Article

Plane Wave Solutions of Weakened Field Equations in a Plane Symmetric Space-time-II

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Abstract

In this paper, we propose to obtain the $Z=(t/z)-$type plane gravitational waves in a set of five vacuum weakened equations in the space-time introduced by us having plane symmetry in the sense of Taub [**Ann.**Math.**53**,472 (1951)]. These vacuum field equations has been suggested as alternatives to the Einstein vacuum field equations of general relativity. Furthermore the physical significance of modified gravitational waves for the space-time is obtained.

Keywords: plane symmetry, plane gravitational waves, curvature tensor, Ricci tensor, weakened field equations.

1. Introduction

The theory of plane gravitational waves in general relativity has been introduced by many investigators like Einstein and Rosen (1937); Bondi, Pirani and Robinson (1959); Takeno (1961). Takeno (1961) has discussed the mathematical theory of plane gravitational waves and classified them into two categories, namely, $(z-t)$ and $Z=(t/z)-$type wave according as the phase function $Z=(z-t)$ and $Z=(t/z)-$type wave respectively. According to him, a plane wave g_{ij} is a non-flat solution of Ricci tensor $R_{ij}=0$ in general relativity and in some suitable coordinate system; all the components of the metric tensor are functions of a single variable $Z=Z(z,t)$ (i.e. a phase function). Takeno (1957) deduce the space-time having plane symmetry characterized by Taub (1951) for $Z=(z-t)$ -type plane gravitational waves as

$$ds^2=-A(dx^2+dy^2)-C(dz^2-dt^2) \tag{1.1}$$

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where A and C are arbitrary functions of Z, $Z = z - t$.

Recently Bhoyar et al. (2011) transform the metric (1.1) to (1.2) using suitable transformations for $Z = (t/z)-$type plane gravitational waves, which take the form

$$ds^2 = -A(dx^2 + dy^2) - Z^2 C dz^2 + B dt^2 \tag{1.2}$$

Where A and B are functions of Z , $Z \equiv (t/z)$.

Lovelock (1967a, b) has considered a set of five weakened field equations (WFE) in vacuum, namely,

$$R^h_{ijk;h} = 0, \tag{1.3}$$

$$(-g)^{\frac{1}{4}}\left[g^{ih}R_{kj;ih} - g^{ih}R_{ij;kh} + \frac{1}{6}R_{;kj} - \frac{1}{6}g_{jk}g^{ih}R_{;ih} - R^{ih}C_{jhik} + \frac{R}{6}g^{ih}C_{jhik}\right] = 0, \tag{1.4}$$

$$(-g)^{\frac{1}{2}}\left[g^{hj}g^{ki}(2R_{jlim}R^{ml} + g^{ml}R_{ij;lm} - R_{;ij}) - \frac{1}{2}g^{hk}(R^l_m R^m_l - g^{lm}R_{;lm})\right] = 0, \tag{1.5}$$

$$(-g)^{\frac{1}{2}}\left[(g^{hk}g^{rs} - \frac{1}{2}g^{hr}g^{ks} - \frac{1}{2}g^{hs}g^{kr})R_{;rs} + R(R^{kh} - \frac{1}{4}g^{kh}R)\right] = 0, \tag{1.6}$$

and $$R^{ij}_{;k} = 0, \tag{1.7}$$

where a semicolon (;) followed by an index denotes covariant differentiation and C_{jhik} is the Weyl curvature tensor defined by

$$C_{jhik} = R_{jhik} - \frac{1}{2}(R_{ji}g_{hk} - R_{hi}g_{jk} - R_{jk}g_{hi} + R_{hk}g_{ij}) + \frac{R}{6}(g_{ij}g_{hk} - g_{hi}g_{jk}). \tag{1.8}$$

Kilmister and Newman (1961) have originally proposed the vacuum weakened field equations (1.2)-(1.6). These field equations are suggested as various alternatives to the Einstein field equation of general relativity in vacuum. The Einstein vacuum field equation of general relativity is given by

$$R_{ij} = 0. \tag{1.9}$$

The solution of (1.3)) together with the trajectories of test particles (geodesics hypothesis) give agreement with experiment. Lovelock (1967a, b) obtained the solutions of WFE field equations (1.2)-(1.6) in a spherically symmetric space-time and he proved to be gravitationally unphysical metric by geodesics hypothesis in the sense that these solutions correspond to the static situation of an isolated mass at origin which repels the test particles.

Consequently the physical aspects of weakened field equations are not well established through many researchers (for examples: Thompson 1963; Kilmister 1966; Rund 1967; Lovelock 1967a, b; Swami 1970; Lal and Singh 1973; Lal and Pandey 1975; Pandey 1975) but have tried to investigate the solutions to interpret the useful results. Thompson (1963) made detailed study of these field equations and concluded that they are too weak.

The various alternative vacuum weakened field equations are weaker than the Einstein vacuum field equations in the sense that they each admit (1.4) as a solutions and hence they have been called WFE. Swami (1970) has solved three solutions of the weakened field equations $R_{ij;k} - R_{ik;j} = 0$ with $R_{ij} \neq 0$, $R_{ij} \neq \lambda g_{ij}$ and has discussed the geometrical and dynamical properties of these solutions.

Furthermore R^h_{ijk} satisfies the Bianchi identities

$$R^h_{ijk;m} + R^h_{ikm;j} + R^h_{imj;k} = 0 \ . \qquad (1.10)$$

From which

$$R^i_{j;i} = \frac{1}{2} R_{;j} \ , \qquad (1.11)$$

where $R^i_j = g^{ih} R_{hj}$.

In this paper, we study the $Z = (t/z) -$type plane gravitational waves of vacuum weakened field equations (1.3)-(1.7) in the metric (1.2). In addition to this, we assume the metric (1.2) has non-conformally flat. Physical significance of modified gravitational waves for metric (1.2) is obtained. We have obtained some useful results in the form of theorems under curvature properties with conclusions.

2. Plane Symmetric metric and curvature properties

The components of contravariant tensor g^{ij} from the metric (1.2) are

$$g^{11} = g^{22} = -\frac{1}{A}\ ,\ g^{33} = -\frac{1}{Z^2 C}\ \text{, and} \qquad g^{44} = \frac{1}{C} \tag{2.1}$$

The non vanishing components of Christoffel symbols are

$$Z\begin{Bmatrix}3\\11\end{Bmatrix} = Z\begin{Bmatrix}3\\22\end{Bmatrix} = \begin{Bmatrix}4\\11\end{Bmatrix} = \begin{Bmatrix}4\\22\end{Bmatrix} = \frac{\overline{A}}{2Cz},$$

$$\frac{1}{Z}\begin{Bmatrix}1\\13\end{Bmatrix} = \frac{1}{Z}\begin{Bmatrix}2\\23\end{Bmatrix} = -\begin{Bmatrix}1\\14\end{Bmatrix} = -\begin{Bmatrix}2\\24\end{Bmatrix} = -\frac{\overline{A}}{2Az},$$

$$\begin{Bmatrix}3\\33\end{Bmatrix} = -Z\begin{Bmatrix}3\\34\end{Bmatrix} = -\frac{1}{Z}\begin{Bmatrix}4\\33\end{Bmatrix} = -\frac{\left(2C + \overline{C}Z\right)}{2Cz},$$

$$\frac{1}{Z}\begin{Bmatrix}4\\34\end{Bmatrix} = -\begin{Bmatrix}4\\44\end{Bmatrix} = Z\begin{Bmatrix}3\\44\end{Bmatrix} = -\frac{\overline{C}}{2Cz} \tag{2.2}$$

Using (1.2), (2.1) and (2.2), the components of curvature tensor R_{ijkl} are as follows:

$$\frac{R_{1313}}{Z^2} = \frac{R_{2323}}{Z^2} = -\frac{R_{2324}}{Z} = -\frac{R_{1314}}{Z} = R_{1414} = R_{2424} = \psi, \tag{2.3}$$

The components of covariant and contra -variant Ricci tensor, from (2.1) and (2.2) are as follows:

$$\frac{R_{33}}{Z^2} = -\frac{R_{34}}{Z} = R_{44} = \frac{\psi}{z^2},\ \text{(Say)}, \tag{2.4a}$$

$$Z^2 R^{33} = ZR^{34} = R^{44} = \frac{\psi}{C^2 z^2} \quad \text{and all other} \quad R_{ij} = R^{ij} = 0\ , \tag{2.4b}$$

where

$$\psi = \psi\left(Z\right) = \frac{\overline{\overline{A}}}{A} - \frac{\overline{A}^2}{2A^2} - \frac{\overline{A}\,\overline{C}}{AC}. \tag{2.5}$$

From(2.2), (2.4b), we obtain

$$\frac{1}{Z^4}R_{33;33} = -\frac{1}{Z^3}R_{33;34} = \frac{1}{Z^2}R_{33;44} = -\frac{1}{Z^3}R_{34;33} = \frac{1}{Z^2}R_{34;34} = -\frac{1}{Z}R_{34;44}$$

$$= \frac{1}{Z^2}R_{44;33} = -\frac{1}{Z}R_{44;34} = R_{44;44} = \mathrm{K} \tag{2.6}$$

where $$\mathrm{K} = \frac{1}{z^4}\left(\overline{\overline{\psi}} - \frac{2\overline{\overline{C}}\psi}{C} - \frac{5\overline{C}\,\overline{\psi}}{C} + \frac{8\overline{C}^2\psi}{C^2}\right). \tag{2.7}$$

Here a bar (−) overhead letter denotes the differentiation with respect to Z (i.e. $\overline{\psi} = \frac{\partial \psi}{\partial Z}$ and

$\overline{\overline{\psi}} = \frac{\partial^2 \psi}{\partial Z^2}$).

Also for metric (1.2), we deduce, from (2.4 a,b), that

a) The scalar curvature R defined by $R = g^{ij}R_{ij}$ is zero i.e. $R = 0$,

b) $g = \det(g_{ij}) = -Z^2A^2C^2$,

c) $R^l_m R^m_l = R_{im}R^{im} = 0$

and d) $R_{jlim}R^{ml} = 0$. (2.8)

3. Solutions of weakened field equations (1.3)-(1.7)

By its very structure the metric (1.1) is non-conformally flat which implies that Weyl curvature tensor (1.2) in view of (2.8a) (i.e. $R = 0$) reduces to

$$C_{jhik} = R_{jhik} - \frac{1}{2}(R_{ji}g_{hk} - R_{hi}g_{jk} - R_{jk}g_{hi} + R_{hk}g_{ij}). \tag{3.1}$$

From Bianchi identities (1.10), we find

$$R^h_{ijk;h} = R_{ij;k} - R_{ik;j} \tag{3.2}$$

where $R_{ij;k} = R_{ij,k} - \Gamma^m_{jk}R_{im} - \Gamma^m_{ik}R_{mj}$.

On substituting the components of $\left\{ {i \atop jk} \right\}$ and R_{ij} from (2.2) and (2.4) respectively in R.H.S. of equation (3.2), it is seen that

$$R_{ij;k} - R_{ik;j} = 0 \ , \tag{3.3}$$

or equivalently, from (3.2),

$$R^{h}_{ijk;h} = 0, \tag{3.4}$$

which is weakened field equation (1.3).
Also it follows from (3.3) that

$$R_{ij;kh} = R_{ik;jh}. \tag{3.5}$$

In view of equations (2.8a) and (3.5), the weakened field equation (1.4) reduces to

$$(-g)^{\frac{1}{4}} R^{ih} C_{jhik} = 0 . \tag{3.6}$$

Using (3.1) in (3.6), we obtain

$$(-g)^{\frac{1}{4}} \left[R^{ih} R_{jhik} - \frac{1}{2} (R_{ji} R^{i}_{k} - R^{ih} R_{hi} g_{jk} - R^{i}_{i} R_{jk} + R^{h}_{j} R_{hk}) \right] = 0 , \tag{3.7a}$$

which reduces to

$$(-g)^{\frac{1}{4}} \left[R^{ih} R_{jhik} - \frac{1}{2} (g_{jk} R^{k}_{i} R^{i}_{k} - g_{jk} R^{ih} R_{hi} - R R_{jk} + g_{jk} R^{j}_{h} R^{h}_{j}) \right] = 0 . \tag{3.7b}$$

Using (2.8a, c, d) in (3.7b), it is seen that no term remains in LHS and hence (3.7b) i.e. (3.6) is identically satisfied.
It is observed that the weakened field equation (1.5) is satisfied by R = 0 (i.e. 2.8a) alone. Hence the following theorem:

Theorem 1: *The g_{ij} given by (1.2) is a solution of weakened field equation (1.3), (1.4) and (1.6).*

Theorem 2: *A necessary and sufficient condition that g_{ij} given by (1.2) be a solutions of WFE (1.5) are i) $\psi = 0$, ii) $\mathrm{K} = 0$, where*

$$\psi = \frac{z^2 R_{33}}{Z^2} \quad and \quad \mathrm{K} = \frac{1}{z^4}\left(\overline{\overline{\psi}} - \frac{2\overline{\overline{C}}\psi}{C} - \frac{5\overline{C}\,\overline{\psi}}{C} + \frac{8\overline{C}^2\psi}{C^2}\right).$$

Proof:

First, let g_{ij} given by (1.2) be the solution of WFE (1.5).

By using equation (2.8), the WFE (1.5) reduces to

$$(-g)^{1/2} g^{hj} g^{ki} g^{ml} R_{ij;lm} = 0. \tag{3.8}$$

Case i) $\psi = 0$, the result is obvious.

Case ii) The above equation (3.8) is identically satisfied for all the values of h, k except for h, k = 3, 4. On simplification, for h, k = 3, 4, by the virtue of (2.4) and (2.6), equation (3.8) gives,

$$\mathrm{K} = \frac{1}{z^4}\left(\overline{\overline{\psi}} - \frac{2\overline{\overline{C}}\psi}{C} - \frac{5\overline{C}\,\overline{\psi}}{C} + \frac{8\overline{C}^2\psi}{C^2}\right).$$

Conversely, if

$$\mathrm{K} = \frac{1}{z^4}\left(\overline{\overline{\psi}} - \frac{2\overline{\overline{C}}\psi}{C} - \frac{5\overline{C}\,\overline{\psi}}{C} + \frac{8\overline{C}^2\psi}{C^2}\right) = 0.$$

By (2.4) and (2.6), we have

$$(-g)^{1/2} g^{hj} g^{ki} g^{ml} R_{ij;lm} = 0.$$

Introducing the result (2.8) and above equation we get,

L.H.S. of (1.5) = 0.

So WFE (1.5) is identically satisfied.

Theorem 3: *A necessary and sufficient condition that* g_{ij} *given by (1.2) is the solution of WFE (1.7) is* $\overline{\psi} = \frac{2\psi\overline{C}}{C}$.

Proof:

Let g_{ij} given by (1.2) be a solution of WFE (1.7).

By the definition of covariant derivative, equation (1.7)

$$\frac{\partial R^{ij}}{\partial x^k} + \Gamma^i_{pk} R^{pj} + \Gamma^j_{pk} R^{ip} = 0 . \tag{3.9}$$

By using the components of Ricci tensors and Christoffel's symbols, the equation (3.9) is identically satisfied for all values of i, j, k except when $i, j, k = 3, 4$,

i.e. $\bar{\psi} = \frac{2\psi \bar{C}}{C}$.

Conversely

If $\bar{\psi} = \frac{2\psi \bar{C}}{C}$, then it is seen that the WFE (1.7) is identically satisfied,

i.e. $R^{ij}_{;k} = 0$.

This implies g_{ij} be the solution of WFE (1.7). Hence the theorem.

4. Physical significance of modified gravitational waves for the space-time metric

To attempt the physical significance of modified gravitational waves for the space-time metric (5.1.1), it is useful to consider the effect of motion of test particles introduced into the system. We assume the motion of test particles in a curved geometry which describe the geodesics. The equations of motion of such particles are given by

$$\frac{d^2 x^k}{ds^2} + \Gamma^k_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0$$

or equivalently

$$\frac{d}{ds}\left(g_{kl} \frac{dx^l}{ds} \right) - \frac{1}{2} g_{ij,k} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0 . \tag{4.1}$$

The metric (1.2) represents a modified gravitational waves i.e. generalized plane gravitational waves. For the metric (1.2), first two integrals of motion from (4.1) gives

$$A\frac{dx}{ds}=\lambda \tag{4.2}$$

and $$A\frac{dy}{ds}=\mu\ , \tag{4.3}$$

where λ , μ are constants. For k = 3, 4 the equation (4.1) yields

$$\left(\frac{1}{Z}\right)\frac{d}{ds}\left[\frac{dz}{ds}\left(Z^2C\right)\right]=\frac{d}{ds}\left[\frac{dt}{ds}\left(C\right)\right]. \tag{4.4}$$

Noting that $Z=t/z$, we take

$$\frac{dZ}{ds}=\gamma = \text{constant on geodesics.}$$

From equations (4.2) and (4.3), we obtain

$$\int\frac{A}{m}\,dZ=\frac{\left(\gamma x-k_1\right)}{\lambda}=\frac{\left(\gamma y-k_2\right)}{\mu} \tag{4.5}$$

where k_1 and k_2 are constants of integration and $m=A^2$.

Equation (1.1) can be put in another form as

$$\gamma^2 z\left[Cz+2CZ\,\frac{dz}{dZ}\right]=1+\frac{A}{m}\left(\lambda^2+\mu^2\right) \tag{4.6}$$

Which on integrating and using equations (4.2) and (4.3) reduces to

$$\gamma^2 t\int Cdz\ =Z+\gamma\left(\lambda x+\mu y\right)-\left(\lambda k_1+\mu k_2\right) \tag{4.7}$$

To ensure the plane character of waves we may consider that Z is a function of x^i (Takeno 1961).Consequently C may be assumed as the linear function of Z . Integrating (4.7) and noting $Z=\left(t/z\right)$ we get,

$$\gamma^2\left(\tau\, tz+\xi\, z^2\right)+\gamma\left(\lambda x+\mu y\right)+Z=\delta, \tag{4.8}$$

where τ,ξ and δ are constants. The right hand side constant can be made zero by a suitable choice of the origin without loss of generality.

Discussions

1] If the test particle is at rest (i.e. at rest means the coordinates of the particle do not change) at the origin i.e. (x, y, z) = (0, 0, 0) $\Rightarrow \delta = 0$ at $t = 0$, then equation reflects that either $t = 0$ or γ is real only in the negative direction of z , hence no other particle with real γ passing through origin will return to the resting particle. Further in case of moving particle no return can take place.

2] For the space-time metric (1.2), the coordinates along the world line of test particles are geodesic. If the curvature of trajectories of test particles is everywhere zero, then the geodesics are straight line. If the particle released from rest in *z*, *t* plane, then, for (1.2), world line of such resting particle bring into a world line of free particle moving in the plane. Hence a particle starting from rest in *z*, *t* plane must necessarily move along the geodesic and the solutions of geodesic equations are unique

3] If two particle perform the general motion then we have to consider two equations of type (4.7) with constants γ and γ' and with $\delta = \delta' = 0$ .This leads to

$\gamma = \gamma'$ .Taking $\tau', \xi', \lambda', \mu'$ as corresponding constants for second particle, we find

$$\gamma = \frac{(\lambda' - \lambda)x + (\mu' - \mu)y}{(\tau - \tau')\, t\, z + (\xi - \xi')\, z^2},$$

as the condition for consecutive meeting of particle.

Conclusions

In this paper:

1] The plane wave g_{ij} given by metric (1.2) representing a non-conformal flat space-time with the scalar curvature zero (R = 0) is a solution of the weakened field equation (1.6). It is also a solution of WFE (1.3), (1.4) and (1.5) under the curvature properties (2.8). In non-conformal flat space-time (1.2), the solution of (1.6) follows from Bianchi identity gives (1.10) and (3.2). When R = 0, then $R_{;j}$ = 0 and (1.6) implies that $R^i_{j;i}$ = 0.

2] The physical significance of modified gravitational waves for the space-time is obtained on the basis of geodesics hypothesis. It is observed that the effects of gravitational field can be changed by purely gravitational effects associated with motion of free particle in a curved space-time.

4] It is observed that the metric (1.2) is a non-flat and it is an exact solution of $R_{ij} = 0$ if and only if $\mathrm{K} = 0$. In such case, plane wave metric satisfied the curvature properties given in (2.8). Hence we have the following results:

a) The g_{ij} given by above Takeno's plane wave metric is a solution of weakened field equations (1.3), (1.4) and (1.6).

b) The g_{ij} given by above Takeno's plane wave metric is a solution of weakened field equation (1.5) if and only if $\overline{\overline{\psi}} = 0$, where $\psi = \frac{z^2 R_{33}}{Z^2}$.

c) The g_{ij} given by above Takeno's plane wave metric is a solution of weakened field equation (1.6) if and only if $\overline{\psi} = \frac{2\psi \overline{C}}{C}$.

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