Essay

Holomorphy Vision, Elliptic Functions & Elliptic Curves in TGD Framework

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Abstract

Holography=holomorphy principle allows to solve the extremely nonlinear partial differential equations for the space-time surfaces exactly by reducing them to algebraic equations involving an identically vanishing contraction of two holomorphic tensors of different types. In this article, space-time counterparts for elliptic curves and doubly periodic elliptic functions, in particular Weierstrass function, are considered as an application of the method.

1 Introduction

Holography = holography principle [5, 2, 3, 4] leads to an explicit construction of the solutions of field equations by reducing the field equations from extremely nonlinear partial differential equations to algebraic equations. In this article, elliptic curves and functions are considered as an application.

1.1 Holography=holomorphy as the basic principle

Holography=holomorphy principle allows to solve the field equations for the space-time surfaces exactly by reducing them to algebraic equations.

1. Two functions f_1 and f_2 that depend on the generalized complex coordinates of $H = M^4 \times CP_2$ are needed to solve the field equations. These functions depend on the two complex coordinates ξ_1 and ξ_2 of CP_2 and the complex coordinate w of M^4 and the hypercomplex coordinate u for which the coordinate curves are light-like. If the functions are polynomials, denote them $f_1 \equiv P_1$ and $f_2 \equiv P_2$.

   Assume that the Taylor coefficients of these functions are rational or in the expansion of rational numbers, although this is not necessary either.

2. The condition $f_1 = 0$ defines a 6-D surface in H and so does $f_2 = 0$. This is because the condition gives two conditions (both real and imaginary parts for f_i vanish). These 6-D surfaces are interpreted as analogs of the twistor bundles corresponding to M^4 and CP_2. They have fiber which is 2-sphere. This is the physically motivated assumption, which might require an additional condition stating that ξ_1 and ξ_2 are functions of w as analogs of the twistor bundles corresponding to M^4 and CP_2. This would define the map mapping the twistor sphere of the twistor space of M^4 to the twistor sphere of the twistor space of CP_2 or vice versa. The map need not be a bijection but would be single valued.

   The conditions $f_1 = 0$ and $f_2 = 0$ give a 4-D spacetime surface as the intersection of these surfaces, identifiable as the base space of both twistor bundle analogies.

3. The equations obtained in this way are algebraic equations rather than partial differential equations. Solving them numerically is child's play because they are completely local. TGD is solvable both analytically and numerically. The importance of this property cannot be overstated.

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4. However, a discretization is needed, which can be number-theoretic and defined by the expansion of rationals. This is however not necessary if one is interested only in geometry and forgets the aspects related to algebraic geometry and number theory.

5. Once these algebraic equations have been solved at the discretization points, a discretization for the spacetime surface has been obtained.

   The task is to assign a spacetime surface to this discretization as a differentiable surface. Standard methods can be found here. A method that produces a surface for which the second partial derivatives exist because they appear in the curvature tensor.

   An analogy is the graph of a function for which the (y, x) pairs are known in a discrete set. One can connect these points, for example, with straight line segments to obtain a continuous curve. Polynomial fit gives rise to a smooth curve.

6. It is good to start with, for example, second-degree polynomials P_1 and P_2 of the generalized complex coordinates of H.

1.2 How could the solution be constructed in practice?

For simplicity, let's assume that $f_1 \equiv P_1$ and $f_2 \equiv P_2$ are polynomials.

1. First, one can solve for instance the equation $P_2(u, w, \xi_1, \xi_2) = 0$ giving for example $\xi_2(u, w, \xi_1)$ as its root. Any complex coordinates w, ξ_1 or ξ_2 is a possible choice and these choices can correspond to different roots as space-time regions and all must be considered to get the full picture. A completely local ordinary algebraic equation is in question so that the situation is infinitely simpler than for second order partial differential equations. This miracle is a consequence of holomorphy.

2. Substitute $\xi_2(u, w, \xi_1)$ in P_1 to obtain the algebraic function $P_1(u, w, \xi_1, \xi_2(u, w, \xi_1)) = Q_1(u, w, \xi_1)$.

3. Solve ξ_1 from the condition $Q_1 = 0$. Now we are dealing with the root of the algebraic function, but the standard numerical solution is still infinitely easier than for partial differential equations.

   After this, the discretization must be completed to get a space-time surface using some method that produces a surface for which the second partial derivatives are continuous.

Very interesting special cases are polynomials with order not larger than 4 since for these the roots can be solved explicitly. I have proposed that P_2 characterizes the cosmological constant as a correspondence between the twistor spheres of M^4 and CP_2 and is characterized by the winding number. In standard cosmology Λ is a constant of Nature but in TGD it is predicted to have a hierarchy of values. The simplest relationship would be $P_2 = \xi_2 - w^n$, n integer. In this case, one can solve $\xi_2(w)$ and substitute it to P_1 to obtain the condition

$$P_1(\xi_1, \xi_2(w), w, u) = 0 \ . \tag{1.1}$$

If P_1 as a polynomial of ξ_1 has order lower than 5, the roots of ξ_1 can be solved explicitly. Elliptic curves satisfy the condition

$$\xi_1^2 - w^3 + aw + b = 0 \ . \tag{1.2}$$

The projections of the w-plane are doubly periodic curves and therefore of special interest. For $P_2 = \xi_2 - w^2$ and $P_1 = \xi_1^2 - w\xi_2 + aw + b$, the space-time surface would be a 4-D analog of an elliptic curve. If a and b depend on u, the 3-surface becomes dynamical.

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2 Elliptic curves as an application

One can test whether the numerical method works when the equation giving ξ_1 in terms of w can be solved analytically. For elliptic curves $\xi_1 = \xi_1(w)$, which I have discussed already earlier [1, 2], this is the case.

2.1 Elliptic curves

The third order polynomial characterizing the elliptic curve (see this) can be be expressed in terms of the root of a third order polynomial $P_3(w)$ as

$$E : \xi_1^2 = 4(w - e_1)(w - e_2)(w - e_3) \ , \tag{2.1}$$

One can choose the complex w in such a manner that the equation contains no term proportional to w^2. This is guaranteed if the condition $e_1 + e_2 + e_3 = 0$ holds true. In this case one obtains the form

$$\begin{aligned} & E : \xi_1^2 = 4w^3 - g_2 w - g_3 \ , \qquad\qquad\qquad\qquad , \\ & g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1) \ , \quad g_3 = 4e_1e_2e_3 \ , \quad e_1 + e_2 + e_3 = 0 \ . \end{aligned} \tag{2.2}$$

2.2 Connection with Weierstrass elliptic functions

There is a connection with Weierstrass elliptic functions, which satisfy the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \ . \tag{2.3}$$

Clearly, By using z as a complex coordinate instead of w, $\xi_1(w)$ and w for the elliptic curve can be expressed in terms of Weierstrass elliptic function, which is a solution of this differential equation

$$\xi_1(w) = \wp'(z) \ , \quad w(z) = \wp(z) \ . \tag{2.4}$$

Elliptic functions are doubly periodic and using $z = \wp^{-1}(w)$ as a complex coordinate instead of w, this periodicity becomes manifest. The solution possesses a discrete conformal symmetry consisting of a discrete subgroup of 2-D translations and this gives rise to a lattice structure. This conforms with the fact that the elliptic curve, as a compact 2-D surface in the space spanned by coordinates (ξ_1, w) has the topology of a torus and therefore can allow translations as conformal symmetries. This is the case for the elliptic curves considered.

One can represent torus in a complex plane with coordinate z in terms of Weierstrass elliptic function $\wp$ having a double periodicity in z-plane as conformal symmetries. The torus corresponds to the fundamental domain (2-D lattice cell) obtained by identifying the opposite boundaries of the lattice cell. The periods ω_1 and ω_2 define non-orthogonal directions and their ratio $\tau = \omega_1/\omega_2$ is conformal invariant.

One can solve the fundamental periods ω_1 and ω_2 in the following way. Define the auxiliary quantities

$$a_0 = \sqrt{e_1 - e_3}, \qquad b_0 = \sqrt{e_1 - e_2}, \qquad c_0 = \sqrt{e_2 - e_3}, \ . \tag{2.5}$$

The condition $e_1 + e_2 + e_3 = 0$ allows to eliminate e_3 so that one has

$$a_0 = \sqrt{-e_2}, \qquad b_0 = \sqrt{e_1 - e_2}, \qquad c_0 = \sqrt{-e_1}, \ . \tag{2.6}$$

The fundamental periods ω_1 and ω_2 for the elliptic curve can be calculated very rapidly by

$$\omega_1 = \frac{\pi}{\mathrm{M}(a_0, b_0)}, \qquad \omega_2 = \frac{\pi}{\mathrm{M}(c_0, ib_0)} \tag{2.7}$$

Or more explicitly

$$\omega_1 = \frac{\pi}{\mathrm{M}(\sqrt{-e_2}, \sqrt{e_1 - e_2})} \ , \quad \omega_2 = \frac{\pi}{\mathrm{M}(\sqrt{-e_1}, i\sqrt{e_1 - e_2})} \ . \tag{2.8}$$

Here $\mathrm{M}(x, y)$ is defined as arithmo-geometric mean of x and y by a geometric iteration (see this). Assuming $x \geq y \geq 0$ one has

$$a_0 = x, g_0 = y \ , \quad a_{n+1} = (a_n + g_n)/2, g_{n+1} = \sqrt{(a_n g_n)} \ . \tag{2.9}$$

At the limit $n \to \infty$ one has a_{n+1}
$simeqa_n \to a$ and $g_{n+1} \simeq g_n \to g$ and one has $a = (a + g)/2$ and $g = \sqrt{ag}$ implying $a = g$ so that arithmetic and geometric means are identical. Care is required to take the correct sign of square root at each step of iteration (positive in the case considered). The iteration generalizes to the complex case and there probably exist tested programs performing the iteration.

2.3 Elliptic functions and planetary orbits

Weierstrass elliptic functions $\wp$ are periodic in complex plane and this inspires the question of how they rate to the formulas for the elliptic planetary orbits in the gravitational potential $V(r) = k/r$. Choose the mass unit so that the mass is $m = 1$. By spherical symmetry, orbits are planar and angular momentum conservation gives $L = r^2 d\phi/dt$ as a constant of motion. In the radial degree of freedom, energy conservation $E = (dr/dt)^2/2 - k/r + L^2/2r^2$ gives $(dr/dt)^2 = E + k/r - L^2/r^2$. By using $u = 1/r$ as variable, one obtains $(du/dt)^2 = Eu^4 + ku^3 - L^2u^2$ giving $du/dt = \sqrt{Eu^4 + ku^3 - L^2u^2}$, which in turn gives $t = \int du/\sqrt{Eu^4 + ku^3 - L^2u^2}$. This gives the planetary orbit as an elliptic integral. The elliptic integral continued to complex values z of the time coordinate t defines explicitly the inverse of a doubly periodic elliptic function.

This integrand gives an elliptic function (see this), which is more general than $\wp$. The integrand $1/\sqrt{(1 - c^2t^2)(1 + E^2t^2)}$ gives Abelian elliptic functions whereas the integrand $1/\sqrt{(1 - t^2)(1 - k^2t^2)}$ gives Jacobi elliptic functions.

The elliptic integral defines the inverse of the Weierstrass elliptic function $\wp$ only for $E = 0$ so that the polynomial under the square root reduces to a third order polynomial. The integrand reduces to $1/u\sqrt{ku - L^2}$. The square root factor vanishes at $u_0 = L^2/k$ which corresponds to the minimal distance r between the two masses and $u = 0$, which corresponds to $r = \infty$. This corresponds to a critical situation in which elliptic orbit transforms to a parabolic orbit. The absence of periodicity at real axis is consistent with the double periodicity of $\wp$ in the complex plane.

One can transform the integrand to a form appearing in $\wp$ by assuming $k = 4$ and by making a linear coordinate change $u \to v$, $u = v - v_0$, and choosing v_0 in such a way that the v^2 term under the square root vanishes. The required value of v_0 is $v_0 = -L^2/6k$. The parameters g_2 and g_3 in $4v^3 - g_2v - g_3$ are given by $g_2 = L^4/24$ and $g_3 = -L^6 2^{-5} 3^{-3}$.

One can calculate the inverse of $z = \wp^{-1}(w)$ (complex analog of time coordinate) for a complex argument w (complex analog of the radial coordinate of a planet at the elliptic orbit) by calculating the complex integral

$$\wp^{-1}(w) = \int_{\gamma(w_0 \to w)} \frac{1}{\sqrt{4w^3 - g_2w - g_3}} dw \ .$$

The integration path γ can be chosen in infinitely many ways and a small deformation does not affect the result. The argument of the square root as a polynomial has three roots and the deformation of the integration path in such a way that the deformed curve passes over a root of $4w^3 - g_2w - g_3$, the integral changes. This gives rise to the infinitely many-valued nature of $\wp^{-1}(\mathrm{w})$. For a root with multiplicity 1, the integrand has $1/\sqrt{w - w_0}$ type singularity as the end point of a cut and since the cut means discontinuity, the integral depends on which side of the cut the integration path goes. For a double root there is a pole.

The connection between planetary dynamics and generalized complex surfaces is intriguing and leads to ask whether the connection is more general so that space-time surfaces defined by the conditions $f_1 = 0, f_2 = 0$ represent some dynamical systems, say periodic systems in spherically symmetric potential. These surfaces should allow interpretation as closed surfaces of CP_2 with coordinates ξ_1 and w. These surfaces are characterized some genus and should correspond to a conformal equivalence class characterized by Teichmueller parameters (in the case of torus assignable to the elliptic functions there is only one modular invariant τ defined by the ratio of complex periods). The condition of being closed might require additional constraints. Could closed surfaces as solutions to the conditions $(f_1, f_2) = (0, 0)$ correspond to nonlinear first order differential equations with $\xi_1 = dE/dz$ and $w = E(z)$ defining higher genus analogs of elliptic curves and elliptic functions?

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