Article

The Z = (t/z)-Type Plane Gravitational Waves and Electromagnetic Waves with Massless Scalar Plane Waves and Massive Scalar Waves in Plane Symmetry

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Abstract

In this paper, we deduce the existence of massless scalar plane waves coupled to Z = (t/z)-type plane gravitational waves and electromagnetic waves and non-existence of massive scalar waves coupled to plane gravitational waves and electromagnetic waves in Plane symmetric space-time introduced by Bhoyar and Deshmukh..

Keywords: Ricci tensor, massless scalar field, massive scalar field, energy momentum tensor, electromagnetic waves.

1. Introduction

In general relativity, Takeno [1] has rigorously studied some properties of purely plane gravitational waves and further he has obtained some plane wave solutions of field equations of relativity containing electromagnetic term and investigated their co-existence with electromagnetic waves. Furthemore he reduced a space-time having plane symmetry in the sense of Taub [4] whose metric is given by

$$ds^2 = -A\left(dx^2 + dy^2\right) - B\left(dz^2 - dt^2\right). \qquad (1.1)$$

In which A and B are functions of z and t , $Z = z - t$.

Recently Bhoyar and Deshmukh [7] transform the metric (1.1a) to (1.1b) using suitable transformations for $Z = (t/z)-$type plane gravitational waves, which take the form

$$ds^2 = -A\left(dx^2 + dy^2\right) - Z^2 B dz^2 + B dt^2 , \qquad (1.1b)$$

where A and B are functions of Z , $Z \equiv (t/z)$.

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Bhoyar and Deshmukh [6] investigate the coupling of $Z=(z-t)$ -type plane–fronted waves and electromagnetic waves with massless scalar plane waves and massive scalar waves in peres space-time. In this paper we add the massless and massive scalar field to the plane symmetric field given by (1.1b) and scrutinize its consequences.

For our investigations we engross the field equations of general relativity

$$R_{ij} = -8\pi(T_{ij} - \frac{1}{2}g_{ij}T), \quad i, j = 1,\ldots, 4 \tag{1.2}$$

The paper is organized as follows: In section 2, we have obtained the solutions of the field equations (1.2) for (1.1b) by introducing massless scalar field described by the tensor

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2}g_{ij}V_s V^s\right], \qquad s = 1,\ldots,4.$$

Where V is scalar function of Z and $V_i = \frac{\partial V}{\partial x^i}$, $\quad x^i = (x, y, z, t)$.

Similarly the section 3 is devoted to obtained the solutions of the field equations (1.2) for (1.1b) by introducing massive scalar field described by the tensor

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2}g_{ij}\left(V_s V^s - m^2 V^2 \right)\right], \qquad s = 1,\ldots,4,$$

where $V_i = \frac{\partial V}{\partial x^i}$, V is a scalar function of $Z = (t/z)$.

The last section 4 gives the conclusions depicted in terms of theorems.

2. SOLUTIONS OF THE FIELD EQUATIONS (1.2) FOR (1.1b) DESCRIBING MASSLESS SCALAR FIELD

Case I : Scalar field coupled to gravitational field.

The scalar field is described by the tensor

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2}g_{ij}V_s V^s\right], \qquad s = 1,\ldots,4. \tag{2.1}$$

Where V is scalar function of Z and $V_i = \frac{\partial V}{\partial x^i}$, $x^i = (x, y, z, t)$.

From (2.1),
$$T = T_i^i = \frac{1}{4\pi}\left[V_i V^i - \frac{1}{2} g_i^i V_s V^s\right],$$
$$= \frac{1}{4\pi}\left[V_i V^i - 2V_s V^s\right],$$
$$= \frac{1}{4\pi}\left[-V_s V^s\right]. \tag{2.2}$$

Now,
$$V_1 = \frac{\partial V}{\partial x^1} = \frac{\partial V}{\partial x} = \frac{\partial V}{\partial Z} \cdot \frac{\partial Z}{\partial x} = \bar{V} \cdot 0 = 0,$$
$$V_2 = \frac{\partial V}{\partial x^2} = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial Z} . \frac{\partial Z}{\partial y} = \bar{V} .0 = 0,$$
$$V_3 = \frac{\partial V}{\partial x^3} = \frac{\partial V}{\partial z} = \frac{\partial V}{\partial Z} \cdot \frac{\partial Z}{\partial z} = \bar{V}\left(-\frac{Z}{z}\right), \tag{2.3}$$
$$V_4 = \frac{\partial V}{\partial x^4} = \frac{\partial V}{\partial t} = \frac{\partial V}{\partial Z} \cdot \frac{\partial Z}{\partial t} = \bar{V}\left(\frac{1}{z}\right).$$

Where bar $(-)$ over a letter means derivative with respect to Z.

So,
$$V_s V^s = V_s g^{sp} V_p$$
$$= \left[V_1 g^{1p} + V_2 g^{2p} + V_3 g^{3p} + V_4 g^{4p} \right] V_p$$
$$= V_p \left[\bar{V}\left(-\frac{Z}{z}\right) g^{3p} + \bar{V}\left(\frac{1}{z}\right) g^{4p} \right]$$
$$= \bar{V} \left[V_p \left(-\frac{Z}{z}\right) g^{3p} + V_p \left(\frac{1}{z}\right) g^{4p} \right]$$
$$= \bar{V} \left[V_3 \left(-\frac{Z}{z}\right) g^{33} + V_4 \left(\frac{1}{z}\right) g^{44} \right]$$
$$= \bar{V} \left[\bar{V}\left(-\frac{Z}{z}\right)\left(-\frac{Z}{z}\right)\left(\frac{-1}{Z^2 B}\right) + \bar{V}\left(\frac{1}{z}\right)\left(\frac{1}{z}\right)\left(\frac{1}{B}\right) \right]$$

$$= 0. \tag{2.4}$$

Expression (2.2) implies T = 0, and then

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j\right] \tag{2.5}$$

And $$R_{ij} = (-8\pi)\left[\frac{1}{4\pi} V_i V_j\right] = (-2) V_i V_j \quad . \tag{2.6}$$

The non-vanishing V_3, V_4 assume the form

$$\begin{aligned} V_3 V_3 &= -2\bar{V}^2 Z^2 / z^2 , \\ V_4 V_3 &= \ \ 2\bar{V}^2 Z / z^2 , \\ V_4 V_4 &= -2\bar{V}^2 / z^2 . \end{aligned} \tag{2.7}$$

For the line element (1.2) we have

$$\frac{R_{33}}{Z^2} = -\frac{R_{34}}{Z} = R_{44} \qquad \text{[Bhoyar and Deshmukh (2011), (4.5)].} \tag{2.8}$$

Equations(2.7) are compatible with (2.8) and they implies the co-existence of scalar waves with plane -fronted gravitational waves.

Now we relate the existence of waves with the Eigen values of the matter tensor. The Eigen values of T_{ij} in this case are given by

$$\left|T_{ij} - \lambda g_{ij}\right| = 0,$$

$$\begin{vmatrix} \lambda A & 0 & 0 & 0 \\ 0 & \lambda A & 0 & 0 \\ 0 & 0 & T_{33} + \lambda Z^2 B & T_{34} \\ 0 & 0 & T_{43} & T_{44} - \lambda B \end{vmatrix} = 0. \tag{2.9}$$

From (2.5) we have,

$$T_{33} = \frac{1}{4\pi} V_3 V_3 \ = \frac{1}{4\pi}\left(-2\overline{V}^2 Z^2 / z^2\right)$$
$$T_{43} = \frac{1}{4\pi} V_3 V_4 = \frac{1}{4\pi}\left(2\overline{V}^2 Z / z^2\right) \tag{2.10}$$
$$T_{44} = \frac{1}{4\pi} V_4 V_4 = \frac{1}{4\pi}\left(-2\overline{V}^2 / z^2\right).$$

Solving (2.9) we get

$$\lambda^2 A^2 \left\{ \left[T_{33} + \lambda Z^2 B\right]\left[T_{44} - \lambda B\right] - \left[T_{34}\right]^2 \right\} = 0,$$

$$\lambda^2 A^2 \left\{ -\lambda^2 Z^2 B^2 \right\} = 0,$$

$$(-Z^2 A^2 B^2)\lambda^4 = 0,$$

$$\lambda^4 = 0. \quad \text{Since } g = -Z^2 A^2 B^2 \neq 0.$$

In this case all four Eigen values are zero and hence principal directions are not unique.Thus the co-existence of plane gravitational waves with scalar waves is related to the Eigen values of T_{ij} which are all zero.

Case II: Scalar field coupled to gravitational and electromagnetic field

The matter tensor in this case is

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2} g_{ij} V_s V^s\right] + E_{ij}. \tag{2.11}$$

Where

$$E_{ij} = \frac{1}{4\pi}\left[\frac{1}{4} g_{ij} F_{sp} F^{sp} - F_{is} F_j^{\ s}\right], \qquad (p = 1,2,3,4).$$

Here F_{is} is an electromagnetic Maxwell skew tensor given by

$$F_{is} = K_{s;i} - K_{i;s} = K_{s;i} - K_{i;s} \qquad K_i = K_i(Z).$$

Semicolon $(;)$ means covariant derivative with respect to g_{ij} and the comma $(,)$ stands for partial derivatives with respect to x^i.

The gravitational field is given by (1.1b).

As $V_s V^s = 0$, by (2.4), $T_{ij} = E_{ij} + \frac{1}{4\pi}\left[V_i V_j\right]$

and $R_{ij} = (-8\pi)\left[E_{ij} + \frac{1}{4\pi}\left(V_i V_j - \frac{1}{2} g_{ij} T\right)\right]$. due to (1.2) (2.12)

Now $T = T_i^i = E_i^i + \frac{1}{4\pi} V_i V^i = E_i^i$, $\because$ $V_i V^i = 0$,

$$= \frac{1}{4\pi}\left[\frac{1}{4} g_{ij} F_{sp} F^{sp} - F_{is} F^{is}\right]$$

$$= \frac{1}{4\pi}\left[\frac{1}{4}\cdot(4) F_{sp} F^{sp} - F_{is} F^{is}\right] = 0.$$

Then $R_{ij} = (-8\pi)\left[E_{ij} + \frac{1}{4\pi} V_i V_j\right]$, using (2.12) and *T*,

$$= (-2)\left[4\pi E_{ij} + V_i V_j\right]. \qquad (2.13)$$

Since electromagnetic waves coexist with plane gravitational waves we have

$$\frac{E_{33}}{Z^2} = -\frac{E_{34}}{Z} = E_{44}, \quad \text{[See Bhoyar and Deshmukh (2011), (6.7)].} \qquad (2.14)$$

Equation (2.14) with (2.7) and (2.14) imply

$$\frac{R_{33}}{Z^2} = -\frac{R_{34}}{Z} = R_{44} \qquad (2.15)$$

Equations (2.15) compatible with (2.8) and hence follows the existence of scalar waves coupled with plane gravitational and electromagnetic waves.

The Eigen value equation

$$\left| \; T_{ij} - \lambda g_{ij} \; \right| = 0$$

$$\begin{vmatrix} \lambda A & 0 & 0 & 0 \\ 0 & \lambda A & 0 & 0 \\ 0 & 0 & T_{33} - \lambda g_{33} & T_{34} \\ 0 & 0 & T_{43} & T_{44} - \lambda g_{44} \end{vmatrix} = 0. \qquad (2.16)$$

Solving (2.16), we have

$$\lambda^2 A^2 \left\{ \frac{1}{16\pi^2} \left(-16\pi^2 \lambda^2 Z^2 B^2 \right) \right\} = 0,$$

$$\lambda^4 = 0. \text{ Since } g = -Z^2 A^2 B^2 \neq 0.$$

In this case also all Eigen values are zero and hence principal directions are not unique.Thus the co-existence of plane gravitational waves with scalar and electromagnetic waves is related to the Eigen values of T_{ij} which are all zero. The inference drawn from all the above cases can be reformulated into the following theorem.

Theorem: *The existence of massless scalar waves coupled to the* $Z = (t/z)$ -type *plane gravitational and electromagnetic waves is indubitable by the four zero eigen values of energy momentum tensor of the combined distribution of scalar and electromagnetic fields*

3 SOLUTIONS OF THE FIELD EQUATIONS (1.2) FOR (1.1b) DESCRIBING MASSIVE SCALAR FIELD

Case I: Massive scalar field with gravitational field.

The matter tensor for massive scalar field is

$$T_{ij} = \frac{1}{4\pi} \left[V_i V_j - \frac{1}{2} g_{ij} \left(V_s V^s - m^2 V^2 \right) \right], \qquad s = 1, \ldots, 4. \qquad (3.1)$$

Where $V_i = \dfrac{\partial V}{\partial x^i}$, V is a scalar function of (t/z).

So,

$$T = T_i^i = \frac{1}{4\pi}\left[V_i V^i - \frac{1}{2}\partial_i^i\left(V_s V^s - m^2 V^2\right)\right]$$

$$= \frac{1}{4\pi}\left[V_i V^i - 2\left(V_s V^s - m^2 V^2\right)\right]$$

$$= \frac{1}{4\pi}\left[-V_s V^s + 2m^2 V^2\right] \quad \text{by (1.4)}$$

$$= \frac{1}{4\pi}\left[2m^2 V^2\right], \quad \text{as } V_s V^s = 0. \tag{3.2}$$

So,

$$R_{ij} = -8\pi\left\{\left[\frac{1}{4\pi}\left(V_i V_j - \frac{1}{2}g_{ij}\left(V_s V^s - m^2 V^2\right)\right)\right] - \frac{1}{2}g_{ij}\frac{1}{4\pi}2m^2 V^2\right\}$$

$$= -8\pi\left\{\frac{1}{4\pi}\left(V_i V_j + \frac{1}{2}g_{ij}m^2 V^2 - g_{ij}m^2 V^2\right)\right\}$$

$$= \frac{-8\pi}{4\pi}\left\{V_i V_j - \frac{1}{2}g_{ij}m^2 V^2\right\}$$

$$= (-2)\left\{V_i V_j - \frac{1}{2}g_{ij}m^2 V^2\right\}, \tag{3.3}$$

and then we deduce,

$$R_{33} = (-2)\left\{\frac{\bar{V}^2 Z^2}{z^2} + \frac{Z^2 B m^2 V^2}{2}\right\}$$

$$R_{34} = (-2)\left(-\frac{\bar{V}^2 Z}{z^2}\right) \tag{3.4}$$

$$R_{44} = (-2)\left\{\frac{\bar{V}^2}{z^2} - \frac{B m^2 V^2}{2}\right\}.$$

Where $V_3 = \left(-\frac{Z\bar{V}}{z} \right), \quad V_4 = \left(\frac{\bar{V}}{z} \right), \quad g_{33} = -Z^2 B, \quad g_{34} = 0, \quad g_{44} = B.$

The $(-)$ over a letter means derivative with respect to Z .

But for (1.3) we have

$$\frac{R_{33}}{Z^2} = -\frac{R_{34}}{Z} = R_{44}, \qquad \text{[Bhoyar and Deshmukh (2011), (4.5)]} \quad (3.5)$$

The equation (3.4) are compatible with (3.5) only when $m^2 = 0,$ expressing the fact that massive scalar waves does not exist with plane gravitational waves. It is analogous result of Ray and Rao (1972) with respect to axially symmetric scalar field.

The Eigen values of T_{ij} are given by

$$\begin{vmatrix} T_{11} - \lambda g_{11} & 0 & 0 & 0 \\ 0 & T_{22} - \lambda g_{22} & 0 & 0 \\ 0 & 0 & T_{33} - \lambda g_{33} & T_{34} \\ 0 & 0 & T_{43} & T_{44} - \lambda g_{44} \end{vmatrix} = 0. \qquad (3.6)$$

Where $T_{11} - \lambda g_{11} = \left(\lambda - \frac{m^2 V^2}{8\pi} \right) A,$

$$T_{22} - \lambda g_{22} = \left(\lambda - \frac{m^2 V^2}{8\pi} \right) A,$$

$$T_{33} - \lambda g_{33} = \frac{1}{4\pi} \left[\frac{\bar{V}^2 Z^2}{z^2} - \frac{1}{2} Z^2 B m^2 V^2 + 4\pi\lambda \right],$$

$$T_{34} - \lambda g_{34} = \frac{1}{4\pi} \left[-\frac{\bar{V}^2 Z}{z^2} \right],$$

$$T_{44} - \lambda g_{44} = \frac{1}{4\pi}\left[\frac{\overline{V}^2}{z^2} + \frac{1}{2}Bm^2V^2 - 4\pi\lambda\right].$$

On simplifying (3.6),

$$\left(-Z^2A^2B^2\right)\left(\lambda - \frac{m^2V^2}{8\pi}\right)^4 = 0,$$

$$\lambda = \frac{1}{8\pi}m^2V^2. \quad \text{Since} \quad g = -Z^2A^2B^2 \neq 0.$$

In this case we obtained four equal non-zero Eigen values equal to $(m^2V^2/8\pi)$ and hence principal directions are not unique. Thus the non-existence of plane-fronted gravitational waves with massive scalar waves is related to non-zero Eigen values of T_{ij}.

Case II: Massive Scalar field coupled to gravitational and electromagnetic field.

In this case, we consider the energy momentum tensor,

$$T_{ij} = \frac{1}{4\pi}\left(V_iV_j - \frac{1}{2}g_{ij}\left(V_sV^s - m^2V^2\right)\right) + E_{ij}. \tag{3.7}$$

Here $$E_{ij} = \frac{1}{4\pi}\left[\frac{g_{ij}F_{sp}F^{sp}}{4} - F_{is}F_j^s\right], \qquad P = 1,\ldots,4.$$

Then Einstein's equations (1.5) become

$$R_{ij} = (-2)\left[4\pi E_{ij} + V_iV_j - \frac{1}{2}g_{ij}m^2V^2\right]\text{.by (3.2) and } E_i^i\,, \tag{3.8}$$

This can be written as

$$R_{33} = (-2)\left\{4\pi E_{33} + \frac{\bar{V}^2 Z^2}{z^2} + \frac{1}{2} Z^2 B m^2 V^2\right\}$$

$$R_{34} = (-2)\left\{4\pi E_{34} - \frac{\bar{V}^2 Z}{z^2}\right\} \tag{3.9}$$

$$R_{44} = (-2)\left\{4\pi E_{44} + \frac{\bar{V}^2}{z^2} - \frac{1}{2} B m^2 V^2\right\}$$

Equations (3.9) are not compatible with (3.5) and hence follow the non-existence of massive scalar waves with plane gravitational and electromagnetic waves.

The Eigen values of the energy momentum tensor are given by

$$\left|T_{ij} - \lambda g_{ij}\right| = 0$$

$$\begin{vmatrix} T_{11} - \lambda g_{11} & 0 & 0 & 0 \\ 0 & T_{22} - \lambda g_{22} & 0 & 0 \\ 0 & 0 & T_{33} - \lambda g_{33} & T_{34} \\ 0 & 0 & T_{43} & T_{44} - \lambda g_{44} \end{vmatrix} = 0. \tag{3.10}$$

Where, $T_{11} - \lambda g_{11} = \left(\lambda - \frac{m^2 V^2}{8\pi}\right) A$, $T_{22} - \lambda g_{22} = \left(\lambda - \frac{m^2 V^2}{8\pi}\right) A$,

$$T_{33} - \lambda g_{33} = \frac{1}{4\pi}\left[\frac{\bar{V}^2 Z^2}{z^2} - \frac{m^2 V^2 Z^2 B}{2} + 4\pi\lambda Z^2 B + 4\pi E_{33}\right],$$

$$T_{34} - \lambda g_{34} = \frac{1}{4\pi}\left[-\frac{\bar{V}^2 Z}{z^2} + 4\pi E_{34}\right],$$

$$T_{44} - \lambda g_{44} = \frac{1}{4\pi}\left[\frac{\bar{V}^2}{z^2} + \frac{m^2 V^2 B}{2} - 4\pi\lambda B + 4\pi E_{44}\right].$$

Simplifying (3.10), $(-Z^2A^2B^2)\left(\lambda-\frac{m^2V^2}{8\pi}\right)^2\left(\lambda-\frac{m^2V^2}{8\pi}\right)^2=0$,

$$(-Z^2A^2B^2)\left(\lambda-\frac{m^2V^2}{8\pi}\right)^4=0,$$

$$\left(\lambda-\frac{1}{8\pi}m^2V^2\right)=0, \text{since} \quad g=-Z^2A^2B^2\neq 0$$

$$\lambda=\frac{1}{8\pi}m^2V^2.$$

In this case also we obtained four equal non-zero Eigen values. Hence it can relate non-existence of massive scalar waves coupled to plane gravitational and electromagnetic waves with non-zero Eigen values T_{ij}.

From the cases discussed above we can form a theorem.

Theorem: *Non-existence of massive scalar waves with* $Z=(z-t)$ *-type plane gravitational and electromagnetic waves is secured by the four non-zero* $\left(=m^2V^2/8\pi\right)$ *Eigen values of the energy momentum tensor for combined distribution of massive scalar field and electromagnetic field.*

4. CONCLUSIONS

It is interesting that for plane symmetric space-time (1.1b),

[1] The existence of massless scalar waves coupled to $Z=(t/z)$ -type plane gravitational waves and electromagnetic waves is **assured** by four zero Eigen values $\left(\lambda^4=0\right)$ of energy-momentum tensor of combined distribution of scalar and electromagnetic field.

[2] The non-existence of massive scalar waves coupled to $Z=(t/z)$ -type plane gravitational and electromagnetic waves is **guaranteed** by the four non-zero $\left(\lambda=m^2V^2/8\pi\right)$ Eigen values of energy -momentum tensor for combined distribution of massive scalar field and electromagnetic field.

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