Article

Study of N-Dimensional Kaluza-Klein String Cosmological Model

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Abstract

In this paper, a new class of cosmological model with Kaluza-Klein metric in f(R) theory of gravity is discussed. We find the exact solution of field equations by assuming f(R)=R and by taking equation of state $\rho = k\lambda$, where ρ is the rest energy density for the cloud of strings and λ is the tension density. The physical and geometrical properties of the model are also discussed. Results obtained by using these conditions are very interesting about the behavior of this universe.

Keywords: Cosmic string, *f*(*R*) theory of gravity, n-dimensional Kaluza-Klein metric.

1. Introduction

Cosmological models are mathematical models which describes the large scale structure and evolution of the universe. Latest evidences coming from cosmology and astrophysics have divulged a quite unpredictable picture of the universe. Dataset coming from different sources (Cosmic Microwave Background Radiations and Supernovae surveys) suggests that the budget of energy of the universe is: 4% ordinary baryonic matter, 20% dark matter and 76% dark energy [1-4]. The string models have very important role in cosmology during the early stage of the universe [5] and can generate density fluctuations which lead to galaxy formation [6]. The study of string theory in cosmological models was initiated by Letelier [7] and Stachel [8].

In cosmological models, f(R) gravity is a class of modified theory of gravity that generalize general relativity of Einstein by replacing the Ricci scalar R in the Einstein-Hilbert action with a general function. Aim of these theories is to address issues related to cosmic acceleration, dark energy and inflation. Modified theories have received very much attention due to the combined motivation of cosmology, astrophysics and high-energy physics. f(R) theory is a generalized version of teleparallel gravity in which relation of Weitzenbock is used in place of relation of

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Levi-Civita. The f(R) theory of gravity is thought to be the most clear and basic extension of Einstein's general theory of relativity (GTR) among the various modified theories of gravity.

In this theory, the functions of the Ricci scalar are the higher-order curvature invariants. Nojiri and Odintsov created the generic formula for reconstructing the modified f(R) gravity for any given FRW metric [9]. In the f(R) gravity, Capozziello et al. [10] investigated the exact solution of cosmological models. The f(R) gravity model was put forth and shown to be a good option by Nojiri and Odintsov [11] for combining early-time inflation with late-time cosmic acceleration. Felice and Tsujikawa [12] addresses various topics in their survey of f(R) theories, including as spherically symmetric solutions in weak and strong gravitational backgrounds, cosmological perturbations, inflation, dark energy, and local gravity restrictions. Tripathy and Mishra [13] have discovered the solutions to the field equations and characterized the dynamics of anisotropic locally rotationally symmetric (LRS) Bianchi type-I models in f(R) gravity.

Using two different forms of the Ricci scalar, Agrawal et al. [14] recently investigated the Bianchi type-I cosmological model in the f(R) theory of gravity with perfect fluid and talked about the behavior of gravitational baryogenesis. Numerous DE substitutes, including holographic dark energy [15–23], tachyon [24,25], k-essence [26], quintom [27], quintessence [28], phantom [29–33], and others, have been proposed to overcome these problems. Scenario of two-fluid dark energy models in Bianchi type-III Universe, Behaviour of the cosmological model with variable deceleration parameter, Behaviour of the cosmological model with variable deceleration parameter and FLRW Cosmological Models with Dynamic Cosmological Term in Modified Gravity is discussed by Tiwari, Beesham and Shukla [34- 37]. Khan et al. discussed Hyperbolic Scenario of Accelerating Universe in Modified Gravity [38]. Tiwari and Sonia discussed Non Existence of Shear model and String Cosmological Models with Bulk Viscosity and Time-Dependent Λ Term [39-40].

Here in the f(R) theory of gravity, a novel class of cosmological models using Kaluza-Klein metrics is examined. By assuming that f(R)=R and using the equation of state $\rho = k\lambda$, where λ is the tension density and ρ is the rest energy density for the cloud of strings, the exact solution of field equations are obtained. Additionally geometrical and physical properties of the model are also discussed. The outcomes of applying these Criteria Provide fascinating insights into how the cosmos behaves.

2. The Metric and Field Equation

We consider Bianchi type-I cosmological model which represents a homogeneous but anisotropic universe with higher-dimensional Kaluza-Klein metric of gravitation [41-42]

$$ds^2 = dt^2 - A^2(t)\sum_{i=1}^{m-2} dx_i^2 - B^2(t)dx_{m-1}^2 \tag{1}$$

where A and B are the cosmic scale factors which are functions of time t only.

Since A and B are functions of time only, let us assume that

$$A = t^{2\alpha}, B = t^{2\beta} \tag{2}$$

Where α and β are arbitrary constants.

Therefore, Ricci scalar corresponding to equation (1) is

$$R = \frac{4\alpha\beta}{t^2}\left[(m-2)\left\{\frac{(2\alpha-1)}{\beta} + 2 + (m-3)\frac{\alpha}{\beta}\right\} + \frac{2\beta-1}{\alpha}\right] \tag{3}$$

For the metric equation (1),

The determinant of metric tensor g is

$$g = -t^{4\{(m-2)\alpha+\beta\}} \tag{4}$$

Spatial volume $V = t^{2\{(m-2)\alpha+\beta\}}$ (5)

Average scale factor $a(t) = t^{\frac{2}{m-1}\{(m-2)\alpha+\beta\}}$ (6)

Hubble parameter $H = \frac{\dot{a}}{a} = \frac{2}{(m-1)t}\{(m-2)\alpha+\beta\}$ (7)

Deceleration parameter $q = \frac{(m-2)(\alpha-\beta)}{2\{\alpha(m-2)+\beta\}^2}t^{1-2\alpha-2\beta} - \frac{\{(m-2)\alpha(2\alpha-1)-\beta(2\beta-1)\}-4\alpha\beta}{2\{(m-2)\alpha+\beta\}^2}$ (8)

Directional Hubble parameters in the direction of x_{m-2} and x_{m-1} are given by

$$H_{m-2} = \frac{2\alpha(m-2)}{t} \tag{9}$$

$$H_{m-1} = \frac{2\beta}{t} \tag{10}$$

Relation between Shear scalar σ and Expansion scalar θ is defined as

$$\sigma^2 = \frac{2}{3}\theta^2 \tag{11}$$

Where expansion scalar $\theta = \{(m-2)\alpha + \beta\}\frac{2}{t}$ (12)

And Shear scalar $\sigma = \sqrt{\frac{2}{3}}\theta = \frac{2\sqrt{2}}{\sqrt{3}t}\{(m-2)\alpha + \beta\}$ (13)

From equation (12) and (13), we get

$$\frac{\sigma}{\theta} = \sqrt{\frac{2}{3}} = \text{constant} \tag{14}$$

Since $\lim_{t\to\infty}\frac{\sigma}{\theta} \neq 0$, the model does not approach isotropy for a large value of t.

i.e. the model leads to an anisotropic model.

The relativistic field equation in f(R) theory of gravity is

$$G_{ij} = \frac{1}{f_R}\left\{\frac{f(R) - Rf_R}{2} g_{ij} + G_{ij} + T_{ij}\right\} \tag{15}$$

i, j=1, 2, 3, 4 and 8π=1(for the sake of simplicity)

where $G_{ij} = \nabla_i \nabla_j f_R - g_{ij}\nabla^k \nabla_k f_R$

$$\text{or } G_{ij} = f_{RR}\left\{R_{ij} - \Gamma^i_{ij}\dot{R} - g_{ij}(\dot{R}\ln\sqrt{-g})_{;t} + \ddot{R}\right\} + f_{RRR}\left\{R_{;i}R_{;j} - g_{ij}\dot{R}^2\right\} \tag{16}$$

where $f_R \equiv \frac{df(R)}{dR}$, R_{ij} is the Ricci tensor, R is the Ricci scalar and ∇_i is the covariant derivative.

The energy momentum tensor for the cosmic string is

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j \tag{17}$$

where u_i is the four-velocity vector for the fluid particles, x_i is the unit vector (space-like), ρ is rest energy density and λ is the tension density for the cloud of strings. The direction of string in co-moving co-ordinate system satisfies

$$u_m u_m = -1 = -x_{m-1} x_{m-1}, \text{ otherwise, zero} \tag{18}$$

Therefore, the value of T_{ij} , the non-vanishing component is

$$T_{11} = 0, T_{22} = 0, T_{33} = 0, \ldots\ldots, T_{mm} = 0, T_{(m-1)(m-1)} = -\lambda, T_{mm} = -\rho \tag{19}$$

By using equations (16)-(19), the field equations (15) for the metric (1) yields

$$(m-2)\frac{\dot{A}\dot{B}}{AB} + \frac{(m-2)(m-3)}{2}\frac{\dot{A}^2}{A^2} = \frac{1}{f_R}\left[\frac{f(R)-Rf_R}{2} - f_{RR}\left\{(m-2)\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right\} - \rho\right] \tag{20}$$

$$(m-2)\frac{A\dot{A}}{A^2} + \frac{(m-2)(m-3)}{2}\frac{\dot{A}^2}{A^2} = \frac{1}{f_R}\left[\frac{f(R)-Rf_R}{2} - f_{RRR} - f_{RR}\left\{(m-2)\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \lambda\right\}\right] \tag{21}$$

$$(m-3)\frac{A\ddot{A}}{A^2} + \frac{B\ddot{B}}{B^2} + \frac{(m-3)(m-4)}{2}\frac{\dot{A}^2}{A^2} + (m-3)\frac{\dot{A}\dot{B}}{AB} = \frac{1}{f_R}\left[\frac{f(R)-Rf_R}{2} - f_{RRR} - f_{RR}(m-2)\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right]$$
(22)

3. Solution of field equations in f(R) theory of gravity

There are three independent equations (20)-(22) in five unknown variables; therefore two more conditions are required to simplify the equations. Let us consider the solution of f(R) gravity in the form

$$f(R)=R \tag{23}$$

and second is the equation of state $\rho = k\lambda$ (24)

where k is arbitrary constant.

Therefore, we get

$$\rho = \frac{2(m-2)\alpha}{t^2}\{\beta + (m-3)\alpha\} + \frac{2}{t}(m-2+\beta) \tag{25}$$

$$\lambda = \frac{2k(m-2)\alpha}{t^2}\{\beta + (m-3)\alpha\} + \frac{2}{t}(m-2+\beta) \tag{26}$$

From the above equations, it is observed that ρ is rest energy density and λ is the tension density for the cloud of strings tends to zero, when time t tends to infinity.

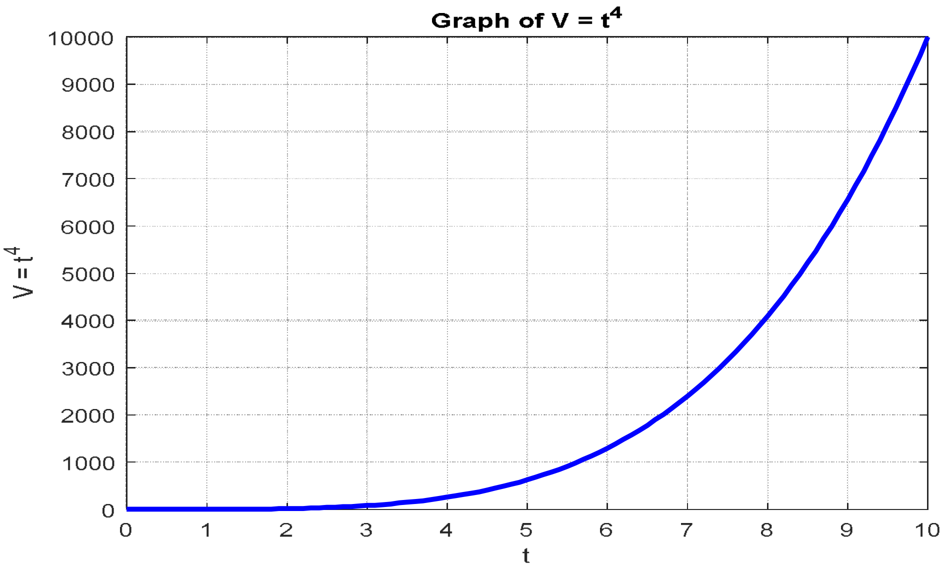

Fig.1. The plot between Spacial volume versus cosmic time.

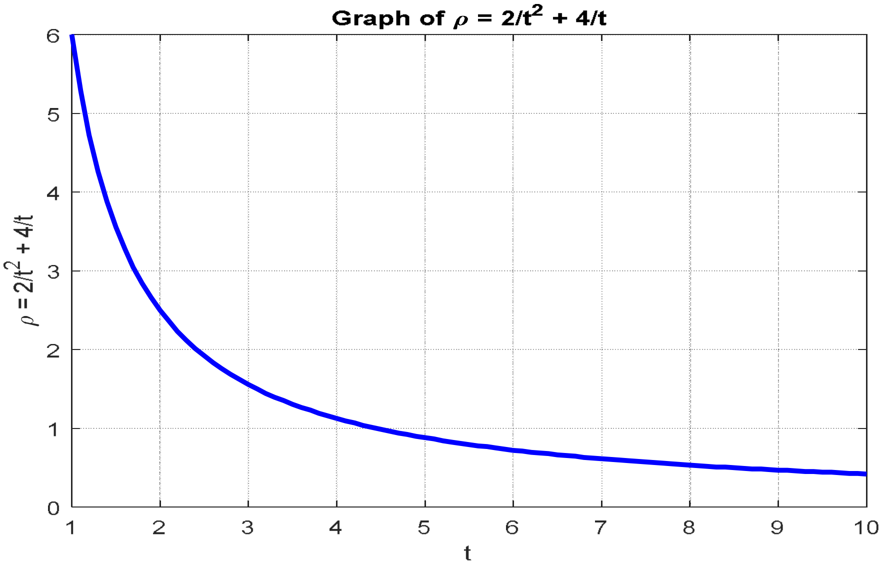

Fig.2. The plot between energy density versus cosmic time

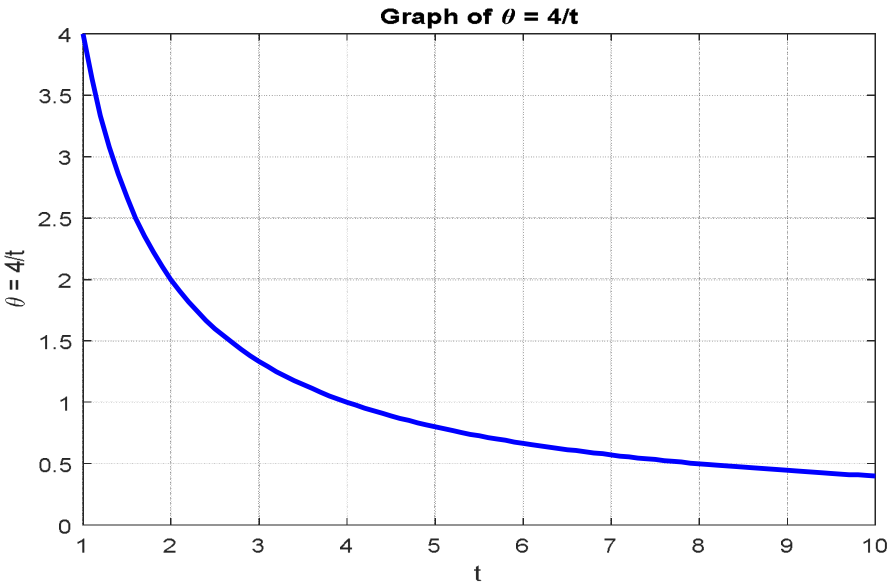

Fig.3. The plot between expansion scalar versus cosmic time

5. Results and Discussion

Here it is observed that, when time t tends to zero i.e. at the time of pre big-bang, the volume of the universe tends to zero i.e. at the moment of big-bang, the volume of the universe is infinitesimally small because all matter and energy were concentrated in a point (whole mass of the universe is in compressed form), where size of universe is not measurable. This point of zero volume is known as singularity. As the time t increases, the volume V of our universe also increases (Fig.1) i.e. space between our galaxies is growing larger with the passage of time.

Also the expansion scalar θ decreases when the time t increases i.e. as the universe expands, rate of expansion slows down with the passage of time (Fig.3). Similarly shear scalar σ and Hubble parameter H also shows a similar trend.

Energy density ρ also decreases with the increase in time (Fig. 2) because as the universe is expanding, same amount of energy is distributed across a large volume of space with the passage of time.

Tension density λ of cloud of strings also decreases with the passage of time i.e. the overall tightness or pulling force exerted by the strings is gradually decreases, when the time increases.

Ratio of shear scalar σ to expansion scalar θ is constant, which means that the amount of anisotropy which is represented by shear scalar σ and the overall expansion rate which is represented by expansion scalar θ remains the same throughout the evolution of this universe, which shows the fixed level of anisotropy relative to the expansion.

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