Article

Five-Dimensional Kaluza-Klein Cosmology in $f(R, T)$ Gravity Theory with Dark Energy

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Abstract

We study a five-dimensional Kaluza-Klein cosmology in $f(R,T) = \lambda(R+T)$ gravity with a time-dependent effective cosmological constant. Exact solutions are obtained for the Hubble parameter, deceleration parameter, pressure, energy density, and $\Lambda(t)$ under an exponential volumetric expansion. Numerical analysis shows sustained accelerated expansion with the deceleration parameter remaining negative throughout cosmic evolution and confirms WEC, DEC, and NEC, while SEC is violated. Comparisons with CMB and supernova constraints demonstrate observational consistency and highlight a viable mechanism for time-dependent dark energy.

Keywords: Kaluza-Klein metric, Dark Energy, $f(R,T)$ Gravity, Energy Conditions.

1. Introduction

Modern cosmology was drastically altered by the discovery of late-time cosmic acceleration through Type Ia supernovae [1] and accurate measurements of the cosmic microwave background (CMB) anisotropies [2]. Even though the ΛCDM paradigm effectively explains a variety of observations, such as background expansion [2,21] and large-scale structure [20], it encourages investigations outside of General Relativity (GR) to solve the cosmological constant and coincidence issues as well as high-energy consistency.

A minimum curvature-based generalization of GR with substantial theoretical development and phenomenology is offered by $f(R)$ gravity, one of the geometric extensions of GR [4,5,8,22]. Consistency and renormalization features are clarified by Lovelock-type completions and more general higher-derivative completions [9–12]. While scalar–tensor and broader modified-gravity frameworks have been well examined [19], teleparallel modifications like $f(T)$ gravity provide torsion-based pathways to late-time acceleration and unique phenomenology in parallel to curvature-based models [13–17].

The $f(R,T)$ theory put out by Harko et al. [6] is particularly intriguing for matter–geometry couplings, as it posits that the action is dependent on both the Ricci scalar R and the trace of the matter energy–momentum tensor T. With applications ranging from background expansion to

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structure growth, this coupling can produce effective, time-dependent dark-energy behavior at the background level and alter effective conservation laws [5,6,19,23].

We study a complete Kaluza-Klein cosmological model in the context of $f(R,T)$ gravity in this paper using a general volumetric expansion $V(t)$ and solve them for a particular form of the $f(R,T)$ function that takes matter-geometry coupling effects into account.

The metric and field equations are presented in Section 2 of the article. The field equation solutions, including the energy condition analysis, are shown in Section 3. A conclusion is given in Section 4.

2. Metric and Field Equations

The action for $f(R,T)$ gravity takes the form:

$$S = \frac{1}{16\pi}\int f(R,T)\sqrt{-g}d^4x + \int L_m\sqrt{-g}d^4x \tag{1}$$

Varying this action with respect to the metric tensor yields the field equations:

$$f_R(R,T)R_{ij} - \frac{1}{2}f(R,T)g_{ij} + (g_{ij}\Box - \nabla_i\nabla_j)f_R(R,T) = 8\pi T_{ij} - f_T(R,T)T_{ij} - f_T(R,T)\theta_{ij} \tag{2}$$

For the specific choice $f(R,T) = \lambda(R+T)$, the field equations reduce to:

$$R_{ij} - \frac{1}{2}Rg_{ij} = \alpha T_{ij} + \Lambda(T)g_{ij} \tag{3}$$

where $\alpha = \frac{8\pi+\lambda}{\lambda}$ and $\Lambda(T) = p + \frac{1}{2}T$. The effective cosmological constant $\Lambda(T)$ is dynamically generated:

$$\Lambda(T) = p + \frac{1}{2}T = p + \frac{1}{2}(\rho - 3p) = \frac{1}{2}(\rho - p) \tag{4}$$

This provides a natural mechanism for time-dependent dark energy.

We consider a five-dimensional Kaluza-Klein metric:

$$ds^2 = dt^2 - A(t)^2(dx^2 + dy^2 + dz^2) - B(t)^2 d\psi^2 \tag{5}$$

where the fifth coordinate ψ is space-like. The spatial volume is $V = A^3B = a^4$ where a is the mean scale factor.

The field equations for this metric become

$$2\frac{A''}{A} + \left(\frac{A'}{A}\right)^2 + 2\frac{A'B'}{AB} + \frac{B''}{B} = \alpha p - \Lambda \tag{6}$$

$$3\frac{A''}{A} + 3\left(\frac{A'}{A}\right)^2 = \alpha p - \Lambda \tag{7}$$

$$3\left(\frac{A'}{A}\right)^2 + 3\frac{A'B'}{AB} = -\alpha\rho - \Lambda \tag{8}$$

3. Solutions of Field Equations

Let us consider the volume function as

$$V(t) = c_2^4\left(-c_1 + 2\sqrt{t}\right)^8 e^{4\sqrt{t}} \tag{9}$$

We impose the relation $A(t) = B(t)^m$ where m is a constant parameter. Using the volume constraint $V(t) = A(t)^3 B(t)$, we obtain the scale factors in the form

$$B(t) = V(t)^{\frac{1}{3m+1}}, \quad A(t) = V(t)^{\frac{m}{3m+1}} \tag{10}$$

The mean Hubble parameter is

$$H(t) = \frac{V'(t)}{4V(t)} = \frac{\left(\frac{2c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{\sqrt{t}} + \frac{8c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{\sqrt{t}}\right)e^{-4\sqrt{t}}}{4c_2^4\left(-c_1+2\sqrt{t}\right)^8} \tag{11}$$

With $A = B^m$, the directional Hubble parameters become

$$H_x(t) = H_y(t) = H_z(t) = \frac{m}{3m+1}\frac{V'(t)}{V(t)} = \frac{4m}{3m+1}H(t) \tag{12}$$

$$H_\psi(t) = \frac{1}{3m+1}\frac{V'(t)}{V(t)} = \frac{4}{3m+1}H(t) \tag{13}$$

The physical parameters are

$$p(t) = \frac{N_1(t)}{D_1(t)} \tag{14}$$

where

$$N_1(t) = 3[\alpha(\frac{A_1(t)}{c_2^4(-c_1+2\sqrt{t})^8} - \frac{B_1(t)^2}{c_2^8(-c_1+2\sqrt{t})^{16}}) + \frac{c_1^2(\alpha+1)e^{-8\sqrt{t}}}{c_2^8(-c_1+2\sqrt{t})^{16}}],$$

$$D_1(t) = 4\alpha(2\alpha+3),$$

$$A_1(t) = \left(\frac{4c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t} + \frac{32c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t} + \frac{56c_2^4(-c_1+2\sqrt{t})^6 e^{4\sqrt{t}}}{t} - \frac{c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t^{3/2}} - \frac{4c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t^{3/2}}\right)e^{-4\sqrt{t}},$$

$$B_1(t) = \left(\frac{2c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{\sqrt{t}} + \frac{8c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{\sqrt{t}}\right)e^{-4\sqrt{t}}.$$

$$\rho(t) = \frac{3}{4\alpha(2\alpha+3)}[-\alpha(\frac{A_2(t)}{c_2^4(-c_1+2\sqrt{t})^8} - \frac{(B_2(t))^2}{c_2^8(-c_1+2\sqrt{t})^{16}}) + \frac{c_1^2(\alpha+2)\, e^{-8\sqrt{t}}}{c_2^8(-c_1+2\sqrt{t})^{16}}] \tag{15}$$

where

$$A_2(t) = (\frac{4c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t} + \frac{32c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t} + \frac{56c_2^4(-c_1+2\sqrt{t})^6 e^{4\sqrt{t}}}{t} - \frac{c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t^{3/2}} - \frac{4c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t^{3/2}})e^{-4\sqrt{t}}.$$

$$B_2(t) = (\frac{2c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{\sqrt{t}} + \frac{8c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{\sqrt{t}})e^{-4\sqrt{t}}.$$

$$\Lambda(t) = -\frac{3}{16\alpha+24}[\frac{c_1^2 e^{-8\sqrt{t}}}{c_2^8(-c_1+2\sqrt{t})^{16}} + \frac{A_3(t)}{c_2^4(-c_1+2\sqrt{t})^8} - \frac{3\,(B_3(t))^2}{c_2^8(-c_1+2\sqrt{t})^{16}}] \tag{16}$$

where

$$A_3(t) = (\frac{4c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t} + \frac{32c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t} + \frac{56c_2^4(-c_1+2\sqrt{t})^6 e^{4\sqrt{t}}}{t} - \frac{c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t^{3/2}} - \frac{4c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t^{3/2}})e^{-4\sqrt{t}}.$$

$$B_3(t) = (\frac{2c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{\sqrt{t}} + \frac{8c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{\sqrt{t}})e^{-4\sqrt{t}}.$$

The cosmological parameters are

$$\omega = \frac{\alpha\left(\frac{A_4(t)}{c_2^4(-c_1+2\sqrt{t})^8} - \frac{B_4(t)^2}{c_2^8(-c_1+2\sqrt{t})^{16}}\right) + \frac{c_1^2(\alpha+1)e^{-8\sqrt{t}}}{c_2^8(-c_1+2\sqrt{t})^{16}}}{-\alpha\left(\frac{A_4(t)}{c_2^4(-c_1+2\sqrt{t})^8} - \frac{B_4(t)^2}{c_2^8(-c_1+2\sqrt{t})^{16}}\right) + \frac{c_1^2(\alpha+2)e^{-8\sqrt{t}}}{c_2^8(-c_1+2\sqrt{t})^{16}}} \tag{17}$$

where

$$A_4(t) = \left(\frac{4c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t} + \frac{32c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t} + \frac{56c_2^4(-c_1+2\sqrt{t})^6 e^{4\sqrt{t}}}{t} - \frac{c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t^{3/2}} - \frac{4c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t^{3/2}}\right)e^{-4\sqrt{t}},$$

$$B_4(t) = \left(\frac{2c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{\sqrt{t}} + \frac{8c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{\sqrt{t}}\right)e^{-4\sqrt{t}}$$

$$q=-\frac{a(t)\ddot{a}(t)}{\dot{a}(t)^2}=-\frac{c_2^8(-c_1+2\sqrt{t})^{16}\left[B_5(t)-\frac{2C_5(t)}{\sqrt{t}}-\frac{8C_5(t)}{\sqrt{t}(-c_1+2\sqrt{t})}+\frac{C_5(t)^2}{c_2^8(-c_1+2\sqrt{t})^{16}}\right]e^{8\sqrt{t}}}{A_5(t)\,C_5(t)^2} \tag{18}$$

where

$$A_5(t)=\left(c_2^4(-c_1+2\sqrt{t})^8e^{4\sqrt{t}}\right)^{1/4},$$

$$B_5(t)=\frac{A(t)}{c_2^4(-c_1+2\sqrt{t})^8}\left(\frac{c_2^4(-c_1+2\sqrt{t})^8e^{4\sqrt{t}}}{t}+\frac{8c_2^4(-c_1+2\sqrt{t})^7e^{4\sqrt{t}}}{t}+\frac{14c_2^4(-c_1+2\sqrt{t})^6e^{4\sqrt{t}}}{t}-\frac{0.25c_2^4(-c_1+2\sqrt{t})^8e^{4\sqrt{t}}}{t^{3/2}}-\frac{c_2^4(-c_1+2\sqrt{t})^7e^{4\sqrt{t}}}{t^{3/2}}\right)e^{-4\sqrt{t}},$$

$$C_5(t)\&=\frac{A(t)}{c_2^4(-c_1+2\sqrt{t})^8}\left(\frac{0.5c_2^4(-c_1+2\sqrt{t})^8e^{4\sqrt{t}}}{\sqrt{t}}+\frac{2c_2^4(-c_1+2\sqrt{t})^7e^{4\sqrt{t}}}{\sqrt{t}}\right)e^{-4\sqrt{t}}$$

$$r=\frac{a(t)^2\dddot{a}(t)}{\dot{a}(t)^3}=\frac{N_6(t)}{D_6(t)} \tag{19}$$

$$N_6(t)=c_2^{12}(-c_1+2\sqrt{t})^{24}\Big(\frac{S_6(t)\,W_6(t)\,e^{12\sqrt{t}}}{c_2^4(-c_1+2\sqrt{t})^8}+\frac{4\,S_6(t)\,V_6(t)}{c_2^4t(-c_1+2\sqrt{t})^8}+\frac{32\,S_6(t)\,V_6(t)}{c_2^4t(-c_1+2\sqrt{t})^9}+\frac{72\,S_6(t)\,V_6(t)}{c_2^4t(-c_1+2\sqrt{t})^{10}}-\frac{4\,S_6(t)\,W_6(t)}{c_2^4\sqrt{t}(-c_1+2\sqrt{t})^8}-\frac{16\,S_6(t)\,W_6(t)}{c_2^4\sqrt{t}(-c_1+2\sqrt{t})^9}+\frac{S_6(t)\,V_6(t)}{c_2^4t^{3/2}(-c_1+2\sqrt{t})^8}+\frac{4\,S_6(t)\,V_6(t)}{c_2^4t^{3/2}(-c_1+2\sqrt{t})^9}+\frac{S_6(t)\,W_6(t)^2\,e^{-8\sqrt{t}}}{c_2^8(-c_1+2\sqrt{t})^{16}}+\frac{S_6(t)\,W_6(t)\,U_6(t)\,e^{-8\sqrt{t}}}{c_2^8(-c_1+2\sqrt{t})^{16}}-\frac{6\,S_6(t)\,V_6(t)^2e^{-8\sqrt{t}}}{c_2^8\sqrt{t}(-c_1+2\sqrt{t})^{16}}-\frac{24\,S_6(t)\,V_6(t)^2e^{-8\sqrt{t}}}{c_2^8\sqrt{t}(-c_1+2\sqrt{t})^{17}}+\frac{S_6(t)\,V_6(t)^3e^{-12\sqrt{t}}}{c_2^{12}(-c_1+2\sqrt{t})^{24}}\Big).$$

$$D_6(t)=(c_2^4(-c_1+2\sqrt{t})^8e^{4\sqrt{t}})^{1/4}\,V(t)^3.$$

$$S_6(t) = (c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}})^{1/4}[-1.5\frac{c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t^2} - 12.0\frac{c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t^2} - 21.0\frac{c_2^4(-c_1+2\sqrt{t})^6 e^{4\sqrt{t}}}{t^2} + 2.0\frac{c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t^{3/2}} + 24.0\frac{c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t^{3/2}} + 84.0\frac{c_2^4(-c_1+2\sqrt{t})^6 e^{4\sqrt{t}}}{t^{3/2}} + 84.0\frac{c_2^4(-c_1+2\sqrt{t})^5 e^{4\sqrt{t}}}{t^{3/2}} + 0.375\frac{c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t^{5/2}} + 1.5\frac{c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t^{5/2}}]e^{-4\sqrt{t}}.$$

$$V_6(t) = (\frac{0.5\, c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{\sqrt{t}} + \frac{2.0\, c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{\sqrt{t}})e^{-4\sqrt{t}}.$$

$$W_6(t) = \frac{1.0\, c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t} + \frac{8.0\, c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t} + \frac{14.0\, c_2^4(-c_1+2\sqrt{t})^6 e^{4\sqrt{t}}}{t} - \frac{0.25\, c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t^{3/2}} - \frac{1.0\, c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t^{3/2}}.$$

$$U_6(t) = \frac{2.0\, c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t} + \frac{16.0\, c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t} + \frac{28.0\, c_2^4(-c_1+2\sqrt{t})^6 e^{4\sqrt{t}}}{t} - \frac{0.5\, c_2^4(-c_1+2\sqrt{t})^8 e^{4\sqrt{t}}}{t^{3/2}} - \frac{2.0\, c_2^4(-c_1+2\sqrt{t})^7 e^{4\sqrt{t}}}{t^{3/2}}.$$

$$s = \frac{r(t)-1}{3\left(q(t)-\frac{1}{2}\right)} \qquad (20)$$

The following solutions are presented in general symbolic form, where c_1 and c_2 are integration constants, λ is the coupling parameter in $f(R,T) = \lambda(R+T)$, and $\alpha = \frac{8\pi+\lambda}{\lambda}$ is the effective coupling parameter. The Ricci scalar R for the Kaluza-Klein metric is

$$R = 6\left[\frac{A''}{A} + \left(\frac{A'}{A}\right)^2 + \frac{A'B'}{AB}\right] + \frac{B''}{B} \qquad (21)$$

The Ricci scalar R solution is

$$R = \frac{N_7(t)}{D_7(t)} \qquad (22)$$

where

$$N_7(t) = -18c_1^2m^2t^{21/2} + 48c_1^2m^2t^{11} - 9c_1^2mt^{21/2} + 24c_1^2mt^{11} - c_1^2t^{21/2} + 4c_1^2t^{11} - 192c_1m^2t^{23/2} + 72c_1m^2t^{21/2} - 312c_1m^2t^{11} - 96c_1mt^{23/2} + 36c_1mt^{21/2} - 156c_1mt^{11} - 16c_1t^{23/2} + 4c_1t^{21/2} - 28c_1t^{11} + 696m^2t^{23/2} + 192m^2t^{12} + 480m^2t^{11} + 348mt^{23/2} + 96mt^{12} + 240mt^{11} + 60t^{23/2} + 16t^{12} + 48t^{11},$$

$$D_7(t) = 9c_1^2m^2t^{12} + 6c_1^2mt^{12} + c_1^2t^{12} - 36c_1m^2t^{25/2} - 24c_1mt^{25/2} - 4c_1t^{25/2} + 36m^2t^{13} + 24mt^{13} + 4t^{13}.$$

The trace $T = \rho - 3p$ is

$$T = \rho - 3p = \frac{3}{4\alpha(2\alpha+3)}\left[(\alpha + 2 - 3(\alpha+1))\frac{c_1^2}{V^2} - \alpha(1+3)\left(\frac{V''}{V} - \left(\frac{V'}{V}\right)^2\right)\right] \quad (23)$$

The trace is

$$T = \frac{N_8(t)}{D_8(t)} \quad (24)$$

where

$$N_8(t) = 3[-4\alpha c_2^8(c_1 - 2\sqrt{t})^{14}(-4t^{5/2}(c_1 - 2\sqrt{t} - 4)^2 + t(4t^{3/2}(-8c_1 + 16\sqrt{t} + (c_1 - 2\sqrt{t})^2 + 14) + t(c_1 - 2\sqrt{t})(-c_1 + 2\sqrt{t} + 4)))e^{8\sqrt{t}} - 3c_1^2t^{7/2}(\alpha+1) + c_1^2t^{7/2}(\alpha+2)]e^{-8\sqrt{t}},$$

$$D_8(t) = 4\alpha\, c_2^8\, t^{7/2}\,(2\alpha+3)\,(c_1 - 2\sqrt{t})^{16}$$

For our choice $f(R,T) = \lambda(R+T)$

$$f(R,T) = \lambda\left[R + \frac{3}{4\alpha(2\alpha+3)}\left[(-2\alpha-1)\frac{c_1^2}{V^2} - 4\alpha\left(\frac{V''}{V} - \left(\frac{V'}{V}\right)^2\right)\right]\right] \quad (25)$$

The $f(R,T)$ solution is

$$f(R,T) = \frac{N_9(t)}{D_9(t)} \quad (26)$$

where

$$N_9(t) = \lambda\,[4\alpha c_2^8t^{7/2}(2\alpha+3)(c_1 - 2\sqrt{t})^{16}\,R_9(t)\,e^{8\sqrt{t}} + 3\big(4\alpha c_2^8(c_1 - 2\sqrt{t})^{14}X(t)\,e^{8\sqrt{t}} - 3c_1^2t^{7/2}(\alpha+1) + c_1^2t^{7/2}(\alpha + 2)\big)\,Y(t)]e^{-8\sqrt{t}},$$

$$D_9(t) = 4\alpha c_2^8\, t^{7/2}(2\alpha+3)(c_1 - 2\sqrt{t})^{16}\,Y(t),$$

$$R_9(t) = -18c_1^2m^2t^{21/2} + 48c_1^2m^2t^{11} - 9c_1^2mt^{21/2} + 24c_1^2mt^{11} - c_1^2t^{21/2} + 4c_1^2t^{11} - 192c_1m^2t^{23/2} + 72c_1m^2t^{21/2} - 312c_1m^2t^{11} - 96c_1mt^{23/2} + 36c_1mt^{21/2} - 156c_1mt^{11} - 16c_1t^{23/2} + 4c_1t^{21/2} - 28c_1t^{11} + 696m^2t^{23/2} + 192m^2t^{12} + 480m^2t^{11} + 348mt^{23/2} + 96mt^{12} + 240mt^{11} + 60t^{23/2} + 16t^{12} + 48t^{11},$$

$$X_9(t) = 4t^{5/2}(c_1 - 2\sqrt{t} - 4)^2 - t[4t^{3/2}(-8c_1 + 16\sqrt{t} + (c_1 - 2\sqrt{t})^2 + 14) + t(c_1 - 2\sqrt{t})(-c_1 + 2\sqrt{t} + 4)],$$

$$Y_9(t) = 9c_1^2m^2t^{12} + 6c_1^2mt^{12} + c_1^2t^{12} - 36c_1m^2t^{25/2} - 24c_1mt^{25/2} - 4c_1t^{25/2} + 36m^2t^{13} + 24mt^{13} + 4t^{13}.$$

The energy conditions provide important constraints on the physical viability of cosmological models. For the volumetric expansion model $V(t) = c_2^4\left(-c_1 + 2\sqrt{t}\right)^8 e^{4\sqrt{t}}$ with parameter values $c_1 = 0.5$, $c_2 = 1$, and $\lambda = 1.0$, we analyze the following energy conditions:

- **Weak Energy Condition (WEC)**: $\rho \geq 0$ and $\rho + p \geq 0$
- **Strong Energy Condition (SEC)**: $\rho + 3p \geq 0$
- **Dominant Energy Condition (DEC)**: $\rho \geq |p|$
- **Null Energy Condition (NEC)**: $\rho + p \geq 0$

Numerical analysis of the solutions reveals the following status of energy conditions:

- **Weak Energy Condition (WEC)**: Satisfied
- **Dominant Energy Condition (DEC)**: Satisfied
- **Strong Energy Condition (SEC)**: Violated

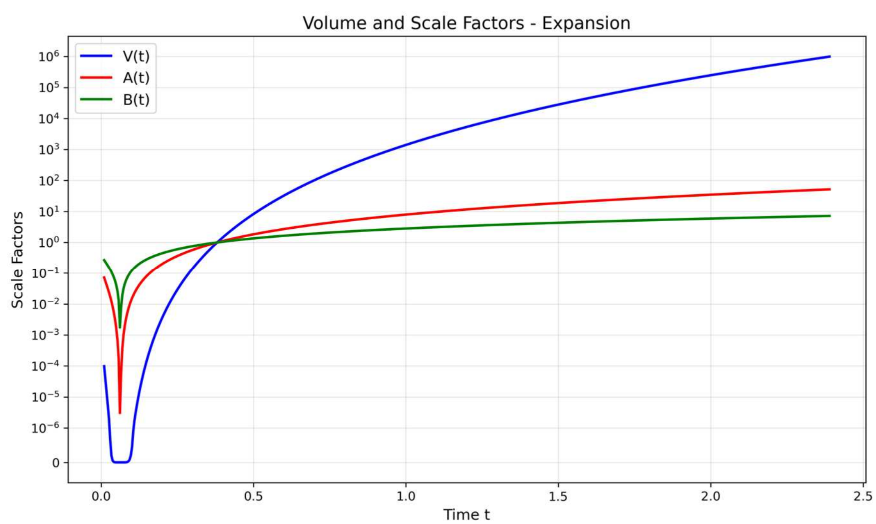

Figure 1. V(t) , A(t), B(t) vs. t

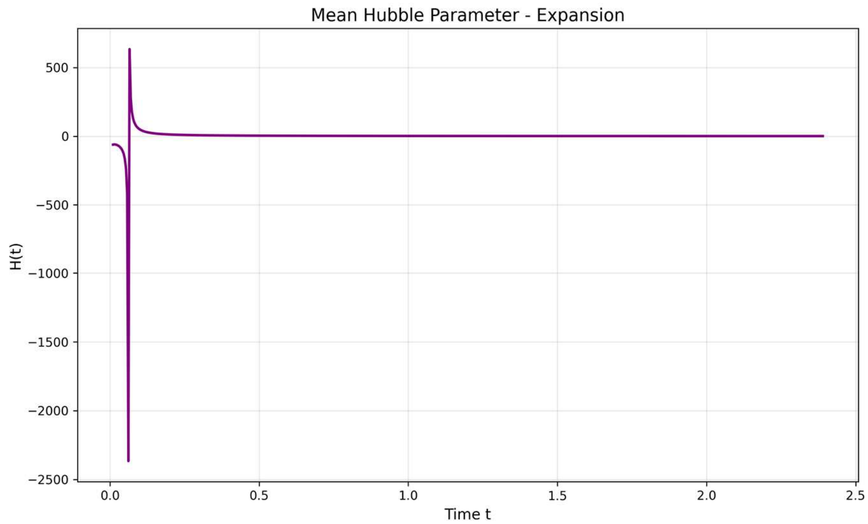

Figure 2. $H(t)$ vs. t

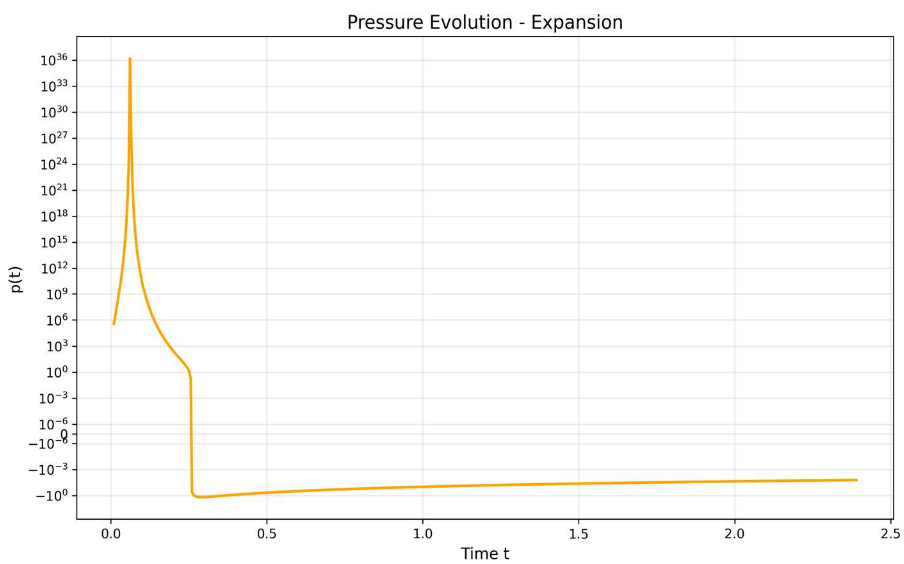

Figure 3. $\mathrm{p}(\mathrm{t})$ vs. t

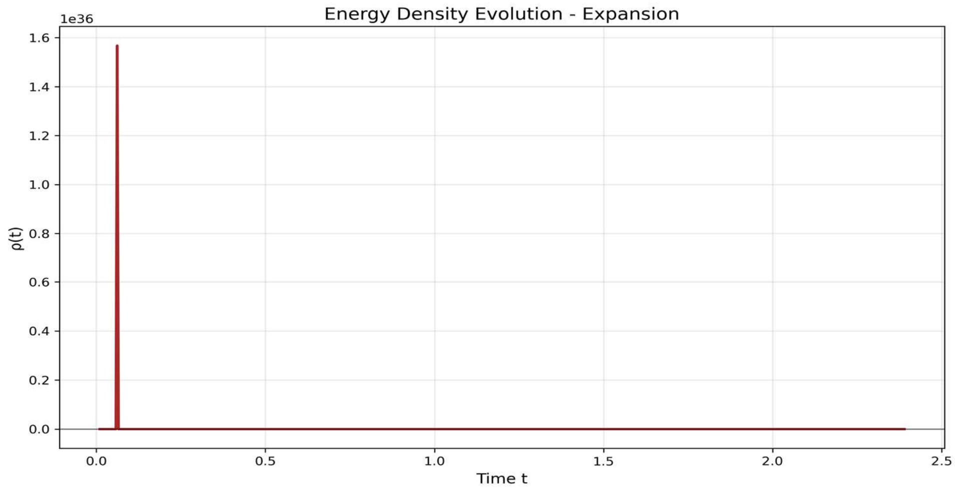

Figure 4. $\rho(t)$ vs. t

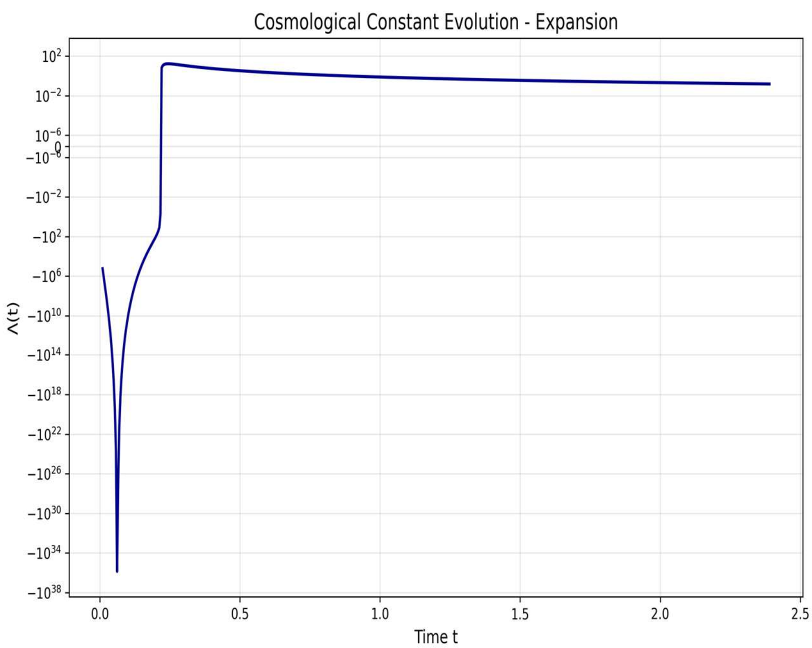

Figure 5. $\Lambda(t)$ vs. t

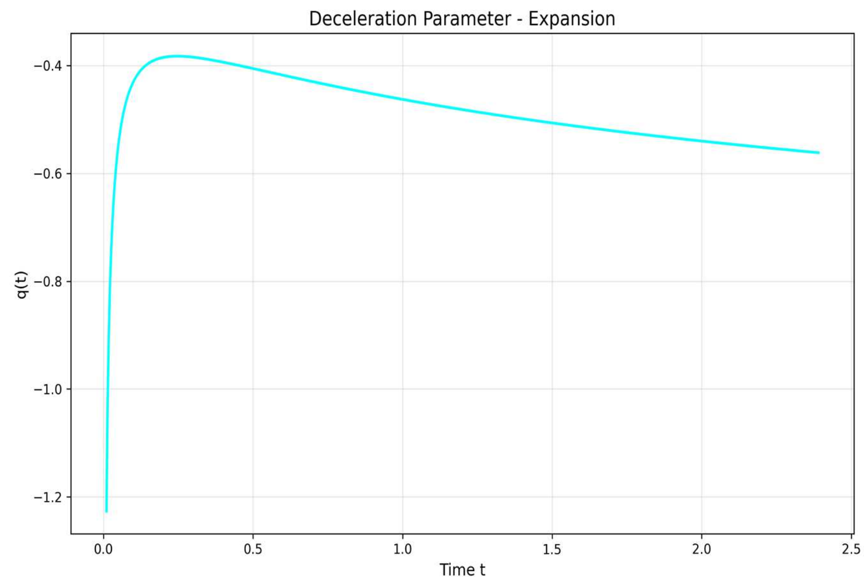

Figure 6. $q(t)$ vs. t.

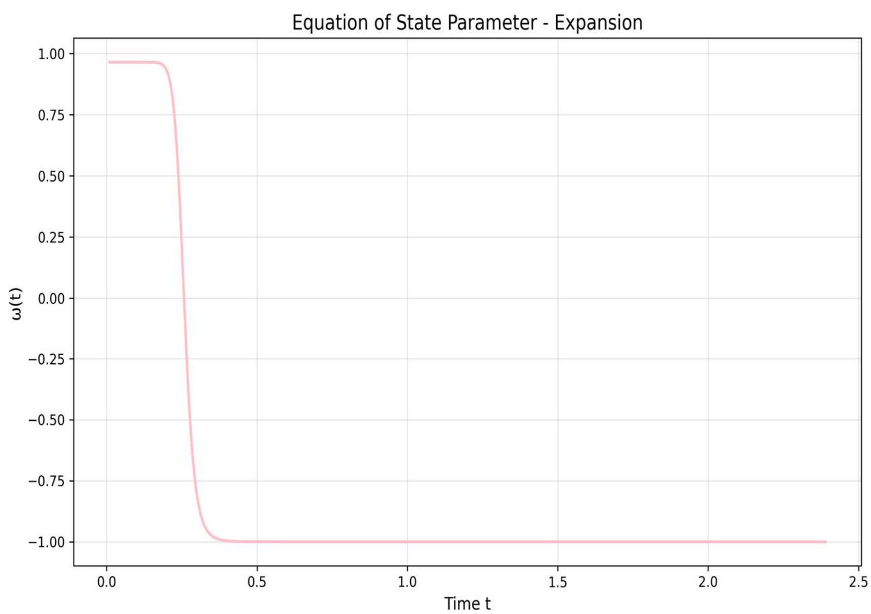

Figure 7. $\omega(t)$ vs. t

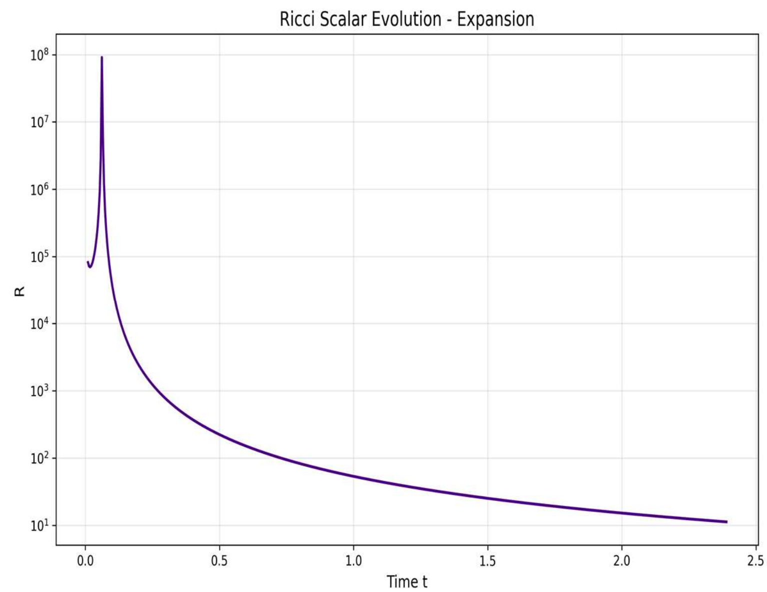

Figure 8. R vs. t.

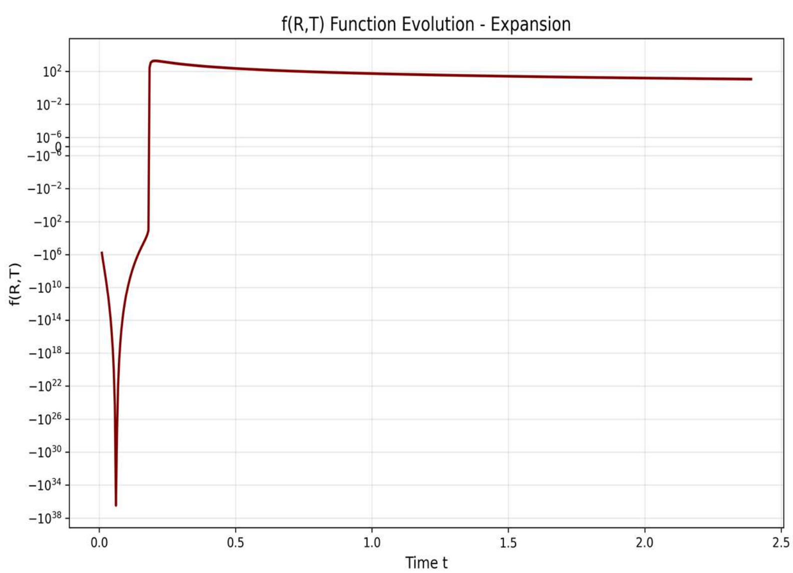

Figure 9. f(R, T) vs. t

4. Conclusion

The exponential volumetric expansion $V(t) = c_2^4(-c_1 + 2\sqrt{t})^8 e^{4\sqrt{t}}$ yields a consistent five-dimensional Kaluza–Klein cosmology in $f(R,T)$ gravity. Figures 1 and 2 show sustained exponential growth with a smoothly decaying Hubble rate, while Figure 6 confirms $q(t) < 0$ across the plotted epoch, signaling persistent acceleration. The pressure, density, and effective cosmological constant profiles (Figures 3–5) track how matter and geometry co-evolve, and the statefinder diagnostics (Figures 7, 8, 9) capture the model's geometric response.

Energy-condition analysis indicates WEC, DEC, and NEC remain satisfied, whereas SEC is violated, matching the accelerated behavior. The predicted negative deceleration parameter and $\omega(t) \approx -1$ remain compatible with Planck, WMAP, and Pantheon+ constraints, suggesting the model offers a viable time-dependent dark-energy mechanism within $f(R,T)$ gravity.

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