**Article**

# Bianchi Type-IX Inflationary Cosmological Models with Flat Potential and Constant Deceleration Parameter in General Relativity

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**Abstract:** In this paper, we have studied about Bianchi type-IX inflationary cosmological models with flat potential and constant deceleration parameter in general relativity. To obtain deterministic solution of these models, we assume that expansion $\theta$ is proportional to shear $\sigma$, which leads to $A = B^n$ and potential $V(\phi)$ as constant. For two different constant value of deceleration parameter, we have obtained two different cosmological models. Behaviour of these models from physical and geometrical aspects is also discussed.

*Keywords: Bianchi type-IX, deceleration parameter, inflationary, general relativity.*

## 1 Introduction

Exponential expansion in the early universe is termed as inflation. This theory comes in the beginning of the 1980s and today receives a lot of attention. It is well-known that self-interacting scalar fields play a vital role in the study of inflationary cosmology. *Guth* [10] has discussed the inflationary universe as a possible natural explanation for the observed large scale homogeneity and near critical density (flatness) of the universal expansion. A universe with only moderate anisotropy will undergo inflation and will be rapidly isotropized.

*Bali* [4] discussed the significance of inflation for isotropization of universe. This inflationary scenario is also confirmed by Cosmic Microwave Background (CMB) observations (*Bassett* et al. [8]). In inflationary models, the universe undergoes a phase transition characterized by the evolution of Higgs field ($\phi$). The inflation will take place if the potential $V(\phi)$ has flat region $\phi$ the field evolves slowly but the universe expands in an exponential way due to the vacuum field energy as suggested by *Stein-Schabes* [23]. The flat part of the potential is naturally associated with a vacuum energy which can be identified as an effective cosmological constant ($\Lambda$) and it makes the universe to enter an inflationary period.

Bianchi type-IX cosmological models are interesting because these models allow not only expansion but also rotation and shear, and in general these are anisotropic. In recent years, many researchers have taken keen interest to study these models because well-known solution like Robertson Walker space-time, the de-Sitter space-time, the Taub-Nut space-time etc. are specific case of Bianchi type-IX universe. Bianchi type-IX inflationary cosmological models in different context have been studied by number of authors viz. *Adhav* et al. [3], *Burd and Barrow* [9], *Henriques* et al. [11], *Sharma and Poonia* [19], *Sharma* et al. [20].

*Jat* et al. [12] calculated the Bianchi type-IX inflationary cosmological model with flat potential for perfect fluid distribution in general relativity. *Singh* et al. [22] studied Bianchi type-IX inflationary cosmological models with flat potential for barotropic fluid distribution. *Bali and Kumari* [5] have studied Bianchi type-V inflationary universe with flat potential and stiff fluid distribution in general relativity. *Bali and Kumari* [6] have obtained Chaotic inflation in spatially homogeneous Bianchi type-V space time. *Bali and Singh* [7] have discussed Bianchi type-V inflationary universe with decaying vacuum energy $\Lambda$. *Reddy* [16] have studied Bianchi type-V inflationary universe in general relativity. *Reddy* et al. [17] have

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discussed Axially symmetric inflationary universe in general relativity. *Shri Ram and Singh* [21] have obtained Bianchi type-II, VIII and IX cosmological models with matter and electromagnetic fields.

Motivated by the above mentioned studies, we investigate Bianchi type-IX inflationary cosmological models with flat potential and constant deceleration parameter in general relativity using the condition expansion ($\theta$) is proportional to shear ($\sigma$), which leads to $A = B^n$ and potential $V(\phi)$ as constant. For two different constant value of deceleration parameter, we have obtained two different cosmological models. We find that spatial volume increases exponentially representing inflationary universe.

## 2 The Metric & Field Equation

We consider Bianchi type-IX line element in form-

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + \left(B^2\sin^2 y + A^2\cos^2 y\right)dz^2 - 2A^2\cos y dxdz \tag{2.1}$$

where $A, B$ are the functions of time $t$ only.
The co-moving coordinate system is chosen as-

$$v^1 = v^2 = v^3 = 0, v^4 = 1$$

The Lagrangian is that of gravity minimally coupled to Higgs scalar field ($\phi$) with effective potential $V(\phi)$ given by Stein-Schabes.

$$S = \int \sqrt{-g}\left[R - \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi - V(\phi)\right]d^4x \tag{2.2}$$

The Einstein's field equations (in the gravitational unit $8\pi G = c = 1$) in case of massless scalar field $\phi$ with potential $V(\phi)$ are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} = -T_{ij} \tag{2.3}$$

with energy momentum tensor ($T_{ij}$) for scalar field in presence of viscosity is given by-

$$T_{ij} = (\rho + p)v_iv_j + pg_{ij} + \partial_i\phi\partial_j\phi - \left[\frac{1}{2}\partial_p\phi\partial^p\phi + V(\phi)\right]g_{ij} \tag{2.4}$$

where $V$ is effective potential, $\phi$ is Higgs field.
The energy conservation law coincides with equation of motion for $\phi$ and we have

$$\frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}\partial^i\phi\right) = -\frac{dV}{d\phi} \tag{2.5}$$

where scalar field $\phi$ is function of $t$ only.
The Einstein field equation (2.3) for metric (2.1) and energy momentum tensor (2.4) leads to following system of equations

$$\frac{2B_{44}}{B} + \frac{B_4^2}{B^2} + \frac{1}{B^2} - \frac{3A^2}{4B^4} = -p - \frac{\phi_4^2}{2} + V(\phi) \tag{2.6}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} + \frac{A^2}{4B^4} = -p - \frac{\phi_4^2}{2} + V(\phi) \tag{2.7}$$

$$\frac{2A_4B_4}{AB} + \frac{B_4^2}{B^2} + \frac{1}{B^2} - \frac{A^2}{4B^4} = \rho + \frac{\phi_4^2}{2} + V(\phi) \tag{2.8}$$

The equation (2.5) for scalar field ($\phi$) leads to

$$\phi_{44} + \left(\frac{A_4}{A} + \frac{2B_4}{B}\right)\phi_4 = -\frac{dV}{d\phi} \tag{2.9}$$

where the overhead symbol dot($\cdot$) with $A$ and $B$ indicates derivative with respect to time $t$.

# 3 Solution of Field Equations

We are interested in inflationary solution so flat region is considered. Thus we have $V(\phi)$ is constant.

$$V(\phi) = K \tag{3.1}$$

From equations (2.9) and (3.1), we get

$$\phi_{44} + \left(\frac{A_4}{A} + \frac{2B_4}{B}\right)\phi_4 = 0 \tag{3.2}$$

Equation (3.2) leads to

$$\phi_4 = \frac{l}{AB^2} \tag{3.3}$$

where $l$ is integration constant. The average scale factor $(a)$ for line element (2.1) is given by

$$a^3 = A^2 B \tag{3.4}$$

To find the deterministic solution, we assume two conditions as follows:

(i). The shear $(\sigma)$ is proportional to expansion $(\theta)$as considered by Throne (1967) which leads to the condition between metric potential

$$A = B^n \tag{3.5}$$

(ii). The deceleration parameter $q$ is constant $(\alpha)$, thus we have

$$q = -\frac{a_{44}a}{a_4^2} = \alpha \tag{3.6}$$

thus two cases arises for the sign of $\alpha$.
**Case(1).** $\alpha > 0$
Equation (3.6) leads to

$$\frac{a_{44}}{a_4} = -\alpha\frac{a_4}{a} \tag{3.7}$$

Now, from equations (3.4), (3.5) and (3.7) we get

$$\frac{B_{44}}{B} = -b\frac{B_4^2}{B^2} \tag{3.8}$$

where $b = \frac{2n-2+\alpha(2n+1)}{3}$

$$\frac{B_{44}}{B_4} + b\frac{B_4}{B} = 0 \tag{3.9}$$

Integrating the above equation (3.9), we get

$$B_4 B^b = c_1 \tag{3.10}$$

where $c_1$ is integration constant.
Again integrate the above equation (3.10), we get

$$\frac{B^{b+1}}{b+1} = c_1 t + c_2 \tag{3.11}$$

where $c_2$ is integrating constant.
After solving above equation (3.11), we find

$$B = (\gamma t + \delta)^{\frac{1}{b+1}} \tag{3.12}$$

$$A = B^n = (\gamma t + \delta)^{\frac{n}{b+1}} \tag{3.13}$$

where $\gamma = c_1(b+1)$ and $\delta = c_2(b+1)$
After suitable transformation of co-ordinates, the metric (2.1) leads to the form

$$ds^2 = -\frac{dT^2}{\gamma^2} + T^{\frac{2n}{b+1}} dX^2 + T^{\frac{2}{b+1}} dY^2 + \left[T^{\frac{2}{b+1}} \sin^2 Y + T^{\frac{2n}{b+1}} \cos^2 Y\right] dZ^2 - 2T^{\frac{2n}{b+1}} \cos Y dX dZ \tag{3.14}$$

where, $X = x, Y = y, Z = z$ and $T = \gamma t + \delta$

**Case(2).** Let $\alpha < 0$
$\alpha = -\alpha_1$, $\alpha_1 > 0$
Now by equation (3.6), we have

$$\frac{a_{44}}{a_4} = \alpha_1 \frac{a_4}{a} \tag{3.15}$$

$$\frac{B_{44}}{B_4} = b_2 \frac{B_4}{B} \tag{3.16}$$

where, $b_2 = \frac{\alpha_1(2n+1)-(2n-2)}{3}$
Integrating above equation (3.16), we get

$$B_4 = C_3 B^{b_2} \tag{3.17}$$

Again integrating the above equation (3.17), we get

$$B^{(1-b_2)} = C_3(1-b_2)t + C_4(1-b_2)$$

where $C_3$ and $C_4$ are integrating constants.

$$B = (\lambda t + \beta)^{\frac{1}{1-b_2}} \tag{3.18}$$

$$A = (\lambda t + \beta)^{\frac{n}{1-b_2}} \tag{3.19}$$

where $\lambda = C_3(1-b_2)$ and $\beta = C_4(1-b_2)$
After suitable transformation of co-ordinates, the metric (2.1) leads to the form ,

$$ds^2 = -d\tau^2 + \tau^{\frac{2n}{1-b_2}} dX^2 + \tau^{\frac{2}{1-b_2}} dY^2 + \left[\tau^{\frac{2}{1-b_2}} \sin^2 Y + \tau^{\frac{2n}{1-b_2}} \cos^2 Y\right] dZ^2 - 2\tau^{\frac{2n}{1-b_2}} \cos Y dX dZ \tag{3.20}$$

where $\tau = \lambda t + \beta$, $X = x, Y = y$ and $Z = z$

# 4 Physical & Geometrical Aspects

Pressure ($p$), energy density ($\rho$), spatial volume ($V$), expansion ($\theta$), directional Hubble parameter ($H_x, H_y, H_z$), Hubble parameter ($H$), anisotropic parameter ($\Delta$), shear ($\sigma$) and Higgs field($\phi$) for the model (3.14) are given by

$$p = \frac{\gamma^2(2b-1)}{(1+b)^2T^2} - \frac{1}{T^{\frac{2}{1+b}}} + \frac{3}{4T^{\frac{4-2n}{1+b}}} - \frac{l^2}{2T^{\frac{2n+4}{1+b}}} + K \tag{4.1}$$

$$\rho = \frac{\gamma^2(1+2n)}{(1+b)^2T^2} + \frac{1}{T^{\frac{2}{1+b}}} - \frac{1}{4T^{\frac{4-2n}{1+b}}} - \frac{l^2}{2T^{\frac{2n+4}{1+b}}} - K \tag{4.2}$$

$$V = a^3 = (\gamma t + \delta)^{\frac{2n+1}{1+b}} = T^{\frac{2n+1}{1+b}} \tag{4.3}$$

$$\theta = \frac{\gamma(n+2)}{(1+b)T} \tag{4.4}$$

$$H_x = \frac{n\gamma}{(1+b)T} \tag{4.5}$$

$$H_y = H_z = \frac{\gamma}{(1+b)T} \tag{4.6}$$

The average Hubble parameter ($H$) is found to be

$$H = \frac{(n+2)\gamma}{3(b+1)T} \tag{4.7}$$

$$\Delta = \frac{2(n-1)^2}{(n+2)^2} \tag{4.8}$$

$$\sigma^2 = \frac{(n-1)^2\gamma^2}{3(b+1)^2T^2} \tag{4.9}$$

$$\frac{\sigma}{\theta} = \frac{n-1}{\sqrt{3}(n+2)}, n \neq -2 \tag{4.10}$$

$$\phi = \frac{l(1+b)T^{\frac{b-n-1}{1+b}}}{(b-n-1)\gamma} + m_1 \tag{4.11}$$

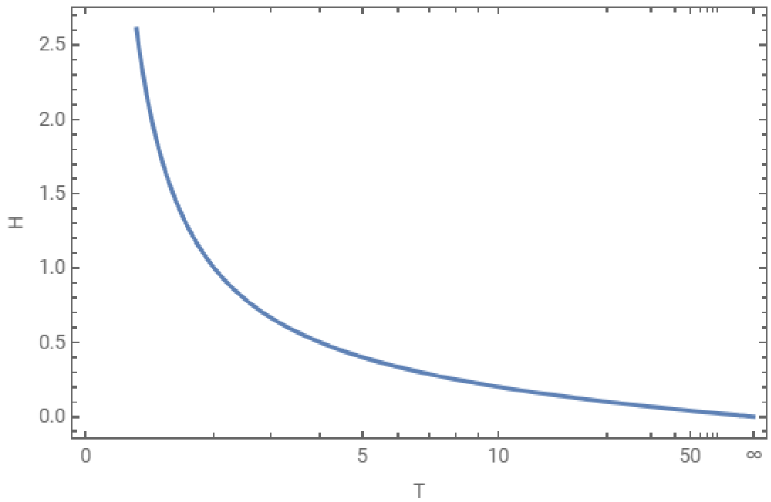


Figure 1: Hubble Parameter $H[T]$ verses $T$

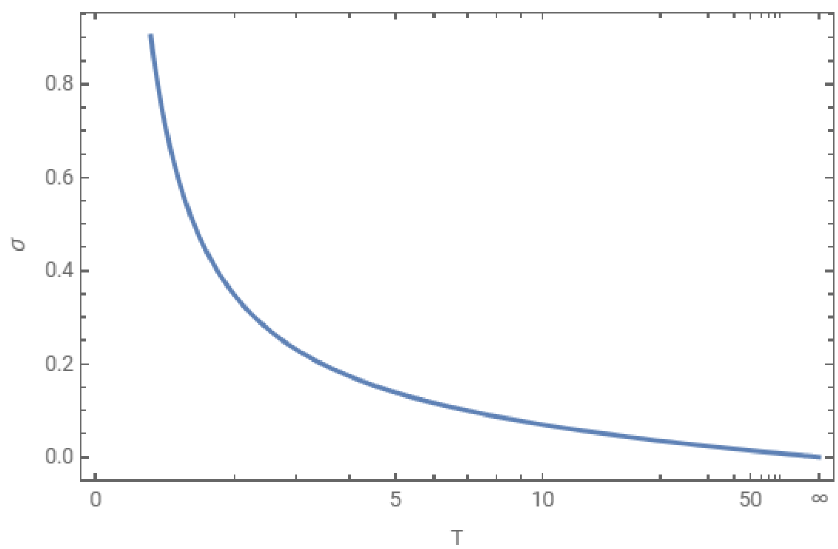


Figure 2: Shear $\sigma[T]$ verses $T$

Figure 1 and Figure 2 represents Hubble Parameter and shear behaviour for model (3.14) at $n = \frac{1}{2}$.

Pressure ($p$), energy density ($\rho$), spatial volume ($V$), expansion ($\theta$), directional Hubble parameter ($H_x, H_y, H_z$), Hubble parameter ($H$), anisotropic parameter ($\Delta$), shear ($\sigma$) and Higgs field($\phi$) for model (3.20) are given by

$$p = \frac{\lambda^2(1+2b_2)}{(1-b_2)^2\tau^2} - \frac{1}{\tau^{\frac{2}{1-b_2}}} + \frac{3}{4\tau^{\frac{4-2n}{1-b_2}}} - \frac{l^2}{2\tau^{\frac{2n+4}{1-b_2}}} + K \tag{4.12}$$

$$\rho = \frac{\lambda^2(1+2n)}{(1-b_2)^2\tau^2} + \frac{1}{\tau^{\frac{2}{1-b_2}}} - \frac{1}{4\tau^{\frac{4-2n}{1-b_2}}} - \frac{l^2}{2\tau^{\frac{2n+4}{1-b_2}}} - K \tag{4.13}$$

$$V = a^3 = (\lambda t + \beta)^{\frac{2n+1}{1-b_2}} = \tau^{\frac{2n+1}{1-b_2}} \tag{4.14}$$

$$\theta = \frac{(n+2)\lambda}{(1-b_2)\tau} \tag{4.15}$$

$$H_x = \frac{n\lambda}{(1-b_2)\tau} \tag{4.16}$$

$$H_y = H_z = \frac{\lambda}{(1-b_2)\tau} \tag{4.17}$$

$$H = \frac{(n+2)\lambda}{3(1-b_2)\tau} \tag{4.18}$$

$$\Delta = \frac{2(n-1)^2}{(n+2)^2} \tag{4.19}$$

$$\sigma^2 = \frac{(n-1)^2\lambda^2}{3(1-b_2)^2\tau^2} \tag{4.20}$$

$$\frac{\sigma}{\theta} = \frac{n-1}{\sqrt{3}(n+2)}, n \neq -2 \tag{4.21}$$

$$\phi = \frac{l(b_2-1)\tau^{\frac{n+b_2+1}{b_2-1}}}{\lambda(n+b_2+1)} + m_2 \tag{4.22}$$

where $m_2$ is integration constant.

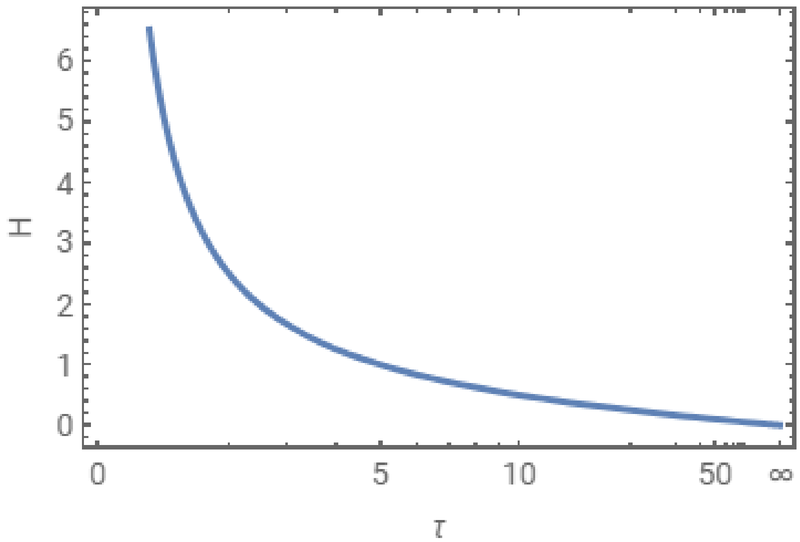


Figure 3: Hubble Parameter $H[\tau]$ verses $\tau$

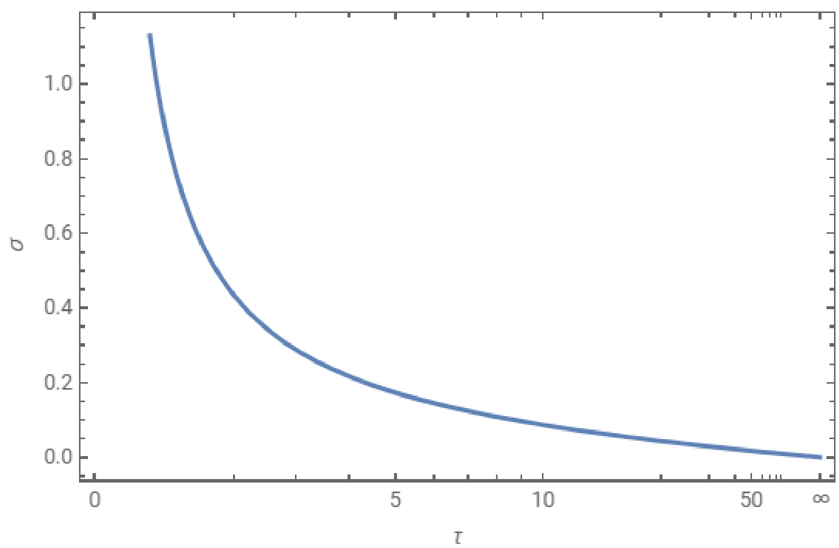


Figure 4: Shear $\sigma[\tau]$ verses $\tau$

Figure 3 and Figure 4 represents Hubble Parameter and shear behaviour for model (3.20) at $n = \frac{1}{2}$.

# 5 Conclusion

The spatial volume ($V$) for the models (3.14) and (3.20) increases with time representing inflationary scenario is observed in Bianchi type-IX space time with flat potential. Pressure ($p$) and density ($\rho$) are decreasing function of time $T$ and $\tau$ for both models respectively (3.14) and (3.20). The Higgs field ($\phi$) for both models evolves slowly but the universe expands. The Hubble parameter ($H$) is initially large but decreases with time. The rate of expansion slows down with the increase of time and finally drops to zero when $T \to \infty$ for model (3.14) and when $\tau \to \infty$ for model (3.20). The ratio of shear and expansion is non-zero (for $n \neq -2$) for both models which represents that both models remain anisotropic throughout the evolution. Both the models have point type singularity at $T = 0$ for model (3.14) when $b < -1$ and $\tau = 0$ for model (3.20) when $b_2 < 1$.

Received November 30, 2024; Accepted March 2, 2025

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