Article

Minimization in Generating Space & Fixed Point

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Abstract

Minimization theorem in generating space and fixed point is studied. A non-convex minimization theorem has been established for generating space of quasi 2-metric family for sequence of mappings with non-commuting weak compatible condition. Also minimization theorem is explained in detail with illustrative example.

Keywords: Quasi 2-metric family, weak compatible mapping, minimization theorem, common fixed point.

1. Introduction

An important area of fixed point theory is the generating space of quasi 2-metric family, because of its involvement and application to fuzzy and probabilistic 2-metric space and a minimization theorem [1], [3] is to obtain fixed point theorem. In 2008 V. B. Dhagat and V. S. Thakur [2] proved non convex minimization theorem for generating space of quasi 2-metric family. In this paper we prove a minimization theorem for sequence of mappings T^{α} for $\alpha \in N$ and further we prove fixed point theorem as an application of minimization theorem with non commuting condition known as weak compatible.

2. Generating Space of Quasi 2-Metric Family

Generating space of quasi 2-metric family already defined[1] and [2] as follows:

Let X be a non empty set and $\{D_{\alpha}: \alpha \in (0,1]\}$ be family of mapping D_{α} from $X \times X \times X$ into R^{+}. $\{X, D_{\alpha}\}$ is called generating space of quasi 2-metric family if it satisfy following axioms:

(GM 1) – For any two distinct points x and y there exit z in X such that

$$D_{\alpha}(x, y, z) \neq \alpha \in (0,1]$$

(GM 2) – $D_{\alpha}(x, y, z) = 0$ if at least two x, y, z are equal and $\alpha \in (0,1]$

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(GM 3) – $D_\alpha(x,y,z) = D_\alpha(x,z,y) D_\alpha(z,y,x) = \cdots \ldots \ldots \ldots$ for all x,y,z in X and $\alpha \in (0,1]$

(GM 4) – for any $\alpha \in (0,1]$ there exists $\alpha_1, \alpha_2, \alpha_3, \in (0,\alpha]$ such that $\alpha_1 + \alpha_2 + \alpha_3, \le (0,\alpha]$ and so $D_\alpha(x,y,z) \le D_{\alpha_1}(x,y,u) + D_{\alpha_2}(x,u,z) + D_{\alpha_3}(u,y,z)$

(GM 5) – $D_\alpha(x,y,z)$ is non increasing and left continuous in α and $\forall\, x,y,z \; in \; X$. Throught this paper, we assume that $k: (0,1] \to (0,\infty)$ is non decreasing function satisfying the condition

$K = Sup\, k(\alpha)$

Let E and F be mappings from generating space of quasi 2-metric family $\{X, D_\alpha\}$ into itself. The mapping E and F are said to be weak compatible if it commute at convergent point. i.e. for sequence x_n in X such that

$\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$ for some t in X then $EFt = FEt$.

3. Main Results

Theorem 3.1. Let $\{X, D_\alpha : \alpha \in (0,1]\}$ and $\{Y, D'_\alpha : \alpha \in (0,1]\}$ be two complete generating space of quasi 2-metric family. $f: X \to Y$ be a closed and $T^a: X \to X$ be continuous mapping satisfying for all $a \in N$

(i) $D_\alpha(T^a x, T^a y, z) \le \max\{D_\alpha(T^a x, y, z).(x, T^a y, z).(x, y, T^a z)\}$ and

(ii) $D'_\alpha\big(f(T^a x).f(T^a y).f(z)\big)$

$$\le max\{D'_\alpha\big(f(T^a x).f(y).f(z)\big), D'_\alpha\big(f(x).f(T^a y).f(z)\big), D'_\alpha\big(f(x).f(y).f(T^a z)\big)\},$$

$\forall\; x,y,z \in X$ and $\alpha \in (0,1]$

(iii) $\Psi: \Re \to \Re$ be non decreasing continuous and bounded below function,

(iv) $\emptyset: f(x) \to \Re$ be a lower semi continuous and bounded below function,

(v) $for\ any\ p \in X$ with $\inf \Psi\Big(\emptyset\big(\mathrm{f}(x)\big)\Big) < \Psi\Big(\emptyset\big(\mathrm{f}(p)\big)\Big)$ there exists q with $p \ne Tq$ and

$$\max[max\{D_\alpha(T^a, q, p, z), D_\alpha(q, T^a p, z), D_\alpha(q, p, T^a z)\}],$$

$$c.max\{D'_\alpha\big(f(T^a q).f(p).f(z)\big), D'_\alpha\big(f(q).f(T^a p).f(z)\big), D'_\alpha\big(f(q).f(p).f(T^a z)\big)\}$$

$\le K(\alpha)\Big[\Psi\Big(\emptyset\big(\mathrm{f}(p)\big)\Big) - \Psi\Big(\emptyset\big(\mathrm{f}(q)\big)\Big)\Big]\; \forall\; x,y,z \in X$ and $\alpha \in (0,1]$

and c is any constant.

Then there exists an x_0 in$X$ such that with $\inf \Psi\Big(\emptyset\big(\mathrm{f}(x)\big)\Big) = \Psi\Big(\emptyset\big(\mathrm{f}(p)\big)\Big)$.

Proof: Let us suppose $\inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) < \Psi\left(\emptyset\left(\mathrm{f}(p)\right)\right)$ for every y in X and choose $r \in X$

For which $\inf\Psi\left(\emptyset\left(\mathrm{f}(r)\right)\right)$ is defined then inductively we define a sequence $\{r_n\} \subset X$ with $r_1 = r$. suppose r_n is know is consider

$$W_n = \left\{\left[w \in X : \max\left[\begin{matrix} max\{D_\alpha(T^a w, r_n, z), D_\alpha(w, T^a r_n, z), D_\alpha(w, r_n, T^a z)\}, \\ c.\,max\{D'_\alpha\left(f(T^a w).f(r_n).f(z)\right), D'_\alpha\left(f(w).f(T^a r_n).f(z)\right), D'_\alpha\left(f(w).f(r_n).f(T^a z)\right)\}\end{matrix}\right]\right]\right\}$$

$$\leq K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) - \Psi\left(\emptyset\left(\mathrm{f}(w)\right)\right)\right] \forall\, x, y, z \in X \text{ and } \alpha \in (0,1]$$

W_n is non empty set and there exists $w \in W_n$ such that $r_n \neq Tw$. We can choose $r_{n+1} \in W_n$ such that

$r_n \neq T(r_{n+1})$ and

$$\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) \leq inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) + 1/3\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) - inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right)\right].$$

Clearly $\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right)$ is a non increasing lower bounded sequence. Hence it is a convergent sequence.

Now we prove $\{r_n\}$ and $\{(r_n)\}$are Cauchy sequences:

$$max\{D_\alpha(T^a r_n, T^a r_{n+1}, w), D'_\alpha\left(f(T^a r_n).f(r_{n+1}).f(w)\right)\}$$

$$\leq max\left[\begin{matrix} max\{D_\alpha(f(T^a r_n), r_{n+1}, w), D_\alpha(r_n, T^a r_{n+1}, w), D_\alpha(r_n, r_{n+1}, T^a w)\}, \\ c.\,max\{D'_\alpha\left(f(T^a r_n).f(r_{n+1}).f(w)\right), D'_\alpha\left(f(r_n).f(T^a r_{n+1}).f(w)\right), D'_\alpha\left(f(r_n).f(r_{n+1}).f(T^a w)\right)\}\end{matrix}\right]$$

$$\leq K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) \leq inf\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right)\right]$$

$\forall n, m \in N, n < m \Longrightarrow$ there exists $\alpha_j = \alpha_j(n,m); \sum \alpha_j \leq \alpha$, such that

$$max\left\{\begin{matrix} max\left\{D_{\alpha_j}(T^a r_n, r_m, w), D_{\alpha_j}(r_n, T^a r_m, w), D_{\alpha_j}(r_n, r_m, T^a w)\right\}, \\ c.\,max\left\{D'_{\alpha_j}\left(f(T^a r_n).f(r_m).f(w)\right), D'_{\alpha_j}\left(f(r_n).f(T^a r_m).f(w)\right), D'_{\alpha_j}\left(f(r_n).f(r_m).f(T^a w)\right)\right\}\end{matrix}\right\}$$

$$\leq \sum_{j=n} max\left\{\begin{matrix} max\left\{D_{\alpha_j}(T^a r_n, r_m, w), D_{\alpha_j}(r_j, T^a r_{j+1}, w), D_{\alpha_j}(r_j, r_{j+1}, T^a w)\right\}, \\ c.\,max\left\{D'_{\alpha_j}\left(f(T^a r_j).f(r_{j+1}).f(w)\right), D'_{\alpha_j}\left(f(r_j).f(T^a r_{j+1}).f(w)\right), D'_{\alpha_j}\left(f(r_j).f(r_{j+1}).f(T^a w)\right)\right\}\end{matrix}\right\}$$

Hence, $\forall\, n, m \in N, n < m$;

$$\leq max\left[\begin{matrix} max\{D_\alpha(T^a r_n, r_m, w), D_\alpha(r_n, T^a r_m, w), D_\alpha(r_n, r_m, T^a w)\}, \\ c.\,max\{D'_\alpha\left(f(T^a r_n).f(r_m).f(w)\right), D'_\alpha\left(f(r_n).f(T^a r_m).f(w)\right), D'_\alpha\left(f(r_n).f(r_m).f(T^a w)\right)\}\end{matrix}\right]$$

$$\leq K(\mu)\sum_{j=n}^{m-1}\left[\Psi\left(\emptyset\left(\mathrm{f}(r_j)\right)\right)-inf\Psi\left(\emptyset\left(\mathrm{f}(r_{j+1})\right)\right)\right]$$

$$\leq K(\alpha)\sum_{j=n}^{m-1}\left[\Psi\left(\emptyset(\mathrm{f}(r_n))\right)-inf\Psi\left(\emptyset(\mathrm{f}(r_m))\right)\right]$$

For some α_j with $0<\alpha_{j+1}<\alpha_k\leq\alpha j=n\ldots\ldots\ldots\ldots.m-1$

$$D_\alpha(r_n,r_{n+1},w)\leq D_{\alpha_1}(r_n,r_{n+1},T^ar_{n+1})+D_{\alpha_2}(r_n,T^ar_{n+1},W)+D_{\alpha_3}(T^ar_{n+1},r_{n+1},W)$$

$$\leq D_{\alpha_1}(r_n,r_{n+1},T^ar_{n+1})+D_{\alpha_2}(r_n,T^ar_{n+1},w)+D_{\alpha_3}(T^ar_{n+1},r_{n+1},T^ar_n)+D_{\alpha_4}(T^ar_{n+1},T^ar_n,w)+D_{\alpha_5}(T^ar_{n+1},r_{n+1},w)$$

For $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\leq\alpha$

$$\leq 3\begin{bmatrix} max\{D_\alpha\left((T^ar_n),r_{n+1},w\right),D_\alpha(r_n,T^ar_{n+1},w),D_\alpha(r_n,r_{n+1},T^aw)\}, \\ c.max\{D'_\alpha\left((T^ar_n).f(r_{n+1}).f(w)\right),D'_\alpha\left(f(r_n).f(T^ar_{n+1}).f(w)\right),D'_\alpha\left(f(r_n).f(r_{n+1}).f(T^aw)\right)\}\end{bmatrix}$$

$$\leq 3K(\alpha)\left[\Psi\left(\emptyset(\mathrm{f}(r_n))\right)-inf\Psi\left(\emptyset(\mathrm{f}(r_{n+1}))\right)\right]$$

Then also we get

$$D_\alpha(r_n,r_{n+1},w)\leq 3K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right)-inf\Psi\left(\emptyset\left(\mathrm{f}(r_m)\right)\right)\right]$$

where $n<m$

In the manner we obtain

$$D'_\alpha\left(f(r_n).f(r_{n+1}).f(w)\right)\leq 3K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right)-inf\Psi\left(\emptyset\left(\mathrm{f}(r_m)\right)\right)\right]$$

where $n<m$

Hence $\{r_n\}$ and $\{f(r_n)\}$ are Cauchy sequences.

Assume that $\lim_{n\to\infty}r_n=A$ and $\lim_{n\to\infty}f(r_n)=B$.

Since f is closed therefore $f(A)=B$.

By the continuity of Ψ and lower semi continuity of $\emptyset$ we have

$$\Psi\left(\emptyset(\mathrm{f}(b))\right)\leq\lim_{n\to\infty}\Psi\left(\emptyset(\mathrm{f}(r_n))\right)=\lim_{n\to\infty}\Psi\left(\emptyset(\mathrm{f}(r_{n+1}))\right)$$

Let $\delta=inf\Psi\left(\emptyset(\mathrm{f}(x))\right)\in\mathrm{R}$

$\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right) \le inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) + 1/3\left[\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right) - inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right)\right]$, we have

$$\lim_{n\to\infty}\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right) \le \left({}^2\!/_3\right)\delta + \frac{1}{3\lim_{n\to\infty}\Psi\left(\emptyset\left(\mathrm{f}(r_n)\right)\right)} = \left({}^2\!/_3\right)\delta + 1/3\lim_{n\to\infty}\Psi\left(\emptyset\left(\mathrm{f}(r_{n+1})\right)\right)$$

Which is contraction, therefore there exists x_0 in X such that

$$inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) = \emptyset\left(\mathrm{f}(x_0)\right)$$

Now we give a fixed point theorem as an application of the above theorem under non commuting condition known as weak compatible.

Theorem 3.2 Let $\{X, D_\alpha : \alpha \in (0,1]\}$ and $\{Y, D'_\alpha : \alpha \in (0,1]\}$ be two complete generating space of quasi 2-metric family. $f : X \to Y$ be a closed and $T^a, S^a : X \to X$ be continuous mapping satisfying

(i) $D_\alpha(T^a x, T^a y, z) \le \max\{D_\alpha(T^a x, y, z).(x, T^a y, z).(x, y, T^a z)\}$ and

(ii) $D'_\alpha\left(f(T^a x).f(T^a y), f(z)\right)$

$$\le max\left\{D'_\alpha\left(f(T^a x).f(y).f(z)\right), D'_\alpha\left(f(x).f(T^a y).f(z)\right), D'_\alpha\left(f(x).f(y).f(T^a z)\right)\right\},$$

$\forall\ x, y, z \in X$ and $\alpha \in (0,1]$

(iii) $\Psi : \mathfrak{R} \to \mathfrak{R}$ be non decreasing continuous and bounded below function,

(iv) $\emptyset : f(x) \to \mathfrak{R}$ be a lower semi continuous and bounded below function,

(v) S^a and T^a are weak compatible and
$\max[max\{D_\alpha(T^a, T^a S^a x, z), D_\alpha(x, T^a S^a x, z), D_\alpha(x, S^a x, T^a z)\}],$

$$c.max\left\{D'_\alpha\left(f(T^a x).f(T^a S^a x).f(z)\right), D'_\alpha\left(f(x).f(T^a S^a x).f(z)\right), D'_\alpha\left(f(x).f(S^a x).f(T^a z)\right)\right\}$$

$\le K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) - \Psi\left(\emptyset\left(\mathrm{f}(S^a x)\right)\right)\right]\ \forall\ x, y, z \in X$ and $\alpha \in (0,1]$

And c is any constant. Then there exists unique common fixed point x_0 in X.

Proof: If $x_0 \in X$ such that $inf\Psi\left(\emptyset\left(\mathrm{f}(x)\right)\right) = \Psi\left(\emptyset\left(\mathrm{f}(x_0)\right)\right)$

then $x_0 = T^a S^a x_0 . S^a x_0 = T^a x_0$ therefore some $\alpha \in (0,1]$

$$0 < max\{D_\alpha(T^a, T^a S^a x, z), D_\alpha(x, T^a S^a x, z), D_\alpha(x, S^a x, T^a z)\}$$

$$\le K(\alpha)\left[\Psi\left(\emptyset\left(\mathrm{f}(x_0)\right)\right) = \Psi\left(\emptyset\left(\mathrm{f}(S^a x_0)\right)\right)\right] \le 0$$

$which\ is\ contraction.$ then $Sx_0 = Tx_0$.

Now by weak compatible of $T^a and\ S^a$

$S^a x_0 = T^a S^a x_0 = S^a T^a x_0 = T^a x_0$.

Also for some $\alpha_1, \alpha_2, \alpha_3 \in (0,1]$ such that $\alpha_1 + \alpha_2 + \alpha_3 \leq \alpha$

$D_\alpha(x_0, T^a x_0, z) \leq D_{\alpha_1}(x_0, T^a x_0, T^a S^a x_0) + D_{\alpha_2}(x_0, T^a S^a x_0, z) + D_{\alpha_3}(T^a S^a x_0, T^a x_0, T^a x_0, z)$

$\leq D_{\alpha_3}(T^a S^a x_0, T^a x_0, T^a x_0, z) = 0$. hence $T^a x_0 = S^a x_0 = x_0$

uniqueness: Let us assume there exists another fixed point y_0 such that

$S^a y_0 = T^a y_0 = y_0$ and by **theorem 3.1** we have $inf\Psi\left(\emptyset\big(\mathrm{f}(x)\big)\right) = \emptyset\big(\mathrm{f}(y_0)\big)$.

But $inf\Psi\left(\emptyset\big(\mathrm{f}(x)\big)\right) = \emptyset\big(\mathrm{f}(x_0)\big)$ hence by uniqueness of infima we get $x_0 = y_0$

Remark: Theorem 3.1 and 3.2 can be proved easily for convergent sequence of mappings.

Corollary: Let $\{X, D_\alpha : \alpha \in (0,1]\}$ and $\{Y, D'_\alpha : \alpha \in (0,1]\}$ be two complete generating space of quasi 2-metric family. $f: X \to Y$ be a closed, $\emptyset: f(X) \to \Re$ be a lower semi continuous and bounded below function. Let $S^a: X \to X$ be a mapping such that $\forall\, x, y, z \in X$ and c is any continuous mapping satisfying

$$\max\left\{D_\alpha(S^a x, x, z). D'_\alpha\big(f(S^a x), f(x), f(z)\big)\right\}$$
$$\leq K(\alpha)[\emptyset(\mathrm{x}) = \emptyset(S^a x)]$$

Proof:Consider $T = 1$ and $\Psi = 1$ we get required result.

Example:

Let $X = [0,1] Y = [0, \infty], D_\alpha = D'_\alpha = D_1$ defined by $D_1(x, y, z) = \frac{D(x,y,z)}{1+D(x,y,z)}$

And $D(x, y, z) = max\{|x - y| + |y - z| + |z - x|\}$,

The mapping defined as follows:

$T^a: X \to X$ as $T^a x = x^{2a} f: X \to X$ as $fx = x, \emptyset: f(x) \to R$ as $\emptyset(x) = 1/(1 - x)$

and Ψ: R → R$\Psi(\mathrm{x}) = x^2/2$ and $K(\alpha) = 3$ satisfy the all conditions of theorem 3.1.

$also\ S^a: X \to X$ is defined $S^a x = \frac{x^{2a}}{2a}$, then (S, T) is weak compatible which satisfying the condition of theorem 3.2, hence 0 is a unique fixed point.

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References

[1] D. Downing and W.A. Kirk, A generalization of Cariti's theorem with applicatioins in nonlinear mapping theory, Pacific j. Math.69(1977),339-346.

[2] V. B. Dhagat and V. S. Thakur, Non convex minimization in generating space, Tamkang Journal of Math,Taiwan vol.39, No 3,13-218, Autumn 2008.

[3] J.S.Jung, Y.J. Cho and J. K. Kim, Minimization theorems for fixed point theorems in fuzzy metric space and application, Fuzzy sets and system 61(1994),199-270.