Article

The Schrödinger-equation Presentation of any Oscillatory Classical Linear System that is Homogeneous and Conservative

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Abstract

The time-dependent Schrödinger equation with time-independent Hamiltonian matrix is a homogeneous linear oscillatory system in canonical form. We investigate whether any classical system that itself is linear, homogeneous, oscillatory and conservative is guaranteed to linearly map into a Schrödinger equation. Such oscillatory classical systems can be analyzed into their normal modes, which are mutually independent, uncoupled simple harmonic oscillators, and the equation of motion of such a system linearly maps into a Schrödinger equation whose Hamiltonian matrix is diagonal, with $\hbar$ times the individual simple harmonic oscillator frequencies as its diagonal entries. Therefore if the coupling-strength matrix of such an oscillatory system is presented in symmetric, positive-definite form, the Hamiltonian matrix of the Schrödinger equation it maps into is $\hbar$ times the square root of that coupling-strength matrix. We obtain a general expression for mapping this type of oscillatory classical equation of motion into a Schrödinger equation, and apply it to the real-valued classical Klein-Gordon equation and the source-free Maxwell equations, which results in relativistic Hamiltonian operators that are strictly compatible with the correspondence principle. Once such an oscillatory classical system has been mapped into a Schrödinger equation, it is automatically in canonical form, making second quantization of that Schrödinger equation a technically simple as well as a physically very interpretable way to quantize the original classical system.

Keywords: Schrödinger equation, oscillatory classical system, normal modes, single-state quantum system, classical Klein-Gordon equation, quantized massive free relativistic particle, source-free Maxwell equations, free photon, second quantization, complementarity.

Introduction

A time-dependent Schrödinger equation, viewed as a real-valued equation of motion that couples the real and imaginary parts of its wave vector, is a homogeneous linear oscillatory canonical classical system (its classical Hamiltonian function is the presentation in the appropriate real canonical variables of the quantum expectation value of its Hamiltonian matrix). Here we shall see that a *general* homogeneous linear oscillatory conservative classical system's equation of motion can always be linearly mapped into a Schrödinger equation, and that this mapping is invertible if the classical system has no zero-frequency normal modes. When the oscillatory classical system's *coupling-strength matrix* is presented in *symmetric form* and is positive definite (i.e., no zero-frequency normal modes), the corresponding Schrödinger equation's *quantum Hamiltonian matrix* comes out to be $\hbar$ times the positive-definite *square root* of that classical coupling-strength matrix. The Schrödinger equation's complex-valued mapped wave vector can be sensibly *normalized* such that the quantum system's energy expectation value equals the oscillatory classical system's energy function. Moreover, that classical system can then be immediately *quantized* by means of the very straightforward *second quantization* of the Schrödinger equation that it maps into, which by its *nature* is in *canonical form*. This is not only technically simple, it is as well automatically accompanied by a detailed physical interpretation—e.g., one has a mathematical depiction of classical-wave/quantum-particle *complementarity* via the linear mapping of the original classical *oscillatory* degrees of freedom (which have Hermitian representation) into the second-quantized Schrödinger-equation wave vector's *annihilation and creation* components (which have non-Hermitian representation). Mapping into a Schrödinger equation of the real-valued classical scalar-field Klein-Gordon equation with mass parameter m yields a complex-valued scalar wave function and the Hamiltonian operator $(|c\widehat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}$, which is in accord with the correspondence-principle prescription for a relativistic free particle of mass m [1]. Such mapping of the classical source-free Maxwell equations yields a complex-valued transverse-vector wave function and the Hamiltonian operator $c|\widehat{\mathbf{p}}|$, which is relativistically appropriate to the massless free photon [2].

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Oscillatory classical linear systems which are homogeneous and conservative are described by second-order equations of motion that have the form,

$$\ddot{q} + Kq = 0, \tag{1a}$$

where K is a nonvanishing real-valued matrix, all of whose eigenvalues are real and nonnegative, and whose real-valued eigenvectors *completely span* the real-valued vector space on which K naturally operates. Note that the use of the terms "vector" and "matrix" in this article is *not* intended to exclude vectors that have a continuum of components (e.g., functions) or matrices that have a continuum of entries (e.g., operators on function spaces). However, in the interest of cutting down on notational clutter, all of the didactic *generic formulas* that are presented in this article which involve vector components or matrix entries display *only* the case that these are discrete—that is notwithstanding the fact that the interesting *examples* which are discussed in the last part of this article all have continuum character.

We note that *first-order* classical equations of motion which have the simple form,

$$\dot{s} = Ws, \tag{1b}$$

also describe homogeneous linear oscillatory conservative classical systems when W is a nonvanishing real-valued matrix that has exclusively *imaginary* eigenvalues whose associated complex-valued eigenvectors *completely span* the *extended* complex-valued vector space on which the real-valued W can operate. This is so because Eq. (1b) implies that,

$$\ddot{s} - W^2 s = 0, \tag{1c}$$

and the matrix $-W^2$ can be shown to conform to all the requirements stipulated for the matrix K below Eq. (1a). To see this, note that if s_ω is *any* complex-valued eigenvector of W, with $i\omega$ its corresponding imaginary eigenvalue, where ω is a real number, then *because* W is *real-valued*, the particular *complex-conjugated* vector s^*_ω is *as well* an eigenvector of W, but with eigenvalue $-i\omega$. Therefore the *real-valued* vector $s_\omega + s^*_\omega$ is an eigenvector of the real-valued matrix $-W^2$ with the real, nonnegative eigenvalue ω^2. In addition, since the complex-valued eigenvectors of W of the form s_ω are assumed to completely span the extended complex-valued vector space on which W can operate, it is apparent that the real-valued eigenvectors of $-W^2$ that have the form $s_\omega + s^*_\omega$ completely span the *real-valued* vector space on which the real-valued matrix $-W^2$ naturally operates—and of course the *eigenvalues* ω^2 of $-W^2$ associated to each member of this *complete set* of its real-valued eigenvectors are themselves real-valued and *nonnegative*. Therefore the nonvanishing real-valued matrix $-W^2$ of Eq. (1c) possesses *all* of the properties that are required of the real-valued matrix K of Eq. (1a).

It is further to be noted at this point that if the nonvanishing real-valued matrix W is *antisymmetric*, then it *automatically* fulfills the remaining requirements that are stipulated below Eq. (1b), and, *in addition*, a linear mapping of Eq. (1b) into a Schrödinger equation is immediately manifest. This is so because if W is real-valued and *antisymmetric*, then iW is *Hermitian* on the extended complex-valued vector space on which W can operate. By virtue of its Hermitian property, iW necessarily possesses a *complete set* of complex-valued eigenvectors, for each of which it has a corresponding *real* eigenvalue. Those real eigenvalues of iW correspond, of course, to *imaginary* eigenvalues of W with the *same* corresponding eigenvectors, and that set of eigenvectors of course completely spans the extended complex-valued vector space on which W can operate. In addition, if we multiply both sides of Eq. (1b) by the factor $i\hbar$, it becomes a Schrödinger equation with the Hermitian Hamiltonian matrix $i\hbar W$.

In the next section we shall show that classical equations of motion given by Eq. (1a), with the restrictions on the matrix K that are stipulated below Eq. (1a), can always be linearly mapped into Schrödinger equations—consequently the same is true for classical equations of motion given by Eq. (1b) with the restrictions on the matrix W that are stipulated below Eq. (1b)). This task will be greatly facilitated by the fact that the oscillatory classical Eq. (1a) can be analyzed into its *normal modes*, which, of course, behave as *mutually independent simple harmonic oscillators*. It turns out that a classical simple harmonic oscillator equation of motion which has the natural angular frequency ω can be linearly mapped into a Schrödinger equation for an ultra-basic *single-state* quantum system whose *one-by-one* Hamiltonian "matrix" is either the real number $\hbar\omega$ or the real number $-\hbar\omega$. The classical equation of motion for a *collection* of such *mutually independent* simple harmonic oscillators (i.e., an oscillatory classical system that has been *analyzed* into its normal modes) correspondingly linearly maps into a Schrödinger equation whose Hamiltonian matrix is *diagonal*, with its diagonal entries corresponding in one-to-one fashion to the angular frequencies of the independent simple harmonic oscillators which comprise that particular collection: each such Hamiltonian-matrix diagonal entry is a unique one of those angular frequencies times one of the two allowed factors $\pm\hbar$.

We now turn to the technical details of the analysis of Eq. (1a) into its normal modes, and the subsequent linear mapping of such collections of independent simple harmonic oscillators into Schrödinger equations.

Analysis into normal modes and their mapping into Schrödinger equations

The real-valued eigenvectors q_j of K in Eq. (1a) completely span the real-valued vector space on which K naturally operates, and each q_j corresponds to a nonnegative eigenvalue ω_j^2, where we take ω_j to be real and nonnegative. Therefore the q_j satisfy eigenvalue equations of the form,

$$Kq_j = \omega_j^2 q_j \tag{2a}$$

It turns out that we can use these eigenvectors q_j to construct a matrix S which is invertible, and for which the composite matrix $S^{-1}KS$ is in *diagonal form*, with all of its nondiagonal entries being equal to zero, while its diagonal entries embrace all the eigenvalues ω_j^2 of K. *Because* of this *diagonal form* of the matrix $S^{-1}KS$, it will be the case that *each* of the *components* $(S^{-1}q)_j$ of the *transformation* $S^{-1}q$ of the dynamical vector q of Eq. (1a) satisfies an *independent* simple harmonic oscillator equation whose natural angular frequency ω_j is the nonnegative square root of one of the eigenvalues ω_j^2 of the matrix K. In short, the *components* of the *transformed vector* $S^{-1}q$ are the *normal modes* of Eq. (1a).

We shall now construct the matrix S by filling its columns with the components of a set of linearly independent q_j, where that set is sufficiently large to completely span the real-valued vector space on which K naturally operates,

$$S_{ij} \stackrel{\text{def}}{=} (q_j)_i. \tag{2b}$$

Because the columns of the matrix S are linearly independent and completely span the real-valued vector space on which S (and K) naturally operate, the matrix S will have an *inverse* S^{-1}. In addition, because of the eigenvalue equations given by Eq. (2a) and the definition of S given by Eq. (2b), it is readily verified that,

$$(KS)_{kj} = K_{kl}(q_j)_l = (Kq_j)_k = \omega_j^2 S_{kj}. \tag{2c}$$

This result permits us to verify that $S^{-1}KS$ is precisely the diagonal form of the matrix K mentioned below Eq. (2a),

$$(S^{-1}KS)_{mj} = (S^{-1})_{mk}(KS)_{kj} = \omega_j^2(S^{-1})_{mk}S_{kj} = \omega_j^2(S^{-1}S)_{mj} = \omega_j^2\delta_{mj}. \tag{2d}$$

It is convenient to denote this diagonal form of K as K_S,

$$K_S \stackrel{\text{def}}{=} S^{-1}KS, \tag{3a}$$

If we now multiply Eq. (1a) through by the matrix S^{-1} and further define,

$$q_S \stackrel{\text{def}}{=} S^{-1}q, \tag{3b}$$

we obtain from Eq. (1a) that,

$$\ddot{q}_S + K_S q_S = 0, \tag{3c}$$

which, from Eqs. (3a) and (2d), reads when written in component form,

$$d^2(q_S)_j/dt^2 + \omega_j^2(q_S)_j = 0, \tag{3d}$$

which is a set of mutually independent simple harmonic oscillator equations whose natural angular frequencies ω_j are given by the *nonnegative square roots* of the nonnegative *eigenvalues* ω_j^2 of the matrix K. From Eq. (3d) we see that the normal-mode simple harmonic oscillator variables are the *components* of the vector q_S.

We wish at this point to further linearly map the set of mutually independent simple harmonic oscillator equations encompassed by Eq. (3c) into a Schrödinger equation. Notwithstanding that they have the *same form*, Eqs. (3c) and (1a) *crucially differ* in that K_S in Eq. (3c) is *known to be diagonal* (with real nonnegative entries on the principal diagonal and uniformly zero entries elsewhere), whereas K in Eq. (1a) is *not* guaranteed to be diagonal. When we now attempt to pass to a Schrödinger equation, we obviously do *not* wish to *undo* the *simplicity* that having *only diagonal matrices present* confers on an equation of motion. Therefore we now make it a rigid rule that *any* attempted further linear mapping of Eq. (3c) into (hopefully) a Schrödinger equation *may only be attempted with diagonal matrices*. This affords an *immediate benefit*: diagonal matrices *all mutually commute*.

Now a Schrödinger equation has the form,

$$i\hbar\dot{\psi} = H\psi, \tag{4a}$$

and, of course, our *cardinal rule* stated above requires that the Hermitian matrix H be *diagonal*.

Eq. (3c) is second-order in time, whereas the Schrödinger Eq. (4a) is first-order in time. To *reconcile* this difference in order, it is *necessary* to take ψ to be a linear mapping of $\dot{q}_S$, and possibly of q_S itself as well. Therefore we now make the *ansatz*,

$$\psi = iN(Wq_S + \dot{q}_S), \tag{4b}$$

where the matrices N and W are of course *both* assumed to be *diagonal*. We make the further assumption that the matrix N is invertible (i.e., has *no* vanishing entries on its principal diagonal), which implies that it simply *factors out* of the linear, homogeneous Schrödinger Eq. (4a), and therefore is *not* determined by it. From Eq. (3c), we know that $\ddot{q}_S = -K_S q_S$. Therefore putting the *ansatz* of Eq. (4b) into the Schrödinger Eq. (4a) results in,

$$i\hbar W\dot{q}_S - i\hbar K_S q_S = HWq_S + H\dot{q}_S, \tag{4c}$$

which yields the two equations,

$$i\hbar W = H, \quad -i\hbar K_S = HW, \tag{4d}$$

that have the solutions,

$$H = \hbar(K_S)^{\frac{1}{2}}, \quad W = -i(K_S)^{\frac{1}{2}}, \tag{4e}$$

which are *consistent* with our *assumption* that H and W are *diagonal matrices*, and *also* imply that H is Hermitian. Putting the results of Eq. (4e) into Eq. (4b) together with the definition of K_S given by Eq. (3a) and that of q_S given by Eq. (3b) yields the desired linear mapping of q and $\dot{q}$ of Eq. (1a) into the Schrödinger equation wave vector ψ, and also yields the associated Hamiltonian matrix H of that Schrödinger equation,

$$\psi = N((S^{-1}KS)^{\frac{1}{2}}S^{-1}q + iS^{-1}\dot{q}), \quad H = \hbar(S^{-1}KS)^{\frac{1}{2}}. \tag{4f}$$

From Eq. (4f), bearing in mind that both N and $(S^{-1}KS)^{\frac{1}{2}}$ are mutually commuting *diagonal* matrices and N is invertible, it can readily be shown that the Schrödinger Eq. (4a) for ψ *follows* from the underlying *classical* Eq. (1a) for q.

We as well note from Eq. (4f) that if all the eigenvalues of K are *positive*, i.e., the classical system is *purely* oscillatory, then the diagonalized matrix $S^{-1}KS$ is *invertible*, as is the diagonal matrix $(S^{-1}KS)^{\frac{1}{2}}$, and therefore the linear mapping between q and ψ is *also* invertible.

An interesting mathematical point is that since the diagonal entries of $S^{-1}KS$ are all real and nonnegative (they are the the eigenvalues of K), $(S^{-1}KS)^{\frac{1}{2}}$ is certainly *defined* as a diagonal matrix, but *multiply* so, i.e., the *signs* of the nonzero diagonal entries of $(S^{-1}KS)^{\frac{1}{2}}$ *can be chosen at will*. So from a strictly mathematical point of view, Eq. (4f) specifies a whole set of distinct linear mappings of the classical q into Schrödinger wave vectors ψ, with *equally distinct* Hamiltonian matrices $H = \hbar(S^{-1}KS)^{\frac{1}{2}}$ to accompany each distinct linear mapping.

Although the Schrödinger Eq. (4a) *does not determine* the invertible diagonal "normalization" matrix N of our Schrödinger wave vector ψ of Eq. (4f), we can ask if there is an *additional physically sensible requirement* which impinges on the value of that "normalization" diagonal matrix N.

Now the behavior of quantum expectation values frequently closely parallels that of their classical counterparts, as Ehrenfest's Theorem attests, and that is particularly the case for simple linear systems. Specifically, the expectation value of the Hamiltonian matrix H, namely $\psi^*H\psi$, is a *real-valued* function of ψ and ψ^* with the dimension of *energy* which is *conserved* because the time evolution of ψ is governed by the Schrödinger Eq. (4a) and the Hamiltonian matrix is Hermitian—this *conservation* of $\psi^*H\psi$ can be explicitly verified. The clear classical analog of $\psi^*H\psi$ is therefore, of course, the *classical conserved energy* that is associated with the Eq. (3c) classical equation of motion. That classical conserved energy is the nonnegative entity,

$$\mathcal{E}_{K_S}(q_S, \dot{q}_S) \stackrel{\text{def}}{=} (\dot{q}_S\dot{q}_S + q_S K_S q_S)/(2\gamma^2), \tag{5a}$$

where the dimension and magnitude of the real positive number γ depends on the dimension and normalization of q_S—note that $\mathcal{E}_{K_S}(q_S, \dot{q}_S)$ is *required* to have the dimension of *energy*. That $\mathcal{E}_{K_S}(q_S, \dot{q}_S)$ is *conserved*, i.e., that its time derivative *vanishes*, follows directly from Eq. (3c) itself and the fact that K_S is diagonal.

Therefore it is *completely sensible physically* to attempt to determine N by *additionally* imposing the utterly natural requirement that,

$$\psi^* H\psi = \mathcal{E}_{K_S}(q_S, \dot{q}_S) = (\dot{q}_S\dot{q}_S + q_S K_S q_S)/(2\gamma^2), \tag{5b}$$

whenever this is possible—we shall see that Eq. (5b) requires the real nonnegative diagonal matrix K_S to be positive definite, i.e., the classical system must be *purely* oscillatory. Furthermore, the strictly *nonnegative character* of the classical energy $\mathcal{E}_{K_S}(q_S, \dot{q}_S)$ now *precludes the possibility* that the diagonal Hamiltonian matrix $H = \hbar(S^{-1}KS)^{\frac{1}{2}}$ can have *anything other than nonnegative entries*. *Unlike* the Schrödinger Eq. (4a), Eq. (5b) is, of course, *neither* linear nor homogeneous in ψ. We now reexpress Eq. (4f) in the more compact form,

$$\psi = N((K_S)^{\frac{1}{2}} q_S + i\dot{q}_S), \quad H = \hbar(K_S)^{\frac{1}{2}}, \tag{5c}$$

and substitute the right-hand sides of both the first and second equalities of Eq. (5c) into the left hand side of Eq. (5b). For the left and right hand sides of Eq. (5b) to then be able to be equal, the following equation involving the diagonal matrices N^*, N and $(K_S)^{\frac{1}{2}}$ must be satisfied,

$$N^* N (K_S)^{\frac{1}{2}} = \mathrm{I}/(2\hbar\gamma^2), \tag{5d}$$

where I is the identity matrix. Of course this is *not possible* if $(K_S)^{\frac{1}{2}}$ has *any* vanishing or negative diagonal entries. If $(K_S)^{\frac{1}{2}}$ indeed has *only positive entries*, which can only be the case if the classical system is *purely* oscillatory, then the *simplest* solution for the diagonal matrix N is one with only real-valued diagonal entries, namely,

$$N = (K_S)^{-\frac{1}{4}}/(2\gamma^2\hbar)^{\frac{1}{2}}. \tag{5e}$$

Putting this determination of N into Eq. (5c) results in the properly normalized Schrödinger wave vector,

$$\psi = ((K_S)^{\frac{1}{4}} q_S + i(K_S)^{-\frac{1}{4}}\dot{q}_S)/(2\gamma^2\hbar)^{\frac{1}{2}}, \quad H = \hbar(K_S)^{\frac{1}{2}}, \tag{5f}$$

which in the more explicit notation used in Eq. (4f) reads,

$$\psi = ((S^{-1}KS)^{\frac{1}{4}} S^{-1}q + i(S^{-1}KS)^{-\frac{1}{4}} S^{-1}\dot{q})/(2\gamma^2\hbar)^{\frac{1}{2}}, \quad H = \hbar(S^{-1}KS)^{\frac{1}{2}}, \tag{5g}$$

where the diagonal matrices $S^{-1}KS$ and $(S^{-1}KS)^{\frac{1}{2}}$ now both need to be *positive definite*, and the real positive constant γ comes from the classical energy function $\mathcal{E}_{S^{-1}KS}(S^{-1}q, S^{-1}\dot{q})$ of Eq. (5a) that is appropriate to the *purely* oscillatory classical equation of motion system of Eq. (3c),

$$\mathcal{E}_{S^{-1}KS}(S^{-1}q, S^{-1}\dot{q}) = ((S^{-1}\dot{q})(S^{-1}\dot{q}) + (S^{-1}q)(S^{-1}KS)(S^{-1}q))/(2\gamma^2), \tag{5h}$$

Because the diagonal matrix $(S^{-1}KS)^{\frac{1}{2}}$ is positive definite, the linear mapping of q and $\dot{q}$ into ψ given in Eq. (5g) is *invertible*,

$$q = ((\gamma^2\hbar)/2)^{\frac{1}{2}} S(S^{-1}KS)^{-\frac{1}{4}}(\psi + \psi^*), \quad \dot{q} = -i((\gamma^2\hbar)/2)^{\frac{1}{2}} S(S^{-1}KS)^{\frac{1}{4}}(\psi - \psi^*). \tag{5i}$$

While the Eq. (5g) route to the desired invertible linear mapping of Eq. (1a) into Schrödinger Eq. (4a) is of great generality in principle, in practice it suffers from the need to explicitly know all the eigenvectors of K in order to be able to *construct* S, and, in addition, from the need to explicitly *invert* S.

We see from Eq. (5g) that one of the consequences of having the matrix S and its inverse S^{-1} in hand is that the Schrödinger equation's Hamiltonian matrix $H = \hbar(S^{-1}KS)^{\frac{1}{2}}$ is presented to us in *already diagonal* form. It is certainly *not essential* that that be the case. In the next section we therefore simply expunge S and its inverse from Eq. (5g), which of course will work if K is *diagonal*. However it quickly becomes clear that the resulting expression *still* works when K is merely *symmetric*.

Schrödinger-equation presentation of symmetrically coupled oscillatory systems

The result of expunging S and S^{-1} from Eq. (5g) is,

$$\psi = (K^{\frac{1}{4}} q + iK^{-\frac{1}{4}}\dot{q})/(2\gamma^2\hbar)^{\frac{1}{2}}, \quad H = \hbar K^{\frac{1}{2}}, \tag{6a}$$

and if K is a real-valued *symmetric* positive-definite matrix, all the expressions in it *still* make sense: in those circumstances $H = \hbar K^{\frac{1}{2}}$ is well defined as a real-valued *symmetric* positive-definite matrix *itself*. Therefore H is *Hermitian*, as required, and $K^{\frac{1}{4}}$ and $K^{-\frac{1}{4}}$ are well-defined as real-valued symmetric invertible matrices. Furthermore, it is straightforwardly verified that in *consequence* of the basic oscillatory classical equation of motion of Eq. (1a), the wave vector ψ of Eq. (6a) *satisfies* the Schrödinger Eq. (4a) with the Hamiltonian matrix $H = \hbar K^{\frac{1}{2}}$ given by Eq. (6a). In addition, when K is a real-valued *symmetric* positive-definite matrix, Eq. (6a) yields,

$$\psi^* H\psi = (\dot{q}\dot{q} + qKq)/(2\gamma^2), \tag{6b}$$

and if γ has been appropriately selected such that,

$$\mathcal{E}_K(q, \dot{q}) \stackrel{\text{def}}{=} (\dot{q}\dot{q} + qKq)/(2\gamma^2), \tag{6c}$$

has the dimension of *energy*, then it is clear that,

$$L_K(q, \dot{q}) \stackrel{\text{def}}{=} (\dot{q}\dot{q} - qKq)/(2\gamma^2), \tag{6d}$$

also has the dimension of energy. Moreover, it is easily verified that the Euler-Lagrange equation which *follows* from the Lagrangian $L_K(q, \dot{q})$ of Eq. (6d) is *precisely* the Eq. (1a) classical equation of motion. Now the *conserved energy* of any classical system that *has* a Lagrangian L is well-known to be *uniquely given* by $(\dot{q}\nabla_{\dot{q}}L - L)$, which, for the particular Eq. (1a) case that L is given by $L_K(q, \dot{q})$ of Eq. (6d), is straightforwardly verified to be $\mathcal{E}_K(q, \dot{q})$, as defined by Eq. (6c). Therefore, Eqs. (6b) and (6c) show that when K is real-valued, *symmetric* and positive definite, then the expectation value of the Hamiltonian matrix which *follows* from Eq. (6a) is *equal* to the conserved energy of the *classical* system of Eq. (1a), as required.

Finally, when K is real-valued, symmetric and positive definite, the *inverse* of the Eq. (6a) linear mapping of q and $\dot{q}$ into ψ is readily calculated to be,

$$q = ((\gamma^2\hbar)/2)^{\frac{1}{2}} K^{-\frac{1}{4}}(\psi + \psi^*), \quad \dot{q} = -i((\gamma^2\hbar)/2)^{\frac{1}{2}} K^{\frac{1}{4}}(\psi - \psi^*), \tag{6e}$$

which is, as expected, the result of expunging S and S^{-1} from Eq. (5i).

What if the matrix K of the classical Eq. (1a) is nonsymmetric? We then *first* need to find a real-valued invertible matrix S such that the *similarity-transformed* $K_S \stackrel{\text{def}}{=} S^{-1}KS$ *is* symmetric. Eq. (6a) is *extended* to cover this situation by replacing K by K_S and q by $q_S \stackrel{\text{def}}{=} S^{-1}q$, *precisely* as in Eq. (5f), *except* that *now* K_S is *merely symmetric and positive definite*, not necessarily diagonal.

In the next section, we use the machinery of Eq. (5f) and its associated Eq. (3c) similarity-transformed version of the Eq. (1a) classical equation of motion (albeit always bearing in mind that K_S is *merely symmetric and positive definite*, not diagonal) to show that the real and imaginary parts of ψ times the factor $(2\hbar)^{\frac{1}{2}}$ obey a simple first-order coupled equation of motion which can immediately be Hamiltonized and then quantized. This *second quantization* of an oscillatory classical system's linear mapping into a Schrödinger equation is a very easy route to that underlying system's quantization, and one which as well automatically yields considerable physical insight.

Hamiltonization and quantization of the Schrödinger-equation presentation

Taking the real-valued similarity-transformed K_S in both Eqs. (3c) and (5f) to now be, as discussed above, *merely symmetric and positive definite* rather than necessarily diagonal, we note that the real and imaginary parts of the wave vector ψ of Eq. (5f), each multiplied (for later convenience) by the factor $(2\hbar)^{\frac{1}{2}}$, are given by,

$$q_c \stackrel{\text{def}}{=} (\hbar/2)^{\frac{1}{2}}(\psi + \psi^*) = (K_S)^{\frac{1}{4}} q_S/\gamma, \qquad p_c \stackrel{\text{def}}{=} -i(\hbar/2)^{\frac{1}{2}}(\psi - \psi^*) = (K_S)^{-\frac{1}{4}} \dot{q}_S/\gamma, \tag{7a}$$

which are readily seen, as a consequence of Eq. (3c), which is a similarity-transformed version of the underlying Eq. (1a) classical equation of motion, to satisfy the simple first-order coupled antisymmetrical equation of motion,

$$\dot{q}_c = (K_S)^{\frac{1}{2}} p_c, \qquad \dot{p}_c = -(K_S)^{\frac{1}{2}} q_c. \tag{7b}$$

With a little effort, it can also be verified that the Eq. (7b) system *implies* the Schrödinger Eq. (4a) with $H = \hbar(K_S)^{\frac{1}{2}}$. Moreover, for $H = \hbar(K_S)^{\frac{1}{2}}$, where $(K_S)^{\frac{1}{2}}$ is real-valued and symmetric, the two equalities of the Eq. (7b) system *follow from* simply the real and imaginary parts of the Schrödinger Eq. (4a). In other words, for the situation that we are concerned with here, namely that $H = \hbar(K_S)^{\frac{1}{2}}$, where $(K_S)^{\frac{1}{2}}$ is real-valued, symmetric and positive definite, the real-valued coupled antisymmetrical system of Eq. (7b) is *completely equivalent* to the complex valued Schrödinger Eq. (4a).

In *addition*, Eq. (7b) *also* follows from a simple bilinear *classical Hamiltonian*, namely,

$$\mathcal{H}_{K_S}(q_c, p_c) = (q_c (K_S)^{\frac{1}{2}} q_c + p_c (K_S)^{\frac{1}{2}} p_c)/2, \tag{7c}$$

via the *classical canonical* Hamiltonian equations of motion, i.e.,

$$\dot{q}_c = \nabla_{p_c} \mathcal{H}_{K_S}(q_c, p_c), \quad \dot{p}_c = -\nabla_{q_c} \mathcal{H}_{K_S}(q_c, p_c), \tag{7d}$$

and the fact that $(K_S)^{\frac{1}{2}}$ is a real *symmetric* matrix.

By putting the definition of (q_c, p_c) given in Eq. (7a) into Eq. (7c) we can reexpress our system's *classical Hamiltonian* in terms of its Schrödinger-equation presentation wave vector ψ and complex conjugate ψ^*,

$$\mathcal{H}_{K_S}(q_c, p_c) = (\psi^* H \psi + \psi H \psi^*)/2 = \psi^* H \psi, \tag{7e}$$

where the last equality in Eq. (7e) follows from the fact that $H = \hbar (K_S)^{\frac{1}{2}}$ is a real, *symmetric* matrix. It is pleasing to once again see the quantum expectation value of the Hamiltonian matrix H come out to be *equal* to the Schrödinger-equation presentation's *classical* energy, i.e., to its *classical Hamiltonian*.

Since Eqs. (7c) and (7d) assure us that the classical equations of motion of Eq. (7b) obeyed by (q_c, p_c) are *indeed* presented in *canonical Hamiltonian form*, we can now safely *quantize* this classical system by imposing first Dirac's canonical commutation rules on the *components* of (q_c, p_c), and next Heisenberg's equations of motion on the now quantized $(\widehat{q}_c, \widehat{p}_c)$. Dirac's canonical commutation rules *promote* the *components* of (q_c, p_c) into noncommuting *Hermitian operators* which obey the commutation relations,

$$[(\widehat{q}_c)_i, (\widehat{q}_c)_j] = [(\widehat{p}_c)_i, (\widehat{p}_c)_j] = 0, \quad [(\widehat{q}_c)_i, (\widehat{p}_c)_j] = i\hbar \delta_{ij}. \tag{8a}$$

The Eq. (8a) commutation relations, in turn, *imply* that the components of the *non-Hermitian* quantized wave vector $\widehat{\psi} = (\widehat{q}_c + i\widehat{p}_c)/(2\hbar)^{\frac{1}{2}}$ satisfy, *in conjunction with* the components of this quantized wave vector's *Hermitian conjugate* $\widehat{\psi}^{\,\dagger} = (\widehat{q}_c - i\widehat{p}_c)/(2\hbar)^{\frac{1}{2}}$, the following commutation relations,

$$[\widehat{\psi}_i, \widehat{\psi}_j] = [\widehat{\psi}_i^{\,\dagger}, \widehat{\psi}_j^{\,\dagger}] = 0, \quad [\widehat{\psi}_i, \widehat{\psi}_j^{\,\dagger}] = \delta_{ij}. \tag{8b}$$

These Eq. (8b) commutation relations are the *fundamental* ones for the components of our Schrödinger-equation presentation *quantized* wave vector, and they give those quantized wave-vector components and their Hermitian conjugates, respectively, their well-known interpretation as *annihilation and creation operators*, which is so often crucial to physical understanding. They are *as well* the key to constructing *physically useful* orthogonal basis sets for the *second-quantized* Hilbert space that is the result of the imposition of Dirac's canonical commutation rules on the components of the dynamical-variable vector (q_c, p_c).

The Hamiltonian *operator* for this quantized (i.e., *second* quantized) Schrödinger-equation presented system is obtained by substituting the *quantized* dynamical-variable vector $(\widehat{q}_c, \widehat{p}_c)$ into the system's *classical* Hamiltonian of Eq. (7c), namely by writing down,

$$\mathcal{H}_{K_S}(\widehat{q}_c, \widehat{p}_c) = (\widehat{q}_c (K_S)^{\frac{1}{2}} \widehat{q}_c + \widehat{p}_c (K_S)^{\frac{1}{2}} \widehat{p}_c)/2, \tag{8c}$$

which could have ambiguities due to *operator-ordering* issues, but it is apparent that *those do not arise* in this case. Noting that the quantized dynamical-variable vector $(\widehat{q}_c, \widehat{p}_c)$ is given in terms of the quantized wave vector $\widehat{\psi}$ and its Hermitian conjugate $\widehat{\psi}^{\,\dagger}$ by the quantized analog of the two definitions in Eq. (7a), namely $\widehat{q}_c = (\hbar/2)^{\frac{1}{2}}(\widehat{\psi} + \widehat{\psi}^{\,\dagger})$ and $\widehat{p}_c = -i(\hbar/2)^{\frac{1}{2}}(\widehat{\psi} - \widehat{\psi}^{\,\dagger})$, we reexpress the *uniquely defined* second-quantized Hamiltonian operator $\mathcal{H}_{K_S}(\widehat{q}_c, \widehat{p}_c)$ of Eq. (8c) in terms of the *quantized wave vector* $\widehat{\psi}$ and its Hermitian conjugate $\widehat{\psi}^{\,\dagger}$,

$$\mathcal{H}_{K_S}(\widehat{q}_c, \widehat{p}_c) = (\widehat{\psi}^{\,\dagger} H \widehat{\psi} + \widehat{\psi} H \widehat{\psi}^{\,\dagger})/2, \tag{8d}$$

where $H = \hbar (K_S)^{\frac{1}{2}}$, a real symmetric positive definite matrix.

If we now apply Heisenberg's equation of motion and the commutation rules for the components of the quantized $\widehat{\psi}$ and $\widehat{\psi}^{\,\dagger}$ that are given by Eq. (8b) to the second-quantized Hamiltonian operator written in the form given by Eq. (8d), we can calculate the time derivative of any component of the Schrödinger-equation presentation *quantized* wave vector $\widehat{\psi}$,

$$d\widehat{\psi}_i/dt = (-i/\hbar)[\widehat{\psi}_i, (\widehat{\psi}^{\,\dagger} H \widehat{\psi} + \widehat{\psi} H \widehat{\psi}^{\,\dagger})/2] = (-i/\hbar)((H\widehat{\psi})_i + (\widehat{\psi} H)_i)/2 = (-i/\hbar)(H\widehat{\psi})_i, \tag{8e}$$

where the last step reflects the real *symmetric* character of the Hamiltonian matrix $H = \hbar(K_S)^{\frac{1}{2}}$. Thus we have shown that,

$$i\hbar d\widehat{\psi}/dt = H\widehat{\psi}, \tag{8f}$$

i.e., the Schrödinger Eq. (4a) which the Schrödinger-equation presentation wave vector ψ satisfies is *also* satisfied by that wave vector's operator *quantization* $\widehat{\psi}$, which itself is, of course, a vector of the *annihilation operators* of the complete set of quantum states which the *components* of the wave vector ψ individually describe.

We next turn to the Schrödinger-equation presentations of specifically the classical Klein-Gordon equation and the source-free Maxwell equations.

The spinless quantum free particle from the classical Klein-Gordon equation

The classical Klein-Gordon equation for the real-valued scalar field ϕ differs from the classical wave equation by a simple mass term [3, 1],

$$\ddot{\phi} + (-c^2\nabla^2 + \omega^2)\phi = 0, \tag{9a}$$

where $\omega = ((mc^2)/\hbar)$. Eq. (9a) has the form of Eq. (1a) with,

$$K = -c^2\nabla^2 + \omega^2, \tag{9b}$$

which, on the space of real-valued scalar fields, is a real-valued, symmetric, positive-definite operator with the dimension of frequency squared. Therefore, starting with Eq. (6a) above and going right through to Eq. (8f), we have results that can all be transcribed for the real-valued classical Klein-Gordon equation. We need to bear in mind that during this exercise K is specifically defined by Eq. (9b) and that the real-valued classical dynamical vector q is defined by the real-valued ϕ, which, as a real-valued vector, of course has a three-dimensional *continuous* index instead of a discrete one. In such a case the *summation* that defines index contraction is willy-nilly supplanted by three-dimensional *integration*, which compels some systematic technical changes in the formalism, for example in the dimension of the variables that one deals with (summation is over dimensionless indices, integration here involves the three space dimensions) and in the fact that Kronecker deltas give way to three-dimensional delta functions. That notwithstanding, most of the results properly transcribed to the case of the real-valued classical Klein-Gordon equation remain very similar in appearance to the formulas that run from Eq. (6a) through Eq. (8f).

In particular, Eq. (6a) needs essentially no modification; one simply bears in mind that the operator K is given by Eq. (9b), and one replaces the occurrences of q and $\dot{q}$ by ϕ and $\dot{\phi}$. The only remaining issue is one of a global reconciliation of dimension, which requires the determination of the parameter γ that appears Eq. (6a) so as to accord with the conventions one intends to adopt for the classical Klein-Gordon theory. Now one *conventional* choice of dimension for ϕ is the *same* as that of the electromagnetic vector potential $\mathbf{A}$ [3, 1], which implies that $\int |\nabla\phi|^2 d^3\mathbf{r}$ has the dimension of energy. A glance at the classical conserved energy given by Eq. (6c) reveals that γ must have the dimension of c, so we choose the value c for γ. With that, Eq. (6a) yields the mapping into the wave function and Hamiltonian operator of the Schrödinger equation that corresponds to the classical Klein-Gordon theory,

$$\psi = (K^{\frac{1}{4}}\phi + iK^{-\frac{1}{4}}\dot{\phi})/(2c^2\hbar)^{\frac{1}{2}}, \quad H = \hbar K^{\frac{1}{2}}, \tag{9c}$$

where the operator K is, of course, given by Eq. (9b). The *inverse* of this mapping from ϕ and $\dot{\phi}$ into the complex-valued Schrödinger wave function ψ is easily calculated, or may be transcribed from Eq. (6e),

$$\phi = c(\hbar/2)^{\frac{1}{2}}K^{-\frac{1}{4}}(\psi + \psi^*), \quad \dot{\phi} = -ic(\hbar/2)^{\frac{1}{2}}K^{\frac{1}{4}}(\psi - \psi^*). \tag{9d}$$

Now let's take a closer look at the Schrödinger equation's Hamiltonian operator,

$$H = \hbar K^{\frac{1}{2}} = \hbar(-c^2\nabla^2 + ((mc^2)/\hbar)^2)^{\frac{1}{2}} \tag{9e}$$

In configuration space the quantum momentum operator $\widehat{\mathbf{p}}$ is well-known to be given by,

$$\widehat{\mathbf{p}} = -i\hbar\nabla, \tag{9f}$$

so that,

$$-\nabla^2 = |\widehat{\mathbf{p}}|^2/\hbar^2, \tag{9g}$$

which, when substituted into the expression for H in Eq. (9e), yields,

$$H = (|c\widehat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}, \tag{9h}$$

which is precisely the quantization of the standard relativistic energy of a free particle of mass m. Thus we have the fascinating state of affairs that the *classical* Klein-Gordon equation (i.e., with *real-valued* ϕ) is *linearly isomorphic* to the very Schrödinger equation with the correspondence-principle *mandated* square-root Hamiltonian for the free particle of mass m that Klein and Gordon were in fact trying to sideline. If Klein and Gordon had but been aware of the Eq. (6a) theorem with its *square-root character* of the Hamiltonian matrix $H = \hbar K^{\frac{1}{2}}$, the history of relativistic quantum mechanics and its second quantization might have taken a different route, one in closer harmony with the correspondence principle.

Second quantization of the Schrödinger wave function ψ for the classical Klein-Gordon theory can be transcribed from Eqs. (8). Here the different dimension of ψ that is imposed by its continuum character results in its basic commutation relations coming out in terms of a three-dimensional delta function instead of in terms of the Kronecker delta of Eq. (8b).

$$[\widehat{\psi}(\mathbf{r}), \widehat{\psi}^\dagger(\mathbf{r}')] = \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \qquad [\widehat{\psi}(\mathbf{r}), \widehat{\psi}(\mathbf{r}')] = 0, \qquad [\widehat{\psi}^\dagger(\mathbf{r}), \widehat{\psi}^\dagger(\mathbf{r}')] = 0. \tag{9i}$$

This promotion of the *Schrödinger wave function* $\psi(\mathbf{r})$ to operator field is the most straightforward and physically transparent route to the *quantization* of the classical Klein-Gordon field $\phi(\mathbf{r})$, which, of course, is explicitly given by Eq. (9d) in terms of the Schrödinger wave function and its complex conjugate. The familiar physical interpretation attached to the commutation relations of Eq. (9i) is that the operator field $\widehat{\psi}^\dagger(\mathbf{r})$ creates a relativistic spinless particle of mass m at location $\mathbf{r}$, while the operator field $\widehat{\psi}(\mathbf{r})$ destroys such a particle. Such particle creation and destruction operator fields are *non-Hermitian*. However, from the first equality of Eq. (9d) we note that the quantized classical Klein-Gordon field $\widehat{\phi}(\mathbf{r})$ *itself* will, on the contrary, turn out to be *Hermitian*, and will be ambiguously capable of *both* particle creation and annihilation. In light of the second equality in Eq. (9d), the same comments apply to the quantization of the time derivative of the classical Klein-Gordon field $d\widehat{\phi}(\mathbf{r})/dt$. A telling characteristic of both of these *Hermitian* fields is that *by themselves* they *only* obey the original *second-order* real-valued *classical* Klein-Gordon equation. Eqs. (9c), (9d) and (9i) thus mathematically depict the *complementarity* of the quantized particle outlook (oriented toward non-Hermitian second-quantized wave-functions that unambiguously either annihilate or create particles, and obey a first-order complex-valued *quantum* Schrödinger equation) to the classical wave outlook (oriented toward Hermitian fields that by themselves *only* obey a real-valued second-order *classical* wave equation).

Finally, we wish to exhibit, in terms of these quantized Schrödinger wave functions that create or destroy particles, the Hamiltonian operator functional that oversees free relativistic spinless particles in the second quantized world (we already met this operator in schematic form in Eq. (8d)),

$$\widehat{H}[\widehat{\psi}, \widehat{\psi}^{\,\dagger}] = \tfrac{1}{2}\int[\widehat{\psi}^{\,\dagger}(\mathbf{r})(-c^2\hbar^2\nabla^2 + m^2c^4)^{\frac{1}{2}}\widehat{\psi}(\mathbf{r}) + \widehat{\psi}(\mathbf{r})(-c^2\hbar^2\nabla^2 + m^2c^4)^{\frac{1}{2}}\widehat{\psi}^{\,\dagger}(\mathbf{r})]d^3\mathbf{r}. \tag{9j}$$

We now turn to the similar Schrödinger equation that corresponds to the real-valued homogeneous linear source-free Maxwell equations. The differences to the Schrödinger-equation results for the classical Klein-Gordon equation are that the resulting relativistic particle is *massless*, and that its wave function is a *vector* field which is *strictly transverse*.

Free-photon quantum mechanics from the source-free Maxwell equations

In the source-free case, the Coulomb and Gauss laws tell us that both the electric and magnetic fields are purely *transverse*, i.e., $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$. The results of the Maxwell law and Faraday's law in the source-free case are,

$$\dot{\mathbf{E}} = c\nabla \times \mathbf{B}, \qquad \dot{\mathbf{B}} = -c\nabla \times \mathbf{E}. \tag{10a}$$

This first-order equation system has the simple antisymmetrical character of Eq. (7b), which readily produces a Schrödinger equation. For example, the extremely simple transverse-vector wave function *ansatz* $\mathbf{\Psi} = \mathbf{E} + i\mathbf{B}$ will in consequence of Eq. (10a) satisfy the Schrödinger equation which has the Hamiltonian operator $\hbar c\,\mathbf{curl}$. Unfortunately this operator has *odd parity*, and therefore is not a physically appropriate Hamiltonian for electromagnetism. The reason that a Hamiltonian of odd parity has manifested itself here is that the transverse vector fields on either side of each of the two equations of Eq. (10a) are of *opposite* intrinsic parity:

namely $\mathbf{E}$ is a *polar* vector field, while $\mathbf{B}$ is an *axial* vector field. So it should be feasible to extract a physically appropriate *even-parity* Schrödinger-equation Hamiltonian operator from source-free electromagnetic theory by *first* recasting its linear homogeneous equations of motion such that they involve *only* transverse vector fields *which all have the same intrinsic parity*. We shall do this here by mapping the transverse axial-vector magnetic field $\mathbf{B}$ into a transverse polar-vector field that is already well-known to electromagnetic theory, namely the vector potential in radiation gauge [4]. Specifically, we define,

$$\mathbf{A} \stackrel{\text{def}}{=} (-\nabla^2)^{-1}(\nabla \times \mathbf{B}), \tag{10b}$$

where by $(-\nabla^2)^{-1}$ we mean the standard real-valued symmetric *integral operator* with the Coulomb kernel. Eq. (10b) implies that,

$$\nabla \cdot \mathbf{A} = 0, \tag{10c}$$

i.e., $\mathbf{A}$ is a *transverse* vector field. Furthermore, since $\mathbf{B}$ is *itself* a transverse vector field, Eq. 10b implies that,

$$\nabla \times \mathbf{A} = \mathbf{B}, \tag{10d}$$

which is, of course, the basic property of a vector potential $\mathbf{A}$. We can further delineate the properties of $\mathbf{A}$ in source-free electromagnetism by using its definition together with Faraday's law (i.e., the second equality in Eq. (10a)) to calculate its time derivative,

$$\dot{\mathbf{A}} = (-\nabla^2)^{-1}(\nabla \times \dot{\mathbf{B}}) = -c(-\nabla^2)^{-1}(\nabla \times (\nabla \times \mathbf{E})) = -c\mathbf{E}, \tag{10e}$$

where the last equality holds when $\mathbf{E}$ is *transverse*, which is, of course the case for source-free electromagnetism. So in that case,

$$\mathbf{E} = -\dot{\mathbf{A}}/c. \tag{10f}$$

Eqs. (10d) and (10f) together imply that for source-free electromagnetism, we can obtain *both* of $\mathbf{B}$ and $\mathbf{E}$ from $\mathbf{A}$, so we *only* need to concern ourselves with calculating the polar transverse vector field $\mathbf{A}$. Therefore we now substitute Eqs. (10d) and (10f) into the Maxwell law, which in the case of source-free electromagnetism is the first equality of Eq. (10a), to obtain a linear homogeneous second-order equation which involves the *polar* transverse vector field $\mathbf{A}$ *alone*,

$$\ddot{\mathbf{A}} - c^2\nabla^2\mathbf{A} = 0. \tag{10g}$$

This is, of course, the *classical wave equation*, and it bears a *marked resemblance* to the classical Klein-Gordon equation of Eq. (9a). The *only* differences are that in Eq. (10g) the parameter ω that appears in Eq. (9a) *vanishes identically*, and, of course, in Eq. (10g) the transverse vector field $\mathbf{A}$ *replaces* the scalar field ϕ of Eq. (9a). Even the *dimension* of the transverse vector field $\mathbf{A}$ is the *same* as the dimension that we *chose* for ϕ by adhering to a common convention [3, 1]. Therefore, for the linear mapping, and its inverse, of the real-valued transverse-vector fields $\mathbf{A}$ and $\dot{\mathbf{A}}$ into a complex-valued transverse-vector Schrödinger-equation wave function $\mathbf{\Psi}$, we can simply transcribe Eqs. (9b), (9c) and (9d) for the real-valued classical scalar Klein-Gordon theory, taking ω (and m) to be zero identically, and replacing ϕ, $\dot{\phi}$, and ψ by, respectively, $\mathbf{A}$, $\dot{\mathbf{A}}$, and $\mathbf{\Psi}$. Thus our basic real, symmetric operator is,

$$K = -c^2\nabla^2, \tag{10h}$$

which, to be sure, is not positive-definite in the broadest sense. However, Fourier transformation methodology indicates that on a sufficiently restricted function space, $-\nabla^2$ can indeed be regarded as positive definite. The operators we *actually require* in the following mapping formulas are $(-\nabla^2)^{\frac{1}{2}}$, $(-\nabla^2)^{-\frac{1}{4}}$ and $(-\nabla^2)^{\frac{1}{4}}$, and they *themselves* have the tractable-looking positive-definite Fourier representations $|\mathbf{k}|$, $|\mathbf{k}|^{-\frac{1}{2}}$ and $|\mathbf{k}|^{\frac{1}{2}}$ respectively.

Transcribing Eq. (9c) as described above, the linear mapping of the real-valued transverse-vector fields $\mathbf{A}$ and $\dot{\mathbf{A}}$ into the complex-valued transverse-vector Schrödinger-equation wave function $\mathbf{\Psi}$, together with the associated Schrödinger-equation Hamiltonian operator, is given by,

$$\mathbf{\Psi} = (K^{\frac{1}{4}}\mathbf{A} + iK^{-\frac{1}{4}}\dot{\mathbf{A}})/(2c^2\hbar)^{\frac{1}{2}}, \quad H = \hbar K^{\frac{1}{2}}. \tag{10i}$$

The *inverse* of this linear mapping from $\mathbf{A}$ and $\dot{\mathbf{A}}$ into the complex-valued Schrödinger-equation wave function $\mathbf{\Psi}$ is,

$$\mathbf{A} = c(\hbar/2)^{\frac{1}{2}}K^{-\frac{1}{4}}(\mathbf{\Psi} + \mathbf{\Psi}^*), \quad \dot{\mathbf{A}} = -ic(\hbar/2)^{\frac{1}{2}}K^{\frac{1}{4}}(\mathbf{\Psi} - \mathbf{\Psi}^*). \tag{10j}$$

In light of Eq. (10h) and the fact that in configuration representation $\widehat{\mathbf{p}} = -i\hbar\nabla$, we have from the second equality in Eq. (10i) that the Schrödinger-equation Hamiltonian operator can be written,

$$H = \hbar K^{\frac{1}{2}} = \hbar(-c^2\nabla^2)^{\frac{1}{2}} = (c^2|\widehat{\mathbf{p}}|^2)^{\frac{1}{2}} = c|\widehat{\mathbf{p}}|. \tag{10k}$$

This Hamiltonian operator is clearly the quantized version of the relativistic energy of a *massless* free particle, which is appropriate to the free photon, and it as well has even parity.

By using Eqs. (10b) and (10e), the vector potential can be removed from the Schrödinger-equation linear mapping of Eq. (10i) in favor of the **E** and **B** fields,

$$\mathbf{\Psi} = (cK^{-\frac{3}{4}}(\nabla\times\mathbf{B}) - iK^{-\frac{1}{4}}\mathbf{E})/(2\hbar)^{\frac{1}{2}}, \quad H = \hbar K^{\frac{1}{2}}. \tag{11a}$$

The mapping of **E** and **B** into $\mathbf{\Psi}$ given in Eq. (11a) has the inverse,

$$\mathbf{B} = c(\hbar/2)^{\frac{1}{2}}K^{-\frac{1}{4}}(\nabla\times(\mathbf{\Psi}+\mathbf{\Psi}^*)), \quad \mathbf{E} = i(\hbar/2)^{\frac{1}{2}}K^{\frac{1}{4}}(\mathbf{\Psi}-\mathbf{\Psi}^*). \tag{11b}$$

We invite the reader to *verify* that the complex-valued linear mapping of the classical **E** and **B** fields into the wave function $\mathbf{\Psi}$ which Eq. (11a) specifies, along with its specified Hamiltonian operator $H = \hbar K^{\frac{1}{2}}$ (where $K = -c^2\nabla^2$), *actually satisfies* the Schrödinger equation. (Hint: use the source-free Maxwell and Faraday laws of Eq. (10a) and the transverse nature of the source-free **E** field.) One should *also verify* that the quantum expectation value of the Hamiltonian operator *agrees* with the *classical energy* of the **E** and **B** field system, i.e., that,

$$\int\mathbf{\Psi}^*(\mathbf{r})\cdot(H\mathbf{\Psi}(\mathbf{r}))\,d^3\mathbf{r} = \tfrac{1}{2}\int\left(|\mathbf{E}(\mathbf{r})|^2 + |\mathbf{B}(\mathbf{r})|^2\right)d^3\mathbf{r}. \tag{11c}$$

Upon their second quantization, Eqs. (11a) and (11b) manifest the expected tantalizing *complementary* interplay of the potential for photon creation and annihilation with the familiar, workaday transverse electric and magnetic fields.

In addition to its zero mass parameter, the second special feature of electromagnetic theory vis-à-vis classical Klein-Gordon theory is, of course, the free photon's *always transverse* polarization (spin) states. This signature free-photon characteristic does not cause much in the way of complications, but there is one formula concerning second quantization which it *notationally* impacts, albeit *no substantive physical effect is involved*. The canonical commutation rule for second quantization of the free photon's transverse vector wave function might naively be expected to read,

$$[(\widehat{\mathbf{\Psi}}(\mathbf{r}))_i, (\widehat{\mathbf{\Psi}}^\dagger(\mathbf{r}'))_j] = \delta_{ij}\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{12a}$$

but this is *not* mathematically consistent with the transverse character of the second-quantized photon wave-functions, i.e., it is mathematically inconsistent with the fact that $\nabla\cdot\widehat{\mathbf{\Psi}} = 0$. The nature of the right-hand of Eq. (12a) is one of completeness, but the transverse wave function creation and annihilation operators are *incomplete* in that they do *not* pertain to vector fields which are the gradients of scalar fields, i.e., they do *not* pertain to vector fields *which fail to be transverse*. Now the ij components of the *projection operator* onto the subspace of such purely gradient vector fields is given by,

$$P_{ij} = -\partial_i(-\nabla^2)^{-1}\partial_j. \tag{12b}$$

We note that P_{ij} is Hermitian, and that its contraction with itself yields itself, which are the two essential properties of the ij components of projection operators. Of course its contraction with the components of any transverse vector field vanishes. Thus $(\delta_{ij} - P_{ij})$ are the ij components of the *projection operator onto the subspace of transverse vector fields*, and therefore,

$$[(\widehat{\mathbf{\Psi}}(\mathbf{r}))_i, (\widehat{\mathbf{\Psi}}^\dagger(\mathbf{r}'))_j] = \langle\mathbf{r}|(\delta_{ij} - P_{ij})|\mathbf{r}'\rangle = (2\pi)^{-3}\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\left(\delta_{ij} - \mathbf{k}_i\mathbf{k}_j|\mathbf{k}|^{-2}\right)d^3\mathbf{k}. \tag{12c}$$

Notwithstanding these fancy maneuvers with projection operators, the *only* issue which is involved here is the simple fact that free-photon creation and annihilation operators (and as well free photon wave functions in the first quantized regime) are *purely transverse*, and therefore *any expression involving these operators*, e.g., the expression which describes their canonical commutation relation, must, of course, *correctly reflect this fact*. There is obviously *no physics implication* which flows from this requirement of *mere notational correctness*.

Conclusion

It is a remarkable fact that any classical system whose equation of motion is linear, homogeneous, *purely* oscillatory and conservative is effectively *already first-quantized*: once its Eq. (1a) coupling-strength matrix K has been similarity-transformed to a symmetric, positive-definite presentation, Eq. (6a) invertibly linearly maps that equation of motion into explicit time-dependent Schrödinger-equation form with Hamiltonian matrix $\hbar K^{\frac{1}{2}}$. Thus we see that Michael Faraday and James Clerk Maxwell were actually the first to effectively elucidate a quantized particle, namely the very important and not exactly simple ultra-relativistic massless transverse-vector free photon.

Any *complex-valued* solution wave function of a time-dependent Schrödinger-equation has the familiar characteristic expansion in terms of the complete set of mutually orthogonal eigenfunctions of that equation's Hamiltonian operator. The one-to-one linear mapping of any purely oscillatory linear classical system that is homogeneous and conservative into a Schrödinger equation thus implies a characteristic *two-component* eigenfunction expansion of such a classical system's solutions. For the case of certain wave equations that fall into the class of Eq. (1a), precisely such a solution expansion has been described in detail by Leung, Tong and Young [5].

The natural *correspondence-principle* version of the relativistic free-particle Schrödinger equation was *iterated* by Klein, Gordon and Schrödinger *for no physically motivated reason*, but merely in an effort to *rid* it of its *calculationally unpalatable* square-root Hamiltonian operator [6, 1, 7]. If this *iterated* equation is *still* regarded as a *complex-valued quantum-mechanical entity*, a large class of *completely extraneous, highly unphysical unbounded-below negative-energy solutions* are *injected* by that iteration. These also destroy its probability interpretation, and the fact that it depends on only the *square* of a Hamiltonian cuts it adrift from the Heisenberg picture and Ehrenfest theorem. However, if this iterated equation is regarded as the description of a *classical, real-valued* field, it thereupon becomes strongly analogous to the *classical wave equation*, and has an eminently sensible *nonnegative* conserved energy [3, 1]. This *classical* Klein-Gordon equation is *as well* one of those classical equation systems which is linearly equivalent to a Schrödinger equation: it quite marvelously *chooses* to be equivalent to *precisely* the Schrödinger equation with the natural correspondence-principle *square-root Hamiltonian operator* which Klein, Gordon and Schrödinger *had tried to sideline by concocting it*.

It is a pity that Klein, Gordon and Schrödinger had no idea of the theorem presented by this paper, and thus were not equipped to unearth this astonishing fact themselves. If they had but grasped the full consequences of the real-valued classical Klein-Gordon equation, they might well have abandoned their physically unmotivated rejection of the *correspondence-principle mandated* relativistic free-particle square-root Hamiltonian operator $(|c\widehat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}$ [7, 1].

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