**Article**

# Proposed Coordinate System in which $g_{aa}$=0 and the Line Element for Plane Gravitational Waves of Type Z=($t_1$-$t_2$+$t_3$)/(√3*z*)

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## Abstract

For the study of $t/z$ type plane gravitational waves, we have considered the six-dimensional space-time with three time axes. We have chosen the plane wave of type $Z=\frac{t_1-t_2+t_3}{\sqrt{3}z}$ and made the essential investigations in the paper [6]. Adopting the same notations and results as in above paper, and using the transformations given in the proof of Theorem 1 below, we have proved that the co-ordinate system in which $g_{a\alpha}=0$, $(a=1,2\ ;\ \alpha=3,4,5,6)$ can be obtained in the six-dimensional space-time, which is our proposed co-ordinate system. In this paper, we have proved the three necessary theorems supporting the existence of such a co-ordinate system. The whole work is on the lines of H. Takeno (1961). The consequent line element in the form $ds^2=g_{mn}dx^m dx^n$ is also obtained for $m,n=1,...,6$.

**Keywords:** general relativity; plane gravitational waves; three time axes; line element.

## 1. Introduction

In general theory of relativity, Takeno (1961) [1] studied rigorously the plane gravitational waves $g_{ij}(Z)$ ; $i,j=1,2,3,4$ ; $Z=Z(z,t)$ and obtained numerous results, one of which is the co-ordinate system in which $g_{a\alpha}=0$, $a=1,2$; $\alpha=3,4$; for $Z=(z-t)$ and $Z=(t/z)$ types of waves. Thengane and Karade (2000) [2] extended the work for five dimensional space-time with two time axes, and obtained the plane wave solutions of the Einstein's field equation by choosing the co-ordinate system in which $g_{a\alpha}=0$; for $a=1,2$; $\alpha=3,4,5$. Khapekar and Deshmukh [3] determined this chosen co-ordinate system. Pawar, Bhaware and Deshmukh [4] obtained the plane wave solutions of the Einstein's field equations in six dimensions with three time axes. Bhaware, Pawar and Deshmukh [5] obtained the plane wave solutions for $Z=\frac{t_1-t_2+t_3}{\sqrt{3}z}$ type

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waves assuming the co-ordinate system in which $g_{a\alpha}=0$; for $a=1,2$; $\alpha=3,4,5,6$. Pawar et al [6] had obtained the line element for $(z-t)$ type waves $Z=\left[z-\frac{1}{\sqrt{3}}(t_1-t_2+t_3)\right]$. In the present paper we have obtained this assumed co-ordinate system and the line element for the plane gravitational waves $Z=\frac{t_1-t_2+t_3}{\sqrt{3}z}$.

$$\left.\begin{aligned} &\text{If } g_{ij} \text{ satisfies, } g_{ij}=g_{ij}(Z),\ Z=Z(x^i),\\ &\phi_1\omega^3+\omega^4+\phi_2\omega^5+\phi_3\omega^6=0,\\ &\rho_a=\bar{g}_{ai}\omega^i=0, \end{aligned}\right\} \quad (1.1)$$

$$\text{and} \quad Z=\frac{t_1-t_2+t_3}{\sqrt{3}\,z}, \quad (1.2)$$

then by suitable co-ordinate transformation we get $g_{a\alpha}=0$.

## 2. Preliminaries

H. Takeno (1961) has defined the plane gravitational waves as follows.

A plane gravitational wave $g_{ij}$ is a non-flat solution of the field equation

$$R_{ij}=0 \quad (2.1)$$

in an empty region of space time with

$$g_{ij}=g_{ij}(Z),\ Z=Z(x^i),\ \text{where } x^i=(x^1,x^2,x^3,x^4)=(x,y,z,t) \quad (2.2)$$

in some suitable co-ordinate system such that,

$$g^{ij}Z,_i\,Z,_j=0 \quad , \quad \left(Z,_i=\frac{\partial Z}{\partial x^i}\right) \quad (2.3)$$

and $Z=Z(z,t)$ , $(Z,_3\neq 0$ , $Z,_4\neq 0)$. (2.4)

The signature convention adopted was

$g_{aa}<0$ , $\begin{vmatrix} g_{aa} & g_{ab} \\ g_{ba} & g_{bb} \end{vmatrix}>0$ , $\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}<0$ , $g_{44}>0$, for $a=1,2$; $\alpha=3,4$ (Not summed for $a$ and $b$; $a,b=1,2$) (2.5)

and accordingly, $g=\det(g_{ij})<0$.

We extended this definition for six dimensional space time with three time axes as:

A plane gravitational wave $g_{ij}$ is a non-flat solution of the field equation

$$R_{ij} = 0 \tag{2.6}$$

in an empty region of space time with

$$g_{ij} = g_{ij}(Z), \quad Z = Z(x^i), \quad \text{where} \quad x^i = (x^1,...,x^6) \equiv (x, y, z, t_1, t_2, t_3) \tag{2.7}$$

in some suitable co-ordinate system such that,

$$g^{ij} Z,_i Z,_j = 0 \quad , \quad \left( Z,_i = \frac{\partial Z}{\partial x^i} \right) \tag{2.8}$$

and $Z = Z(z, t_1, t_2, t_3)$ , $(Z,_3 \neq 0 \ , \ Z,_4 \neq 0 \ , \ Z,_5 \neq 0 \ , \ Z,_6 \neq 0)$. (2.9)

The signature convention adopted is

$$g_{aa} < 0 \ , \ \begin{vmatrix} g_{aa} & g_{ab} \\ g_{ba} & g_{bb} \end{vmatrix} > 0 \ , \ \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} < 0 \ , \ g_{44} > 0 \ , \ g_{55} > 0 \ , \ g_{66} < 0$$

for $a = 1,2$ ; $\alpha = 3,4,5,6$ (Not summed for $a$ and $b$ ; $a, b = 1,2$ )
and accordingly, $g = \det(g_{ij}) > 0$. (2.10)

## 3. Co-ordinate system:

**Theorem 1:** If $g_{ij}$ satisfies (1.1) with $Z = \dfrac{t_1 - t_2 + t_3}{\sqrt{3}\, z}$ then we can transform the co-ordinates so that $g_{ij}$ satisfies

$$g_{14} = g_{24} = g_{15} = g_{25} = g_{16} = g_{26} = 0 \tag{3.1}$$

and (1.1) is kept invariant.

Proof: We use the following transformations satisfying the conditions in the hypothesis:

$$x = x' + \alpha z' \ , \quad y = y' + \beta z' \ , \quad z = z' \ , \quad t_1 = t_1' \ , \quad t_2 = t_2' \ , \quad t_3 = t_3' \tag{3.2}$$

and hence $\det\left( \dfrac{\partial x^i}{\partial x'^j} \right) = 1$ , $Z = Z'$ , (3.3)

where $\alpha, \beta$ are the functions of $Z = Z'$ satisfying

$$\begin{aligned} g_{11}\overline{\alpha} + g_{12}\overline{\beta} &= -\sqrt{3} g_{14} = \sqrt{3} g_{15} = -\sqrt{3} g_{16} \ , \\ g_{12}\overline{\alpha} + g_{22}\overline{\beta} &= -\sqrt{3} g_{24} = \sqrt{3} g_{25} = -\sqrt{3} g_{26} \ . \end{aligned} \tag{3.4}$$

With these transformations we obtained $\dfrac{d}{dZ} = \dfrac{d}{dZ'}$ and

$$\omega'^1 = \omega^1 - \alpha\omega^3 \ ,$$
$$\omega'^2 = \omega^2 - \beta\omega^3 \ ,$$

where
$$\left.\begin{aligned} &\omega^i = \phi_1 g^{3i} + g^{4i} + \phi_2 g^{5i} + \phi_3 g^{6i}, \\ &\sqrt{3} z\omega^3 - \omega^4 + \omega^5 - \omega^6 = 0, \\ &\omega'^{\alpha} = \omega^{\alpha}, \qquad (\alpha = 3,4,5,6). \end{aligned}\right\} \quad (3.5)$$

Using $g'_{ab} = g_{mn} \dfrac{\partial x^m}{\partial x'^a} \dfrac{\partial x^n}{\partial x'^b}$, it can be proved that

$$\left.\begin{aligned} &g'_{ab} = g_{ab}, \\ &g'_{a3} = g_{a1}\alpha + g_{a2}\beta + \sqrt{3} Z g_{a4} + g_{a3}, \\ &g'_{a4} = g'_{a5} = g'_{a6} = 0, \qquad (a,b = 1,2) \\ &g'_{33} = (\alpha^2 g_{11} + 2\alpha\beta g_{12} + \beta^2 g_{22}) + 2(\alpha g_{13} + \beta g_{23}) + g_{33} \\ &\qquad - Z\left[\alpha\bar{\alpha} g_{11} + \beta\bar{\beta} g_{22} + \left(\bar{\alpha}\beta + \alpha\bar{\beta}\right) g_{12} + 2\left(\bar{\alpha} g_{13} + \bar{\beta} g_{23}\right)\right] \\ &g'_{34} = \frac{1}{\sqrt{3}}\left(g_{13}\bar{\alpha} + g_{23}\bar{\beta}\right) + g_{34}, \\ &g'_{35} = -\frac{1}{\sqrt{3}}\left(g_{13}\bar{\alpha} + g_{23}\bar{\beta}\right) + g_{35}, \\ &g'_{36} = \frac{1}{\sqrt{3}}\left(g_{13}\bar{\alpha} + g_{23}\bar{\beta}\right) + g_{36}, \\ &g'_{44} = \frac{1}{\sqrt{3}}\left(g_{14}\bar{\alpha} + g_{24}\bar{\beta}\right) + g_{44}, \\ &g'_{45} = -\frac{1}{\sqrt{3}}\left(g_{14}\bar{\alpha} + g_{24}\bar{\beta}\right) + g_{45}, \\ &g'_{46} = \frac{1}{\sqrt{3}}\left(g_{14}\bar{\alpha} + g_{24}\bar{\beta}\right) + g_{46}, \end{aligned}\right\} \quad (3.6)$$

$$\left.\begin{aligned} &g'_{55} = -\frac{1}{\sqrt{3}}\left(g_{15}\bar{\alpha} + g_{25}\bar{\beta}\right) + g_{55}, \\ &g'_{56} = \frac{1}{\sqrt{3}}\left(g_{15}\bar{\alpha} + g_{25}\bar{\beta}\right) + g_{56}, \\ &g'_{66} = \frac{1}{\sqrt{3}}\left(g_{16}\bar{\alpha} + g_{26}\bar{\beta}\right) + g_{66} \end{aligned}\right\} \quad (3.6)$$

which proves the theorem.

**Theorem 2:** If $g_{ij}$ satisfies (1.1) and (3.1) with $Z = \dfrac{t_1 - t_2 + t_3}{\sqrt{3}\, z}$ then

$$g_{13} = c_1 g_{11} + c_2 g_{12}, \; g_{23} = c_1 g_{21} + c_2 g_{22} \qquad (3.7)$$

where *c*s are constants.

Proof: By using theorem 3.1, we have

$$\left.\begin{aligned} &u_1 = u_2 = u_3 = 0, \\ &s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 0, \\ &p_a = -\overline{g}_{a3}, \\ &q_a = -g_{a3} \end{aligned}\right\} \quad (3.8)$$

where

$$\left.\begin{aligned} &u_1 = g_{14}g_{23} - g_{13}g_{24}, \\ &u_2 = g_{15}g_{23} - g_{13}g_{25}, \\ &u_3 = g_{16}g_{23} - g_{13}g_{26}, \\ &s_1 = g_{12}g_{24} - g_{22}g_{14}, \\ &s_2 = g_{11}g_{24} - g_{12}g_{14}, \\ &s_3 = g_{12}g_{25} - g_{22}g_{15}, \\ &s_4 = g_{11}g_{25} - g_{12}g_{15}, \\ &s_5 = g_{12}g_{26} - g_{22}g_{16}, \\ &s_6 = g_{11}g_{26} - g_{12}g_{16}. \end{aligned}\right\} \quad (3.9)$$

From (1.1), $\rho_a = \overline{g}_{ai}\omega^i = 0$, and noting $p_i = \phi g_{4i} - g_{3i}$ and $q_i = \phi \overline{g}_{4i} - \overline{g}_{3i}$ we get

$$\left.\begin{aligned} &\omega^3 q_1 = \omega^1 \overline{g}_{11} + \omega^2 \overline{g}_{12}, \\ &\omega^3 q_2 = \omega^1 \overline{g}_{12} + \omega^2 \overline{g}_{22}. \end{aligned}\right\} \quad (3.10)$$

Solving equations (3.9) we get

$$\left.\begin{aligned} &m\overline{g}_{13} = -k_1\overline{g}_{11} + k_2\overline{g}_{12}, \\ &m\overline{g}_{23} = -k_1\overline{g}_{12} + k_2\overline{g}_{22}, \end{aligned}\right\} \quad (3.11)$$

where $m = g_{11}g_{22} - g_{12}{}^2$,

$k_1 = g_{12}g_{23} - g_{13}g_{22}$,

$k_2 = g_{11}g_{23} - g_{12}g_{13}$.

From equations (3.7) and (3.10), we have

$$m\overline{k}_1 - \overline{m}k_1 = 0, \qquad m\overline{k}_2 - \overline{m}k_2 = 0,$$

$$\text{where } \frac{k_1}{m} = c_1, \qquad \frac{k_2}{m} = -c_2 \quad (3.12)$$

On integrating equation (3.11), we get

$$\left.\begin{aligned} &g_{13} = c_1 g_{11} + c_2 g_{12} + c_3 \\ &g_{23} = c_1 g_{21} + c_2 g_{22} + c_4. \end{aligned}\right\} \quad (3.13)$$

where $c_1, c_2, c_3, c_4$ are integration constants.

Using the definition of $k$'s in (3.11), we get equation (3.7).

Hence the theorem is proved.

**Theorem 3:** If $g_{ij}$ satisfies (1.1) and (3.1) with $Z=\dfrac{t_1-t_2+t_3}{\sqrt{3}\,z}$ and theorem 2, then there exists a co-ordinate transformation by which $g_{a\alpha}=0$ , $(a=1,2\ ;\ \alpha=3,4,5,6)$. Moreover by this transformation, (1.1) is kept invariant.

Proof: If we transform the co-ordinate by

$$x=x'-c_1z',\ y=y'-c_2z',\ z=z',\ t_1=t_1',\ t_2=t_2',\ t_3=t_3'. \tag{3.14}$$

where *c*s are constants as in equation (3.7),

then $\det\left(\dfrac{\partial x^i}{\partial x'^j}\right)=1,\quad Z=Z'.$

By this transformation $\dfrac{d}{dZ}=\dfrac{d}{dZ'}$ and from equation (3.7) we get

$$\left.\begin{aligned}&\omega'^1=\omega^1+c_1\omega^3,\\ &\omega'^2=\omega^2+c_2\omega^3,\\ &\omega'^\alpha=\omega^\alpha,\qquad(\alpha=3,4,5,6),\\ \text{where}\quad &\omega^i=\phi_1g^{3i}+g^{4i}+\phi_2g^{5i}+\phi_3g^{6i},\\ &\rho_i'=\rho_i,\qquad (i=1,\ldots,6).\end{aligned}\right\} \tag{3.15}$$

It can be shown that

$$\left.\begin{aligned}&g'_{ab}=g_{ab},\qquad (a,b=1,2),\\ &g'_{a\alpha}=0,\qquad (a=1,2),\ (\alpha=3,4,5,6)\\ &g'_{33}=g_{33}-(c_1^{\,2}g_{11}+2c_1c_2g_{12}+c_2^{\,2}g_{22}),\\ &g'_{34}=g_{34},\quad g'_{35}=g_{35},\quad g'_{36}=g_{36},\\ &g'_{44}=g_{44},\quad g'_{45}=g_{45},\quad g'_{46}=g_{46},\\ &g'_{55}=g_{55},\quad g'_{56}=g_{56},\quad g'_{66}=g_{66}.\end{aligned}\right\} \tag{3.16}$$

In this new co-ordinate system we have

$$\begin{aligned}\omega^i&=(\omega^1,\omega^2,\omega^3,\omega^4,\omega^5,\omega^6)\\ &=(0,\ 0,\ \phi_1g^{33}+g^{43}+\phi_2g^{53}+\phi_3g^{63},\ \phi_1g^{34}+g^{44}+\phi_2g^{54}+\phi_3g^{64},\\ &\qquad \phi_1g^{35}+g^{45}+\phi_2g^{55}+\phi_3g^{65},\ \phi_1g^{36}+g^{46}+\phi_2g^{56}+\phi_3g^{66})\end{aligned} \tag{3.17}$$

which proves that (1.1) is invariant. Hence the theorem is proved.

# 4. The line element

Considering equations (2.8), (2.9) and noting $\phi_1 = Z,_3 / Z,_4$ , $\phi_2 = Z,_5 / Z,_4$ and $\phi_3 = Z,_6 / Z,_4$ and using the co-ordinate system, determined in section 3 above in which $g_{a\alpha} = 0$ , $a = 1,2$ , $\alpha = 3,4,5,6$ , we get

$$\phi_1^{\,2} g^{33} + 2\phi_1 g^{34} + 2\phi_1\phi_2 g^{35} + 2\phi_1\phi_3 g^{36} + g^{44} + 2\phi_2 g^{45} + 2\phi_3 g^{46} + \phi_2^{\,2} g^{55} + 2\phi_2\phi_3 g^{56} + \phi_3^{\,2} g^{66} = 0. \tag{4.1}$$

Noting $\phi_1 = -\sqrt{3}Z$ , $\phi_2 = -1$ and $\phi_3 = 1$ , the matrix $(g_{ij})$ can be written as

$$(g_{ij}) = \begin{bmatrix} -P & -Q & 0 & 0 & 0 & 0 \\ -Q & -R & 0 & 0 & 0 & 0 \\ 0 & 0 & 3Z^2(-S+T) & -\sqrt{3}ZT & \sqrt{3}Z(-S+U) & -\sqrt{3}ZU \\ 0 & 0 & -\sqrt{3}ZT & (S+T) & -U & (S+U) \\ 0 & 0 & \sqrt{3}Z(-S+U) & -U & (S+T) & -T \\ 0 & 0 & -\sqrt{3}ZU & (S+U) & -T & (-S+T) \end{bmatrix} \tag{4.2}$$

where $P,Q,R,S,T,U$ are functions of $Z$ satisfying (2.5),

i.e. $P,R > 0$ , $m > 0$ , $S > |T|$ , $S > |U|$. (4.3)

The corresponding matrix $(g^{ij})$ is

$$(g^{ij}) = \begin{bmatrix} \frac{-R}{m} & \frac{Q}{m} & 0 & 0 & 0 & 0 \\ \frac{Q}{m} & \frac{-P}{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-2S^2(S+T+U)}{3Z^2 n} & \frac{-2S^2(T+U)}{\sqrt{3}Zn} & \frac{-2S^3}{\sqrt{3}Zn} & 0 \\ 0 & 0 & \frac{-2S^2(T+U)}{\sqrt{3}Zn} & \frac{2S^2(S-T-U)}{n} & 0 & \frac{2S^3}{n} \\ 0 & 0 & \frac{-2S^3}{\sqrt{3}Zn} & 0 & \frac{2S^2(S-T+U)}{n} & \frac{2S^2(U-T)}{n} \\ 0 & 0 & 0 & \frac{2S^3}{n} & \frac{2S^2(U-T)}{n} & \frac{2S^2(U-S-T)}{n} \end{bmatrix} \tag{4.4}$$

where $m = PR - Q^2 > 0$ and $n = 12Z^2S^4$.

Therefore, $g = mn = 12Z^2mS^4 > 0$ , so that the condition (2.5) is satisfied. Also by using the $g^{ij}$ s from (4.4), condition (4.1) is satisfied. Hence the line element obtained from (4.2) is

$$\begin{aligned} ds^2 = & -Pdx^2 - 2Qdxdy - Rdy^2 + 3Z^2(T-S)dz^2 - 2\sqrt{3}ZTdzdt_1 + 2\sqrt{3}Z(U-S)dzdt_2 - 2\sqrt{3}ZUdzdt_3 \\ & + (S+T)dt_1^{\,2} - 2Udt_1dt_2 + 2(S+U)dt_1dt_3 + (S+T)dt_2^{\,2} - 2Tdt_2dt_3 + (T-S)dt_3^{\,2} \end{aligned} \tag{4.5}$$

where $P,Q,R,S,T,U$ are the functions of $Z$ satisfying (2.5), (4.1) and (4.3).

## 5. Conclusion

Thus if $g_{ij}$ satisfies (1.1) with $Z = \frac{t_1 - t_2 + t_3}{\sqrt{3}\, z}$, then there exists a co-ordinate system in which $g_{a\alpha} = 0$ , $(a = 1,2\ ;\ \alpha = 3,4,5,6)$ ; and (1.1) is kept invariant. Also we have obtained the line element for $Z = \frac{t_1 - t_2 + t_3}{\sqrt{3}z}$ type plane gravitational waves as given by (4.5).

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