Article

On Certain Number Theoretical Aspects of TGD

Matti Pitkänen [1]

Abstract

Recently a considerable progress has occurred in the understanding of number theoretic aspects of quantum TGD:

1. There are reasons to think that TGD could be formulated purely number theoretically without introduction of any action principle. This would conform with the $M^8 - H$ duality and the generalization of the geometric Langlands correspondence to dimension $D = 4$.

   Number theoretic vision however gives extremely powerful constraints on the vacuum functional suggesting even an explicit formula for it. The condition that this expression corresponds to the exponent of Kähler function expressible as Kähler action fixes the coupling constant evolution for the action.

2. Extensions of rationals, the corresponding Galois groups and ramified primes assignable to polynomials and identifiable as p-adic primes assigned to elementary particles are central notions of quantum TGD. In the recent formulation based on holography = holomorphy principle, it is not quite clear how to assign these notions to the space-time surfaces. The notion of Galois group has a 4-D generalization but can one obtain the ordinary Galois groups and ramified primes? Two ways to achieve this are discussed in this article.

   One could introduce a hierarchy of 4-polynomials (f_1, f_2, f_3, f_4) instead oly (f_1, f_2) and the common roots all 4 polynomials as a set of discrete points would give the desired basic notions assignable to string world sheets.

   One can also consider the maps $(f_1, f_2) \to G \circ (f_1, f_2) = (g_1(f_1, f_2), g_2(f_1, f_2))$ and assign these notions to the surfaces $(g_1(f_1, f_2), g_2(f_1, f_2)) = (0, 0)$.

3. Number theoretical universality is possible if the coefficients of the analytic functions (f_1, f_2) of 3 complex coordinates and one hypercomplex coordinate of $H = M^4 \times CP_2$ are in an algebraic extension of rationals. This implies that the solutions of field equations make sense also in p-adic number fields and their extensions induced by extensions of rationals.

In this article the details of the adelicization boiling to p-adicization for various p-adic number fields, in particular those assignable to ramified primes, are discussed. p-Adic fractals and holograms emerge very naturally and the iterations of $(f_1, f_2) \to G \circ (f_1, f_2) = (g_1(f_1, f_2), g_2(f_1, f_2)$ define hierarchical fractal structures analogs to Mandelbrot and Julia fractals and p-adically mean exponential explosion of the complexity and information content of cognition. The possible relationship to biological and cognitive evolution is highly interesting. Both p-adicization and hyperfinite finite factors of type II_1 (HFFs)[**?**]vNeumann,vNeumannnew, which both allow a description of finite measurement resolution. The relationship between these two building bricks of TGD is discussed.

1 Introduction

Recently a considerable progress has occurred in the understanding of number theoretic aspects of quantum TGD [7, 9]. These steps of progress are buried in the separate articles and chapters to that it is appropriate to collect them to a single article.

[1] Correspondence: Matti Pitkänen http://tgdtheory.fi/. Address: Valtatie 8 as 2, 03600, Karkkila, Finland. Email: matpitka6@gmail.com.

1. There are reasons to think that TGD could be formulated using only number theoretic concepts without the introduction of any action principle. The reason is that the holomorphy= holography principle implies that field equations are satisfied for any general coordinate invariant action constructible in terms of the induced geometry. This would conform with the $M^8 - H$ duality and the generalization of the geometric Langlands correspondence to dimension $D = 4$.

   Number theoretic vision however gives extremely powerful constraints on the vacuum functional suggesting even an explicit formula for it in terms of a discriminant assignable to a polynomial. The condition that this expression corresponds to the exponent of Kähler function expressible as Kähler action fixes the coupling constant evolution for the action defining the space-time surface. At those singularities, in which the space-time surface is branched, manifold property requires selection of a single branch and boundary conditions expressing the conservation laws must be satisfied. This poses constraints on the action and the coupling parameters appearing in the action. Also the modified Dirac action determined by supersymmetry must satisfy these constraints.

2. Extensions of rationals, the corresponding Galois groups and ramified primes assignable to the polynomials defining the extensions and identifiable as p-adic primes assigned to elementary particles are central notions of quantum TGD. The recent formulation is based on holography = holomorphy principle giving space-time surfaces as 4-D roots for a pair (f_1, f_2) of analytic functions of 3 complex coordinates and 1 hypercomplex coordinate of H with Taylor coefficients in an extension of rationals. In this approach it is however not quite clear how to assign these notions to the space-time surfaces. The notion of Galois group has a generalization to 4-D generalization but can one obtain the ordinary Galois groups and ramified primes? Two ways to achieve this are discussed in this article.

   One could introduce a hierarchy of 4-polynomials (f_1, f_2, f_3, f_4) instead oly (f_1, f_2) giving rise to 4-D, 2-D and 0-D geometric objects. The common roots all 4 polynomials as a set of discrete points would give the desired basic notions assignable to string world sheets.

   One can also consider the maps $(f_1, f_2) \rightarrow G \circ (f_1, f_2) = (g_1(f_1, f_2), g_2(f_1, f_2))$ and assign these notions to the surfaces $(g_1(f_1, f_2), g_2(f_1, f_2)) = (0, 0)$.

3. Number theoretical universality is possible if the coefficients of the analytic functions (f_1, f_2) of 3 complex coordinates and one hypercomplex coordinate of $H = M^4 \times CP_2$ are in an algebraic extension of rationals. This implies that the solutions of field equations make sense also in the extensions of p-adic number fields induced by extensions of rationals.

In this article the details of the adelicization boiling to p-adicization for various p-adic number fields, in particular those assignable to ramified primes, are discussed. p-Adic fractals and holograms emerge very naturally and the iterations of $(f_1, f_2) \rightarrow G \circ (f_1, f_2) = (g_1(f_1, f_2), g_2(f_1, f_2))$ define hierarchical fractal structures analogs to Mandelbrot and Julia fractals and p-adically mean exponential explosion of the complexity and information content of cognition. The possible relationship to biological and cognitive evolution is highly interesting.

Both p-adicization and hyperfinite finite factors of type II_1 (HFFs) [**?**, **?**], which both allow a description of finite measurement resolution. The relationship between these two building bricks of TGD is discussed.

2 Does the universality of the holomorphy-holography principle make the notion of action un-necessary in the TGD framework?

It is gradually becoming clear that in the TGD framework the holography-holomorphy principle could make the notion of action defining the space-time surfaces un-necessary at the fundamental level. Only the Dirac action for the second quantized free spinors of H and the induced Dirac action would be needed.

The geometrization of physics would reduce to its algebraic geometrization and number theoretical universality would allow to describe correlates of cognition. The four-dimensionality of space-time surfaces would be essential in making the theory non-trivial by allowing to identify vertices for fermion pair creation in terms of defects of the standard smooth structure of the space-time surface making it an exotic smooth structure.

2.1 Holography=holomorphy as the basic principle

Holography=holomorphy principle allows to solve the field equations for the space-time surfaces exactly by reducing them to algebraic equations.

1. Two functions f_1 and f_2 that depend on the generalized complex coordinates of H=$M^4 \times CP_2$ are needed to solve the field equations. These functions depend on the two complex coordinates ξ_1 and ξ_2 of CP_2 and the complex coordinate w of M^4 and the hypercomplex coordinate u for which the coordinate curves are light-like. If the functions are polynomials, denote them $f_1 \equiv P_1$ and $f_2 \equiv P_2$.

   Assume that the Taylor coefficients of these functions are rational or in the expansion of rational numbers, although this is not necessary either.

2. The condition $f_1 = 0$ defines a 6-D surface in H and so does $f_2 = 0$. This is because the condition gives two conditions (both real and imaginary parts for f_i vanish). These 6-D surfaces are interpreted as analogs of the twistor bundles corresponding to M^4 and CP_2. They have fiber which is 2-sphere. This is the physically motivated assumption, which might require an additional condition stating that ξ_1 and ξ_2 are functions of w as analogs of the twistor bundles corresponding to M^4 and CP_2. This would define the map mapping the twistor sphere of the twistor space of M^4 to the twistor sphere of the twistor space of CP_2 or vice versa. The map need not be a bijection but would be single valued.

   The conditions $f_1 = 0$ and $f_2 = 0$ give a 4-D spacetime surface as the intersection of these surfaces, identifiable as the base space of both twistor bundle analogies.

3. The equations obtained in this way are algebraic equations rather than partial differential equations. Solving them numerically is child's play because they are completely local. TGD is solvable both analytically and numerically. The importance of this property cannot be overstated.

4. However, a discretization is needed, which can be number-theoretic and defined by the expansion of rationals. This is however not necessary if one is interested only in geometry and forgets the aspects related to algebraic geometry and number theory.

5. Once these algebraic equations have been solved at the discretization points, a discretization for the spacetime surface has been obtained.

   The task is to assign a spacetime surface to this discretization as a differentiable surface. Standard methods can be found here. A method that produces a surface for which the second partial derivatives exist because they appear in the curvature tensor.

   An analogy is the graph of a function for which the (y, x) pairs are known in a discrete set. One can connect these points, for example, with straight line segments to obtain a continuous curve. Polynomial fit gives rise to a smooth curve.

6. It is good to start with, for example, second-degree polynomials P_1 and P_2 of the generalized complex coordinates of H.

2.2 How could the solution be constructed in practice?

For simplicity, let's assume that $f_1 \equiv P_1$ and $f_2 \equiv P_2$ are polynomials.

1. First, one can solve for instance the equation $P_2(u, w, \xi_1, \xi_2) = 0$ giving for example $\xi_2(u, w, \xi_1)$ as its root. Any complex coordinates w, ξ_1 or ξ_2 is a possible choice and these choices can correspond to different roots as space-time regions and all must be considered to get the full picture. A completely local ordinary algebraic equation is in question so that the situation is infinitely simpler than for second order partial differential equations. This miracle is a consequence of holomorphy.
2. Substitute $\xi_2(u, w, \xi_1)$ in P_1 to obtain the algebraic function $P_1(u, w, \xi_1, \xi_2(u, w, \xi_1)) = Q_1(u, w, \xi_1)$.
3. Solve ξ_1 from the condition $Q_1 = 0$. Now we are dealing with the root of the algebraic function, but the standard numerical solution is still infinitely easier than for partial differential equations.

   After this, the discretization must be completed to get a space-time surface using some method that produces a surface for which the second partial derivatives are continuous.

Very interesting special cases are polynomials with order not larger than 4 since for these the roots can be solved explicitly. I have proposed that P_2 characterizes the cosmological constant as a correspondence between the twistor spheres of M^4 and CP_2 and is characterized by the winding number. In standard cosmology Λ is a constant of Nature but in TGD it is predicted to have a hierarchy of values. The simplest relationship would be $P_2 = \xi_2 - w^n$, n integer. In this case, one can solve $\xi_2(w)$ and substitute it to P_1 to obtain the condition

$$P_1(\xi_1, \xi_2(w), w, u) = 0 \ .$$

If P_1 as a polynomial of ξ_1 has order lower than 5, the roots of ξ_1 can be solved explicitly. Elliptic curves satisfy the condition

$$\xi_1^2 - w^3 + aw + b = 0 \ .$$

The projections of in w-plane are doubly periodic curves and therefore of special interest. For $P_2 = \xi_2 - w^2$ and $P_1 = \xi_1^2 - w\xi_2 + aw + b$, the space-time surface would be a 4-D analog of an elliptic curve. If a and b depend on u, the 3-surface becomes dynamical.

2.3 Algebraic universality

What is so remarkable is that the solutions of $(f_1, f_2) = (0, 0)$ to the variation of any action if the action is general coordinate invariant and depends only on the induced geometry. Metric and the tensors like curvature tensor associated with it and induced gauge fields and tensors associated with them.

The reason is that complex analyticity implies that in the equations of motion there appears only contractions of complex tensors of different types. The second fundamental form (external curvature) defined by the trace of the tensor with respect to the induced metric defined by the covariant derivatives of the tangent vectors of the space-time surfaces is as a complex tensor of type (2,0)+(0,2) and the tensors contracted with it are of type (1,1). The result is identically zero.

The holography-holomorphy principle provides a nonlinear analogy of massless field equations and the four surfaces can be interpreted as trajectories for particles that are 3-surfaces instead of point particles, i.e. as generalizations of geodesics. Geodesics are indeed 1-D minimal surfaces. We obtain a geometric version of the field-particle duality.

2.4 Number-theoretical universality

If the coefficients of the function f_1 and f_2 are in an extension of rationals, number-theoretical universality is obtained. The solution in the real case can also be interpreted as a solution in the p-adic cases $p = 2, 3, 5, 7, ...$ when we allow the expansion of the p-adic number system as induced by the rational expansions.

p-adic variants of space-time surfaces are cognitive representations for the real surfaces. The so-called ramified primes are selected for a special position, which can be associated with the discriminant as its prime factors. A prime number is now a prime number of an algebraic expansion. This makes possible adelic physics as a geometric correlate of cognition. Cognition itself is assignable to quantum jumps.

2.5 Is the notion of action needed at all at the fundamental level?

The universality of the space-time surfaces solving the field equations determined by holography=holomorphy principle forces us to ask whether the notion of action is completely unnecessary. Does restricting geometry to algebraic geometry and number theory replace the principle of action completely? This could be the case.

1. The vacuum functional $exp(K)$, where the Kähler function corresponds to the classical action , could be identified as the discriminant D associated with a polynomial. It would therefore be determined entirely by number theory as a product of differences of the roots of a polynomial P or in fact, of any analytic function. The problem is that the space-time surfaces are determined as roots of two analytic functions f_1 and f_2, rather than only one.

2. Could one define the 2-surfaces by allowing a third analytic function f_3 so that the roots of $(f_1, f_2, f_3) = (0, 0, 0)$ would be 2-D surfaces. One can solve 3 complex coordinates of $M^4 \times CP_2$ as functions of the hypercomplex coordinate u whereas its dual remains free. One would have a string world sheet with a discrete set of roots for the 3 complex coordinates whose values depend on time. By adding a fourth function f_4 and substituting the 3 complex coordinates, $f_4 = 0$ would allow as roots values of the coordinate u. Only real roots would be allowed. A possible interpretation of these points of the space-time surface would be as loci of singularities at which the minimal surface property, i.e. holomorphy, fails.

   Note that for quadratic equations $ax^2 + bx + c = 0$, the discriminant is $D = b^2 - 4ac$ and more generally the product of the differences of the roots. This formula also holds when f_1 and f_2 are not polynomials.

   The assumptions that some power of D corresponds to $exp(K)$ and that K corresponds to the action imply additional conditions for the coupling constants appearing in the action , i.e. the coupling constant evolution.

3. This is not yet quite enough. The basic question concerns the construction of the interaction vertices for fermions. These vertices reduce to the analogs of gauge theory vertices in which induced fermion current assignable to the volume action is contracted with the induced gauge boson.

   The volume action is a unique choice in the sense that in this case the modified gamma matrices defined as contractions of the canonical momentum currents of the action with the gamma matrices of H reduce to induced gamma matrices, which anticommute to the induced metric. For a general action this is not the case.

   The vertex for fermion pair creation corresponds to a defect of the standard smooth structure for the space-time surface and means that it becomes exotic smooth structure. These defects emerge in dimension D=4 and make it unique. In TGD, bosons are bound states of fermions and antifermions so that this also gives the vertices for the emission of bosons.

For graviton emission one obtains an analogous vertex involving the second fundamental form at the partonic orbit. The second fundamental form would have delta function singularity at the vertex and vanish elsewhere. If field equations are true also in the vertex, the action must contain an additional term, say Kähler action. Could the singularity of the second fundamental form correspond to the defect of the standard smooth structure?

4. If this view is correct, number theory and algebraic geometry combined with the geometric vision would make the notion of action un-necessary at the fundamental level. Geometrization of physics would be replaced by its algebraic geometrization. Action would however be a useful tool at the QFT limit of TGD.

Physical intuition suggests that the various general coordinate invariant actions could have physical relevance. For branched space-time surfaces manifold property forces to select one branch. The natural condition is that classical conservation laws are satisfied at the lower dimension surface at which branching would occur. This could select the classical action and the coupling constant evolution of various coupling parameters associated with it. This would add to the modified gamma matrices determined as contractions of canonical momentum currents and gamma matrices of H an additional term so that the anticommutators would not give the induced metric anymore. Could this replacement of the induced metric with a modified metric containing terms quadratic in canonical momentum currents be regarded as some kind of radiative correction?

3 About adelization of TGD

p-Adicization of TGD relates real number based physics as physics of sensory perceptions to the physics of cognition based on number theoretic notions, in particular p-adic number fields which combine with reals to adeles forming a number theoretical hierarchy.

3.1 How to assign ordinary Galois groups, and ramified primes to space-time surfaces?

Extensions of rationals, Galois groups, and ramified primes assignable to polynomials of a single complex variable are central in the number theoretic vision. It is not however completely clear how they should emerge from the holography= holomorphy vision.

1. If the functions $g_i \equiv P_i$ are polynomials, which vanish at the origin $(0,0)$ (this is not a necessary condition), the surfaces $(f_1, f_2) = (0,0)$ are roots of $(P_1(f_1, f_2), P_2(f_1, f_2)) = (0,0)$. Besides these roots, there are roots for which (f_1, f_2) does not vanish. One can solve the roots $f_2 = h(f_1)$ from $g_2(f_1, f_2) = 0$ and substitute to $P_1(f_1, f_2) = 0$ to get $P_1(f_1, h(f_1)) \equiv P_1 \circ H(f_1)) = 0$. The values of $H(f_1)$ are roots of P_1 and are algebraic numbers if the coefficients of P_1 are in an extension of rationals. One can assign to the roots discriminant, ramified primes, and Galois group. This is just what the phenomenological number theoretical picture requires.

2. In the earliest approach to $M^8 - H$ duality summarized in [3, 4, 6] polynomials P of a single complex coordinate played a key role. Although this approach was a failure, it added to the number theoretic vision Galois groups and ramified primes as prime factors of the discriminant D, identified as p-adic primes in p-adic mass calculations. Note that in the general case the ramified primes are primes of algebraic extensions of rationals: the simplest case corresponds to Gaussian primes and Gaussian Mersenne primes indeed appear in the applications of TGD [1, 2].

   The problem was to assign a Galois group and ramified primes to the space-time surfaces as 4-D roots of $(f_1, f_2) = (0,0)$. One can indeed define the counterpart of the Galois group defined as analytic flows permuting various 4-D roots of $(f_1, f_2) = (0,0)$ [7].

Since the roots are 4-D surfaces, it is far from clear whether there exists a definition of discriminant as an analog for the product of root differences. Also it is unclear what the notion of ramified prime could mean. However, the ordinary Galois group plays a key role in the number theoretic vision: can one identify it? An possible identification of the ordinary Galois group and ramified primes would by the assignment to maps defined by $(f_1, f_2) \to (P_1(f_1, f_2), P_2(f_1, f_2))$ would be in terms of $(P_1(f_1, f_2), P_2(f_1, f_2)) = (0, 0)$ giving the roots $P_1(f_1, h(f_1)) = 0$ as values of $h(f_1)$. The roots belong to extension rationals even when f_i are arbitrary analytic functions of H coordinates but correspond geometrically to 4-surfaces.

3. The earlier proposal is that the ordinary Galois group can be assigned to the partonic 2-surfaces so that points of the partonic 2-surface as roots of a polynomial give rise to the Galois group and ramified primes. The most elegant way to realize this is to introduce 4 polynomials (P_1, P_2, P_3, P_4). The roots of (P_1, P_2, P_3) allow to solve the 3 complex coordinates as a function of the hypercomplex coordinate u. This surface can be identified as a string world sheet.

   The additional condition $P_4(u) = 0$ gives roots which are algebraic numbers if the coefficients of P_4 are in an extension of rationals. Note that only real roots are allowed.

   The interpretation of the roots for u would be as singularities of the space-time surface located at the partonic 2-surfaces where the minimal surface property fails and the trace of the second fundamental form diverges. These points would correspond to vertices for the creation of a fermion pair and would represent defects of the standard smooth structure giving rise to an exotic smooth structure [10, 5, 11].

Cognition always requires a discretization.

1. The space of space-time surfaces ("world of classical worlds", WCW) allows a hierarchy of discretizations. The Taylor coefficients of the two analytic functions f_1, f_2 defining space-time belong to some extension E of rationals forming a hierarchy. Therefore a given space-time surface corresponds to a discrete set of integers/rationals in an extension of rationals so that also WCW is discretized for given E. For polynomials and rational functions this set is discrete. This gives a hierarchy. At the level of the space-time surface an analogous discretization in terms of E takes place.

2. Gödel number for a given theorem as almost deterministic time evolution of 3-surface would be parametrized by the Taylor coefficients in a given extension of rationals. Polynomials are simplest analytic functions and irreducible polynomials define polynomial primes having no decomposition to polynomials of a lower degree. Polynomial primes might be seen as counterparts of axioms. General analytic functions are analogous to transcendentals.

3. One can form analogs of integers as products of polynomials inducing products of space-time surfaces as their roots. The space-time surfaces are unions for the space-time surfaces defined by the factors but an important point is that they have a discrete set of intersection points. Fermionic n-point functions defining scattering amplitudes are defined in terms of these intersection points and give a quantum physical realization giving information of the quantum superpositions of space-time surfaces as quantum theorems.

3.2 p-Adicization, assuming holography = holomorphy principle, produces p-adic fractals and holograms

The recent chat with Tuomas Sorakivi, a member of our Zoom group, was about the concrete graphical representations of the spacetime surfaces as animations. The construction of the representations is shockingly straightforward, because the partial differential equations reduce to algebraic equations that are easy to solve numerically. For the first time, it seems that GPT has created a program without obvious bugs. The challenges relate to how to represent time=constant 2-D sections of the 4-surface most conveniently and how to build animations about the evolution of these sections.

Tuomas asked how to construct p-adic counterparts for space-time surfaces in $H = M^4 \times CP_2$. I have been thinking about the details of this presentation over the years. Here is my current vision of the construction.

1. By holography = holomorphy principle, space-time surfaces in H correspond to roots $(f_1, f_2) = (0, 0)$ for two analytic (holomorphic) functions f_i of of 3 complex coordinates and one hypercomplex coordinate of H. The Taylor coefficients of f_i are assumed to be rational or in an algebraic extension of rationals but even more general situations are possible. A very important special case are polynomials $f_i = P_i$.

2. If we are talking about polynomials or analytic functions with coefficients that are rational or in algebraic extension to rationals, then a purely formal p-adic equivalent can be associated with every real surface with the same equations.

3. However, there are some delicate points involved.

   (a) The imaginary unit sqrt(-1) is in algebraic expansion if p modulo 4=3. What about p modulo 4=1. In this case, sqrt(-1) can be multiplied as an ordinary p-adic number by the square root of an integer that is only in algebraic expansion. So the problem is solved.

   (b) In p-adic topology, large powers of p correspond to small p-adic numbers, unlike in real topology. This eventually led to the canonical concept of identification. Let's translate the powers of p in the expansion of a real number into powers of p (the equivalent of the decimal expansion).

   $$\sum x_n p^n \leftrightarrow \sum x_n p^{-n} \quad .$$

   This map of p-adic numbers to real numbers is continuous, but not vice versa. In this way, real points can be mapped to p-adic points or vice versa. In p-adic mass calculations, the map of p-adic points to real points is very natural. One can imagine different variants of the canonical correspondence by introducing, for example, a pinery cutoff analogous to the truncation of decimal numbers. This kind of cutoff is unavoidable.

   (c) As such, this correspondence from reals to p-adics is not realistic at the level of H because the symmetries of the real H do not correspond to those of p-adic H. Note that the correspondence at the level of spacetime surfaces is induced from that at the level of the embedding space.

4. is forces number theoretical discretization, i.e. cognitive representations (p-adic and more generally adelic physics is assumed to provide the correlates of cognition). The symmetries of the real world correspond to symmetries restricted to the discretization. The lattice structure for which continuous translational and rotational symmetries are broken to a discrete subgroup is a typical example.

   Let us consider a given algebraic extension of rationals.

   (a) Algebraic rationals can be interpreted as both real and p-adic numbers in an extension induced by the extension of rationals. The points of the cognitive representations correspond to the algebraic points allowed by the extension and correspond to the intersection points of reality as a real space-time surface and p-adicity as p-adic space-time surface.

   (b) These algebraic points are a series of powers of p, but there are only a *finite* number of powers so that the interpretation as algebraic integers makes sense. One can also consider rations of algebraic integers if canonical identification is suitably modified. These discrete points are mapped by the canonical identification or its modification to the rational case from the real side to the p-adic side to obtain a cognitive representation. The cognitive representation gives a discrete skeleton that spans the spacetime surface on both the real and p-adic sides.

Let's see what this means for the concrete construction of p-adic spacetime surfaces.

1. Take the same equations on the p-adic side as on the real side, that is $(f_1, f_2) = (0, 0)$, and solve them around each discrete point of the cognitive representation in some p-adic sphere with radius p^{-n}.

   The origin of the generalized complex coordinates of H is **not** taken to be the origin of p-adic H, but this canonical identification gives a discrete algebraic point on the p-adic side. So, around each such point, we get a p-adic scaled version of the surface $(f_1, f_2) = (0, 0)$ inside the p-adic sphere. This only means moving the surface to another location and symmetries allow it.

2. How to glue the versions associated with different points together? This is not necessary and not even possible!

   The p-adic concept of differentiability and continuity allows fractality and holography. These are closely related to the p-adic non-determinism meaning that any function depending on finite number of pinary digits has a vanishing derivative. In differential and partial differential equations this implies non-determinism, which I have assumed corresponds to the real side of the complete violation of classical determinism for holography.

   The definition of algebraic surfaces does not involve derivatives but also for algebraic surfaces the roots of $(f_1, f_2) = (0, 0)$ can develop branching singularities at which several roots as space-time regions meet and one must choose one representative [8].

   (a) Assume that the initial surface is defined inside the p-adic sphere, whose radius as the p-adic norm for the points is p^{-n}, n integer. One can even assume that a p-adic counterpart has been constructed only for the spherical shell with radius p^{-n}.
   The essential thing here is that the interior points of a p-adic sphere cannot be distinguished from the points on its surface. The surface of a p-adic sphere is therefore more like a shell. How do you proceed from the shell to the "interiors" of a p-adic sphere?

   (b) The basic property of two p-adic spheres is that they are either point strangers or one of the two is inside the other. A p-adic sphere with radius p^{-n} is divided into point strangers p-adic spheres with radius p^{-n-1} and in each such sphere one can construct a p-adic 4-surface corresponding to the equations $(f_1, f_2) = (0, 0)$. This can be continued as far as desired, always to some value n=N. It corresponds to the shortest scale on the real side and defines the measurement resolution/cognitive resolution physically.

   (c) This gives a fractal for which the same $(f_1, f_2) = (0, 0)$ structure repeats at different scales. We can also go the other way, i.e. to longer scales in the real sense.

   (d) Also a hologram emerges. All the way down to the smallest scale, the same structure repeats and an arbitrarily small sphere represents the entire structure. This strongly brings to mind biology and genes, which represent the entire organism. Could this correspondence at the p-adic level be similar to the one above or a suitable generalization of it?

3. Many kinds of generalizations can be obtained from this basic fractal. Endless repetition of the same structure is not very interesting. p-Adic surfaces do not have to be represented by the same pair of functions at different p-adic scales.

   Of particular interest are the 4-D counterparts to fractals, to which the names Feigenbaum, Mandelbrot and Julia are attached. They can be constructed by iteration

$$(f_1, f_2) \to G(f_1, f_2) = (g_1(f_1, f_2), g_2(f_1, f_2)) \to G(G(f_1, f_2)) \to ...$$

so that at each step the scale increases by a factor p. At the smallest scale p^{-n} one has $(f_1, f_2) = (0, 0)$. At the next, longer scale p^{-N+1} one has $G(f_1, f_2) = (0, 0)$, etc.... One can assign to this kind of hierarchy a hierarchy of extensions of rationals and associated Galois groups whose dimension increases exponentially meaning that algebraic complexity, serving as a measure for the level of conscious intelligence and scale of quantum coherence also increases in the same way.

The iteration proceeds with the increasing scale and the number-theoretic complexity measured the dimension of the algebraic extension increases exponentially. Cognition becomes more and more complex. Could this serve as a possible model for biological and cognitive evolution as the length scale increases?

The fundamental question is whether many-sheeted spacetime allows for a corresponding hierarchy at the real side? Could the violation of classical determinism interpreted as p-adic non-determinism for holography allow this?

3.3 How could p-adicization and hyper-finite factors relate?

Factors of type I are von Neumann algebras acting in the ordinary Hilbert space allowing a discrete enumerable basis. Also hyperfinite factors of type II_1 (HFFs in the sequel) play a central role in quantum TGD [**?**, **?**]. HFFs replace the problematic factors of type III encountered in algebraic quantum field theories. Note that von Neumann himself regarded factors of type III pathological.

3.3.1 HFFs and p-adic physics as key notions of TGD

HFFs have rather unintuitive properties, which I have summarized in [**?**, **?**].

1. The Hilbert spaces associated with HFFs do not have a discrete basis and one could say that the dimension of Hilbert spaces associated with HFFs corresponds to the cardinality of reals. However, the dimension of the Hilbert space identified as a trace $Tr(Id)$ of the unit operator is finite and can be taken equal to 1.

2. HFFs have subfactors and the inclusion of sub-HFFs as analogs of tensor factors give rise to subfactors with dimension smaller than 1 defining a fractal dimension. For Jones inclusions these dimensions are known and form a discrete set algebraic numbers. In the TGD framework, the included tensor factor allows an interpretation in terms of a finite measurement resolution. The inclusions give rise to quantum groups and their representations as analogs of coset spaces.

p-Adic numbers represent a second key notion of TGD.

1. p-Adic number fields emerged in p-adic mass calculations [**?**, 1, 2] [**?**]. Their properties led to a proposal that they serve as correlates of cognition. All p-adic number fields are possible and can be combined to form adele and the outcome is what could be called adelic physics [**?**, **?**].

2. Also the extensions of p-adic number fields induced by the extensions of rationals are involved and define a hierarchy of extensions of adeles. The ramified primes for a given polynomial define preferred p-adic primes. For a given space-time region the extension is assignable to the coefficients for a pair of polynomials or even Taylor coefficients for two analytic functions defining the space-time surface as their common root.

3. The inclusion hierarchies for the extensions of rationals accompanied by inclusion hierarchies of Galois groups for extensions of extensions of .... are analogous to the inclusion hierarchies of HFFs.

3.3.2 Holography = holomorphy vision

Before discussing how p-adic and real physics relate, one must summarize the recent formulation of TGD based on holography = holography correspondence.

1. The recent formulation of TGD allows to identify space-time surfaces in the imbedding space $H = M^4 \times CP_2$ as common roots for the pair (f_1, f_2) of generalized holomorphic functions defined in H. If the Taylor coefficients of f_i are in an extension of rationals, the conditions defining the space-time surfaces make sense also in an extension of p-adic number fields induced by this extension. As a special case this applies to the case when the functions f_i are polynomials. For the completely Taylor coefficients of generalized holomorphic functions f_i, the p-adicization is not possible. The Taylor series for f_i must also converge in the p-adic sense. For instance, this is the case for $exp(x)$ only if the p-adic norm of x is not smaller than 1.

2. The notion of Galois group can be generalized when the roots are not anymore points but 4-D surfaces [7]. However, the notion of ramified prime becomes problematic.

   The notion of ramified primes makes sense if one allows 4 polynomials (P_1, P_2, P_3, P_4) instead of two. The roots of 3 polynomials (P_1, P_2, P_3) give rise to 2-surfaces as string world sheets and the simultaneous roots of (P_1, P_2, P_3, P_4) can be regarded as roots of the fourth polynomial and are identified as physical singularities identifiable as vertices [10].

   Also the maps defined by analytic functions g in the space of function pairs (f_1, f_2) generate new space-time surfaces. One can assign Galois group and ramified primes to h if it is a polynomial P in an extension of rationals. The composition of polynomials P_i defines inclusion hierarchies with increasing algebraic complexity and as a special case one obtains iterations, an approach to chaos, and 4-D analogs of Mandelbrot fractals.

3.3.3 Canonical identification and cognitive representations

Consider now the relationship between real and p-adic physics.

1. The connection between real and p-adic physics is defined by common points of reals and p-adic numbers defining a discretization at the space-time level and therefore a finite measurement resolution. This correspondence generalizes to the level of the space-time surfaces and defines a highly unique discretization depending only on the pinary cutoff for the algebraic integers involved. The discretization, I call it cognitive representation, is not completely unique since the choice of the generalized complex coordinates for H is not completely unique although the symmetries of H make it highly unique.

2. This picture leads to a vision in which reals and various p-adic number fields and their extensions induced by rationals form a gigantic book in which pages meet at the back of the book at the common points belonging to rationals and their extensions.

What it means to be a point "common" for reals and p-adics, is not quite clear. These common numbers belong to an algebraic extension of rationals inducing that of p-adic numbers. Since a discretization is in question, one can require that these common numbers have a *finite* pinary expansion in powers of p. For points with coordinates in an algebraic extension of rationals and having p-adic norm equal to 1, a direct identification is possible. In the general case, one can consider two options for the correspondence between p-adic discretization and its real counterpart.

1. The real number and the number in the extension have the same finite pinary expansions. This correspondence is however highly irregular and not continuous at the limit when an infinite number of powers of p are allowed.

2. The real number and its p-adic counterpart are related by canonical identification I. The coefficients of the units of the algebraic extension are finite real integers and mapped to p-adic numbers by $x_R = I(x_p) = \sum x_n p^{-n} \to x_p = \sum x_n p^n$. The inverse of I has the same form. This option is

favored by the continuity of I as a map from p-adics to reals at the limit of an infinite number of pinary digits.

Canonical identification has several variants. In particular, rationals m/n such that m and n have no common divisors and have finite pinary expansions can be mapped their p-adic counterparts and vice versa by using the map $m/n \to I(m)/I(n)$. This map generalizes to algebraic extensions of rationals.

The detailed properties of the canonical identification deserve a summary.

1. For finite integers I is a bijection. At the limit when an infinite number of pinary digits is allowed, I is a surjection from p-adics to reals but not a bijection. The reason is that the pinary expansion of a real number is not unique. In analogy with 1=.999...for decimal numbers, the pinary expansion $[(p-1)/p]\sum_{k\geq 0} p^{-k}$ is equal to the real unit 1. The inverse images of these numbers under canonical identification correspond to $x_p = 1$ and $y_p = (p-1)p\sum_{k\geq 0} p^k$. y_p has p-adic norm $1/p$ and an infinite pinary expansion.

   More generally, I maps real numbers $x = \sum_{n<N} x_n p^{-n} + x_N p^{-N}$ and $y = \sum_{n<N} x_n p^{-n} + (x_N - 1)p^{-N} + p^{-N-1}(p-1)\sum_{k\geq 0} p^{-k}$ to the same real number so that at the limit of infinite number of pinary digits, the inverse of I is two value for finite real integers if one allows the two representations. For rationals formed from finite integers there are 4 inverse images for $I(m/n) = I(m)/I(n)$.

2. One can consider 3 kinds of p-adic numbers. p-Adic integers correspond to finite ordinary integers with a finite pinary expansion. p-Adic rationals are ratios of finite integers and have a periodic pinary expansion. p-Adic transcendentals correspond to reals with non-periodic pinary expansion. For real transcendentals with infinite non-periodic pinary expansion the p-adic valued inverse image is unique since x_R does not have a largest pinary digit.

3. Negative reals are problematic from the point of view of canonical identification. The reason is that p-adic numbers are not well-ordered so that the notion of negative p-adic number is not well-defined unless one restricts the consideration to finite p-adic integers and the their negatives as $-n = (p-1)(1-p)n = (p-1)(1+p+p^2+...)n$. As far as discretizations are considered this restriction is very natural. The images of n and $-n$ under I would correspond to the same real integer but being represented differently. This does not make sense.

   Should one modify I so that the p-adic $-n$ is mapped to real $-n$? This would work also for the rationals. The p-adic counterpart of a real with infinite and non-periodic pinary expansion and its negative would correspond to the same p-adic number. An analog of compactification of the real number to a p-adic circle would take place.

3.3.4 Analogies between p-adicization and HFFs

Both hyperfinite factors and p-adicization allow a description of a finite measurement resolution. Therefore a natural question is whether the strange properties of hyperfinite factors, in particular the fact that the dimension D of Hilbert space equals to the cardinality of reals on one hand and to a finite number ($D = 1$ in the convention used) on the other hand, could have a counterpart in the p-adic sector. What is the cardinality of p-adic numbers defined in terms of canonical identification? Could it be finite?

1. Consider real finite real integers $x = \sum_{n=0}^{N-1} x_n p^n$ but with $x = 0$ excluded. Each pinary digit has p values and the total cardinality of these numbers of this kind is $p^N - 1$. These real integers correspond to two kinds of p-adic integers in canonical identification so that the total number is $2p^N - 2$. One must also include zero so that the total cardinality is $M = 2p^N - 1$. Identify M as a p-adic integer. Its padic norm equals 1.

2. As a p-adic number, M corresponds to $M_p = 2p^N + (p-1)(1+p+p^2+...) = p^N + p^{N+1} + (p-1)(1+p+...-p^N))$. One can write $M_p = p^N + p^{N+2} + (p-1)(1+p+...-p^N-p^{N+1})$. One can continue in this way and obtains at the limit $N \to \infty$ $p^{N\to\infty}(1+p+...) + (p-1)(1+p+...+p^{N-1})$. The first term has a vanishing p-adic norm. The canonical image of this number equals p at the limit $N \to \infty$. The cardinality of p-adic numbers in this sense would be that of the corresponding finite field! Does this have some deep meaning or is it only number theoretic mysticism?

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