = -J_{11}(dx^1)^2 - 2J_{12}(dx^1)(dx^2) - 2J_{13}(dx^1)(dx^3) - 2J_{14}(dx^1)(dx^4) \ldots - 2J_{1(n-2)}(dx^1)(dx^{n-2}) \\
- 2J_{22}(dx^2)^2 - 2J_{23}(dx^2)(dx^3) - 2J_{24}(dx^2)(dx^4) \ldots - 2J_{2(n-2)}(dx^2)(dx^{n-2}) \\
\ldots \\
- J_{(n-2)(n-2)}(dx^{n-2})^2 - (dx^{n-1})^2 + (dx^n)^2. \tag{9} \]

The components of metric tensor \(g_{ij}\) for (9) are as follows

\[
[g_{ij}] = \begin{bmatrix}
-J_{11} & -J_{12} & \ldots & -J_{1(n-2)} & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & 0 & 0 & 0 \\
-J_{(n-2)1} & -J_{(n-2)2} & \ldots & -J_{(n-2)(n-2)} & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix},
\]

Where \(g = \det (g_{ij})\)
\[ g = \begin{pmatrix}
- J_{11} & - J_{12} & \cdots & - J_{1(n-2)} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
- J_{(n-2)1} & - J_{(n-2)2} & \cdots & - J_{(n-2)(n-2)} & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \] (10)

Simplifying, we get \( g = mn = \begin{cases} > 0, & \text{when } n \text{ is even} \\ < 0, & \text{when } n \text{ is odd.} \end{cases} \)

Where \( m = \begin{pmatrix}
- J_{11} & - J_{12} & \cdots & - J_{1(n-2)} \\
\vdots & \vdots & \ddots & \vdots \\
- J_{(n-2)1} & - J_{(n-2)2} & \cdots & - J_{(n-2)(n-2)}
\end{pmatrix} \) and \( n = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1. \) (11)

**Some Useful Formulae**

For metric (9), we made following formulae which are useful to calculate the components of pseudo tensors.

\[ \sqrt{-g} = \sqrt{m} \left( \sqrt{-g} \right)_i = \left( 0,0,0,0,\ldots,\Psi \sqrt{m}, -\Psi \sqrt{m} \right), \] where \( \Psi = \left( m/2m \right) \) and \( m \) means derivative of \( m \) with respect to \( Z \).

\[ \begin{cases}
\frac{i}{(n-2)i} = 0, & \frac{i}{(n-1)i} = \frac{i}{ni} = \Psi, \\
\frac{a}{b(n-1)} = \frac{b}{a(n-1)} = \frac{r}{pq} = 0, & g_{pq} = g_{pr} = g_{qr} = 0, \\
\frac{p}{ac} = 0, & \frac{d}{bp} = 0, \quad \frac{a}{b(n-1)a} = \frac{b}{(n-1)b} = \frac{q}{nb} = \frac{b}{na} = \frac{a}{nb}, \\
\frac{q}{(n-1)p} = \frac{p}{(n-1)q} = \frac{p}{nq} = \frac{q}{np} = \frac{p}{np} = \frac{p}{nq}. \\
\end{cases} \] (12)

Where

\[ i = 1,2,3,\ldots,n; \quad a, b, c, d = 1, 2, 3, \ldots, (n-2); \quad p, q, r, s = (n-1)n \text{ and summation convention is used with respect to these indices}. \]
Pseudo-tensor of Einstein

Using expressions (9)-(12), we calculate the components of energy-momentum pseudo-tensor $t^i_j$, introduced by Einstein

$$16\pi \sqrt{-g} \ t^i_j = \left\{ \begin{array}{c} \left( \frac{j}{m} \right) \left( \sqrt{-g} g^{mn} \right)_{,i} - \left( \log \sqrt{-g} \right)_{,m} + \\
\delta^j_i \left[ \begin{array}{c} h \\ m \\
\begin{array}{c} k \\ n \end{array} \\
\begin{array}{c} g^{mn} \sqrt{-g} - g^{mn} \left( -g \right) \end{array} \\
\begin{array}{c} k \\ h \end{array} \end{array} \right] \\
\end{array} \right\}.$$  \ 
(13)

With the components of Christoffel’s symbol made from (9) (calculations are omitted for brevity sake), (11) and (12), expression (13) gives,

$$t^{n-1}_{n-1} = -t^n_{n-1} = -t^n_n = t^n_n = \Omega, \ \text{Other } t^i_j = 0,$$  \ 
(14)

Where

$$\Omega = \frac{\tau}{16\pi m} \ \text{and}$$

$$\tau = \begin{vmatrix}
-\bar{J}_{11} & -\bar{J}_{12} & \ldots & -\bar{J}_{1(n-2)} \\
\ldots & \ldots & \ldots & \ldots \\
-\bar{J}_{(n-2)1} & -\bar{J}_{(n-2)2} & \ldots & -\bar{J}_{(n-2)(n-2)}
\end{vmatrix}.$$  

Here $\Omega$ is a function of $Z$ and does not vanish in general.

Pseudo-Tensor of Landau and Lifshitz

Now we calculate the components of the symmetric energy momentum pseudo-tensor $t^i_j$ proposed by Landau and Lifshitz given by expression

$$16\pi t^i_j = \left( \frac{g^{ik} g^{jl} - g^{ij} g^{kl}}{2} \right) \left[ \begin{array}{c} h \\ k \\
\begin{array}{c} m \\ l \end{array} \\
\begin{array}{c} h \\ k \end{array} \end{array} \right] + \left[ \begin{array}{c} i \\ k \\
\begin{array}{c} m \\ n \end{array} \\
\begin{array}{c} i \\ k \end{array} \end{array} \right] + \left[ \begin{array}{c} j \\ h \\
\begin{array}{c} k \\ m \end{array} \\
\begin{array}{c} j \\ h \end{array} \end{array} \right] - \left[ \begin{array}{c} i \\ h \\
\begin{array}{c} k \\ m \end{array} \\
\begin{array}{c} i \\ h \end{array} \end{array} \right] - \left[ \begin{array}{c} j \\ k \\
\begin{array}{c} m \\ n \end{array} \\
\begin{array}{c} j \\ k \end{array} \end{array} \right]$$  \ 
(15)

With the components of Christoffel’s symbol made from (9), (11) and (12), expression (15) we obtained the following result:

$$t^{(n-1)(n-1)} = t^{n(n-1)} = t^{*nn} = \Omega^*, \ \text{other } t^i_j = 0.$$  \ 
(16)
\[ \Omega^* = -\frac{\tau + \frac{m^2}{2m}}{16\pi m}, \]

\( \tau = \begin{vmatrix} -J_{11} & -J_{12} & \cdots & -J_{1(n-2)} \\ \vdots & \ddots & \vdots & \vdots \\ -J_{(n-2)1} & -J_{(n-2)2} & \cdots & -J_{(n-2)(n-2)} \end{vmatrix} \]

Again \( \Omega^* \) is not zero in general.

\[ -16\pi \Omega = \frac{-m + \frac{m^2}{2m}}{m} \quad \text{and} \quad 16\pi \Omega^* = -\frac{m}{2m}. \]

Again these values are functions of \( Z \) and do not vanish in general.

**Conclusions**

We conclude that:

i] If the assertion of the energy momentum pseudo tensors (13) or (15) that \( t_i^j \) or \( t^{ij} \) expresses the energy- momentum due to the gravitational field is correct, hence the gravitational waves given by (9) carry some energy and momentum in the direction of their propagation in \( V_n \).

ii] From our investigations the results deduced by Bhoyar et.al [7], Gawande et.al [6] and H.Takeno [1] were easily obtained by taking \( n=6, n=5 \) and \( n=4 \) respectively.

**References**


