

Article

In Defense of Octonions

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Abstract

Various authors have observed that the unit of the imaginary numbers, i , has a special significance as a quantity whose existence predates our discovery of it. It gives us the ability to treat degrees of freedom in the same way mathematically that we treat degrees of fixity. Thus; we can go beyond the Real number system to create or describe Complex numbers, which have a real part and an imaginary part. This allows us to simultaneously represent quantities like tension and stiffness with real numbers and aspects of vibration or variation with imaginary numbers, and thus to model something like a vibrating guitar string or other oscillatory systems. But if we take away the constraint of commutativity, this allows us to add more degrees of freedom, and to construct Quaternions, and if we remove the constraint of associativity, what results are called Octonions. We might have called them super-Complex and hyper-Complex numbers. But we can go no further, to envision a yet more complicated numbering system without losing essential algebraic properties. A recent Scientific American article by John Baez and John Huerta suggests that Octonions provide a basis for the extra dimensions required by String Theory and are generally useful for Physics, but others disagree. We examine this matter.

Key Words: octonions, quaternions, string theory, quantum gravity.

Introduction

The Octonions are numbers that possess seven degrees of freedom and one of fixity, representing the fourth normed division algebra, and thus having three less-complex cousins, the Quaternions, Complex numbers, and the Reals. A recent article and papers by Baez and Huerta [1] suggest there may be a basis for the extra dimensions needed by String theory in the additional degrees of freedom afforded us by Octonions. And there may be some truth to the idea that a quantity must be free to vary, before it can assume a particular value that is real. This lends credence to the notions expressed in the article, but not all agree that the connection suggested by Baez and Huerta is reasonable or valid, in explaining origins of the Math and Physics relating to String theory. Luboš Motl has been especially vocal in his protest of the idea that octonions are central to String theory, claiming they are peripheral at best and unrelated to its important questions [2]. But people outside that field feel that,

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in the absence of more concrete results, at least the String theorists should be able to clearly explain how those extra dimensions originate.

This question tends to be avoided by folks in String theory, but it has been the driving force to develop Background Independent theories such as Spin foams, Loop Quantum Gravity, and so on. Similarly; constructivist Math and Geometry insist that it is inadmissible to simply posit a particular set of dimensions or a background space with specified properties. To show it is real, we must be able to construct it. And of course; administrators, grant writers, or people with funds to donate, often have legitimate questions about why there are extra dimensions, or how they came to be. So; it appears admirable that Baez and Huerta, who are outside that field, have given considerable thought to how the conditions which allow String theory to work might arise. And we likewise assert that the Octonions might indeed help dictate the mechanism by which dimensions come to be – at the extreme microscale – and thus create the specific dimensionality for String theory to operate. However, the utility of higher-order number types is not limited to Strings, as they are useful in several areas – both in Physics and elsewhere. The remainder of this paper discusses why the Octonions and Quaternions are of special importance to Physics, and should perhaps be regarded as part of the fundamental mathematical background in which reality and the principles of Physics have their origin.

Fundamental Realities of Octonion Math

The existence of i , the root or unit of the imaginary numbers, is almost senseless to dispute. Though we call it imaginary, it is not simply a fictitious entity, as its utility shows it is a marvelous discovery. Nor is its value arbitrary. As with π , the ratio between the diameter and circumference of a circle, the exact numerical value of i is fixed by its definition. Specifically, i^2 must equal -1 . But this condition implies a very interesting behavior, where i is a quantity that appears to oscillate between two values, rather than having a specific or fixed value. Thus, its appearance in an equation of motion indicates the freedom to vary by a specified amount, rather than indicating a fixed quantity. When imaginary elements appear in Complex numbers, Quaternions, or Octonions, this quantifies both the extent and the ways in which a numerical system is free to vary. And this is precisely the property which makes these mathematical constructions so useful in Physics, as a way to model how *physical* systems naturally vary. The higher-order number types simply extend the range or add to our

repertoire of useful tools. Therefore the applicability of Math using quaternions and octonions in Physics hinges on whether that usage is more natural or appropriate, to simplify our reckoning.

Complex numbers are commonly viewed on the Argand plane, with the Real axis left to right, and the Imaginary axis up and down. In this framework, multiplication of any complex number by i is equivalent to rotating its location by 90 degrees. We can carry this idea of imaginary excursions depicting rotation forward to describe the Quaternions as points on a sphere whose radius is the scalar value. That is; if we wish, we can view any quaternion as a 3-d polar coordinate where the Real value defines a sphere of a certain size about the origin and the three imaginary components are rotations about orthogonal axes on that sphere. If we want to extend this metaphor to the Octonions, one might think that this would be hard to visualize because it requires a hypersphere, but it turns out that there is a very nice correspondence if we use a torus instead of a sphere. In that generalization, the scalar is again the size of the figure and three of the imaginary components are again rotational excursions on the surface of the figure, but the remaining components become the proper rotations of the torus itself. Lest you think we have one too many rotational components that way, I remind the reader that a torus can rotate by turning inside out such that its inner rim becomes its outer, and vice versa. So we actually have exactly the correct number of rotational components or imaginaries.

The torus is connected with the Octonions in a variety of ways, including the 7-color map problem, which we discuss later. We note that a string can wrap around a torus and be constrained to create various winding modes. The torus also has an interesting property where it stands alone, as it is a form which can exist as a self-contained propagation within a medium of a single density or viscosity. No boundary or discontinuity is required for it to arise. But it clearly has a center and defines a sense of interiority and exteriority by its existence. It can serve as a boundary and occupy or define a location in dimensional space. Topological distinctions may thus arise in the form of a torus. By defining a location, the concepts of a neighborhood and proximal or distal directions in space (toward or away) are also given meaning. These concepts are seen as foundational in constructive geometry, and may also be a foundational underpinning of nature itself. Concepts like size and distance, or duration and intervals of time, likely were loosely-defined or undefined at the inception of the cosmos – and emerged later – requiring that we assume the most general case at the outset, which is non-associative geometry. If we further assume that the specific geometry adopted by nature was Octonion near the Planck scale, a lot of things work out very nicely.

As stated earlier, the Octonions are the most complex of four normed division algebras (or NDAs). The essential property of NDAs is reversibility. Any series of algebraic operations can be performed in the reverse order and sense (dividing where one had multiplied or subtracting where one had added), when using any NDA, to arrive at the original value. We can add, subtract, multiply, and divide – in any sequence – yet later reverse the steps to return to where we were. This implies or engenders the property which is known as Symmetry. Modern Physics is deeply connected with Symmetry Groups, which are the product of such symmetries within the associated algebras. As the highest order NDA, the Octonion algebra directly spawns the largest and most complex of the Exceptional Groups, E_8 , which is equivalent to the Gosset lattice (see Figure 1 below) that Garrett Lisi [3] and others have explored for its connection to Physics. While Lisi’s ‘exceptionally simple’ first try was apparently not exactly on target, subsequent attempts by Lisi and others to refine or extend the ideas he explored [4] are still regarded as promising. Further; it is widely regarded as true that any effective GUT or TOE will be closely connected with some combination of the known symmetry groups, and we note that any of these can be derived beginning with Octonion algebra as a starting place.

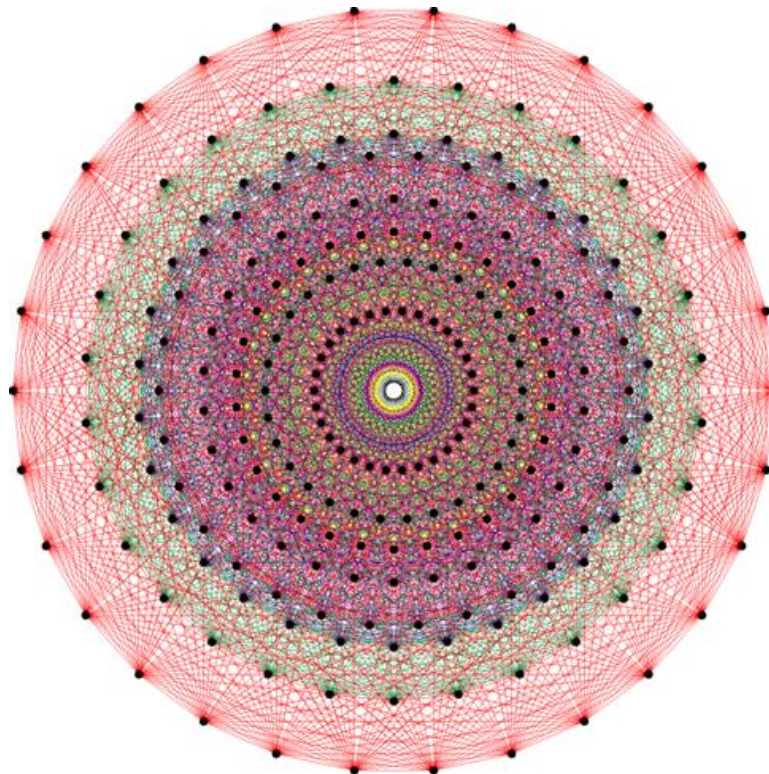


Figure 1 – A 2-D Projection of the Minimally 8-D Gosset Lattice

The property of non-associativity makes Octonion algebra the most demanding of all the NDAs, because the steps in any algebraic process must be applied in exact sequence, in order to obtain the correct result. This is not to say that there is no room for variation with octonions. In fact; there are 480 different multiplication tables which all work, provided that you adopt just one version and use it throughout your calculations [5]. But this variety arises directly from the great range of possibilities octonions can encode or represent, and coincidentally, corresponds to the 480 roots of an $E_8 \times E_8^*$ algebra. It could be said that the Octonions are simply the most general of all the number types. However, where all those extra degrees of freedom or rotations are not needed; it is far easier to leave the additional terms aside, and to deal with numbers as real-valued scalars, real and complex vectors, or quaternions. On the other hand; as our focus moves toward the Planck scale it may require that we leave some of these simplifications behind. One approach that until recently was quite promising is Noncommutative geometry (or NCG). It may be true that the shortcomings of this approach arise from the fact that it does not go far enough, and that to explore closer to the Planck domain we must assume that the underlying geometry is non-associative as well.

This idea has some interesting implications. One intriguing statement about NCG by Alain Connes is that “Noncommutative measure spaces evolve with time!” [6]. He follows this remark by saying that they have a ‘god-given’ automorphism group driving their time evolution. Connes implies that spaces with a noncommutative geometry evolve into something else, presumably sometimes or eventually into spaces possessing a geometry where movement *does* commute, and the familiar concepts of distance and scale *do* apply. As we move from noncommutative to non-associative spaces and geometries though, his statement becomes even more profoundly true, making Octonion geometry quite fecund. The property of non-associativity forces one to adopt an almost process-theoretic view of reality as the step-wise sequential nature of calculations is much like the procedural steps in a computer program. A comment made by Jason Wolfe on Munroe’s FQXi essay forum page [7] suggests that terms in parentheses in some Mathematical expressions are like the nested loops in a program, and with octonions this is exactly correct. One must resolve or calculate all of the inner loops first, and then combine those intermediate results to obtain the final answer. One could say that a quaternion is a qubit times a scalar. This makes an octonion a string of seven (entangled) qubits [8] times a scalar. Perhaps most interesting, in this connection, is that Octonion algebra thus results in the maximal connectivity or inter-connectedness which can be possessed by a computing network, as realized in a figure known as the heptaverton.

Though one can go on with the process of generalization (via the Cayley-Dickson construction) to even higher dimensioned algebras such as the Sedenions (16-D) and Pathions (32-D) [9] the maximum of flexibility and utility is reached with the Octonions, so these yet higher-order number types are of interest mainly to number theorists, and are not of as much use for Physics. Thus; it is seen that octonions represent the richest starting place and offer the most opportunities or possibilities for the evolution of form of any numbering system. Therefore it seems natural that reality would utilize this broad palette of possibilities to help create the universe of form we observe. Though to say that the Octonions represent something that was a ‘given’ at the outset of the universe may seem like an exercise in radical Platonism, it is not without a basis, and assuming this is true leads to some interesting consequences which can produce a cosmos much like the one we observe. Various individuals including Wigner and Tegmark [10, 11] have asserted that regular patterns and unchanging realities within Mathematics may be what generate the regular and unchanging aspects of physical law. Wolfram’s “A New Kind of Science” [12] suggests that instead of the Math, it is the ability to compute or calculate that defines reality. If we combine these two notions, we arrive at the idea that the Mathematics or Computing strategy which allows for the greatest efficiency of determination may be the route nature chooses whenever possible.

If this ‘principle of least computational action’ holds true, then the usefulness of quaternions and octonions may be far more than just an aid to physicists. Their efficiency may be one of the major determining factors which made reality turn out the way that it did, helping our world and the cosmos to become the way that we observe it today. In that scenario; the Octonions take on a role as an important creative force and a shaper of the universe.

Example 1 – Maxwell’s Equations as Quaternion Formalism

Quaternion Grassman multiplication and differentiation is defined as [13, 14]:

$$\begin{aligned}
 qq' &\equiv (\tau, \mathbf{R})(\tau', \mathbf{R}') \equiv (\tau\tau' - \mathbf{R} \cdot \mathbf{R}', \tau \mathbf{R}' + \mathbf{R} \tau' + \mathbf{R} \times \mathbf{R}') \\
 (\partial_t, \nabla)(\tau, \mathbf{R}) &\equiv (\partial_t \tau - \nabla \cdot \mathbf{R}, \partial_t \mathbf{R} + \nabla \tau + \nabla \times \mathbf{R})
 \end{aligned}
 \tag{1}$$

Note that a comma separates the scalar “time-like” and vector “space-like” components, and the derivative operator includes time derivatives, a gradient, a divergence and a curl. Quaternion operations are non-commutative because of that last cross product: $\mathbf{R} \times \mathbf{R}' = -\mathbf{R}' \times \mathbf{R}$. Although this property may seem to be disadvantageous in describing Classical Physics, it may be an advantage in

describing Quantum Physics that utilizes commutator $[a, b] = ab - ba$ and anti-commutator $\{a, b\} = ab + ba$ operators. Defining commutator and determinant forms of the quaternion derivative:

$$\begin{aligned} \frac{1}{2} [(\partial_t, \nabla), (\tau, \mathbf{R})] &\equiv \frac{1}{2} \begin{vmatrix} (\partial_t, \nabla) & (\tau, \mathbf{R}) \\ (\partial_t, \nabla) & (\tau, \mathbf{R}) \end{vmatrix} \equiv (0, \nabla \times \mathbf{R}) \\ \frac{1}{2} \{(\partial_t, \nabla), (\tau, \mathbf{R})\} &\equiv \frac{1}{2} \begin{vmatrix} (\partial_t, \nabla) & -(\tau, \mathbf{R}) \\ (\partial_t, \nabla) & (\tau, \mathbf{R}) \end{vmatrix} \equiv (\partial_t \tau - \nabla \cdot \mathbf{R}, \partial_t \mathbf{R} + \nabla \tau) \end{aligned} \quad (2)$$

And we may write Maxwell's Equations in terms of these quaternion and commutator operators:

$$\frac{1}{2} [(\partial_t, \nabla), (0, \mathbf{E})] + \frac{1}{2} \{(\partial_t, \nabla), (0, \mathbf{B})\} = \frac{1}{2} \begin{vmatrix} (\partial_t, \nabla) & (0, \mathbf{E} - \mathbf{B}) \\ (\partial_t, \nabla) & (0, \mathbf{E} + \mathbf{B}) \end{vmatrix} = (-\nabla \cdot \mathbf{B}, \partial_t \mathbf{B} + \nabla \times \mathbf{E}) = (0, \vec{0}) \quad (3)$$

$$\frac{1}{2} [(\partial_t, \nabla), (0, \mathbf{B})] - \frac{1}{2} \{(\partial_t, \nabla), (0, \mathbf{E})\} = \frac{1}{2} \begin{vmatrix} (\partial_t, \nabla) & (0, \mathbf{B} + \mathbf{E}) \\ (\partial_t, \nabla) & (0, \mathbf{B} - \mathbf{E}) \end{vmatrix} = (\nabla \cdot \mathbf{E}, -\partial_t \mathbf{E} + \nabla \times \mathbf{B}) = 4\pi(\rho, \mathbf{J}) \quad (4)$$

in Gaussian units. Equation (3) yields the Homogeneous components of Maxwell's Equations by decomposing the scalar components into Gauss's Law for Magnetism: $\nabla \cdot \mathbf{B} = 0$, and the vector components into Faraday's Law: $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$. Similarly, equation (4) yields the Inhomogeneous components of Maxwell's Equations by decomposing the scalar components into Gauss's Law for Electricity: $\nabla \cdot \mathbf{E} = 4\pi \rho$, and the vector components into Ampere's Law with Maxwell's Displacement Current: $-\partial_t \mathbf{E} + \nabla \times \mathbf{B} = 4\pi \mathbf{J}$. Now we may also use a combination of quaternion and commutator operators to define our scalar and vector potentials:

$$\begin{aligned} (0, \mathbf{E}) &= -\frac{1}{2} \{(\partial_t, \nabla), (\varphi, \mathbf{A})\} = \begin{vmatrix} (\partial_t, \nabla) & -(\varphi, \mathbf{A}) \\ (\partial_t, \nabla) & (\varphi, \mathbf{A}) \end{vmatrix} = (-\partial_t \varphi + \nabla \cdot \mathbf{A}, -\partial_t \mathbf{A} - \nabla \varphi) \\ (0, \mathbf{B}) &= \frac{1}{2} [(\partial_t, \nabla), (\varphi, \mathbf{A})] = \begin{vmatrix} (\partial_t, \nabla) & (\varphi, \mathbf{A}) \\ (\partial_t, \nabla) & (\varphi, \mathbf{A}) \end{vmatrix} = (0, \nabla \times \mathbf{A}) \end{aligned} \quad (5)$$

where the scalar component of \mathbf{E} equals zero because of the Lorentz Gauge: $\partial_t \varphi - \nabla \cdot \mathbf{A} = 0$. Combining Equations (4) and (5) yields the Riemann-Sommerfeld Equation [15] for the Inhomogeneous components of Maxwell's Equations:

$$\square A^\mu = 4\pi J^\mu \text{ in Gaussian units} \quad (6)$$

where $A^\mu = (\varphi, \mathbf{A})$, $J^\mu = (\rho, \mathbf{J})$, and the d'Alembertian: $\square = \partial_\alpha \partial^\alpha = \partial_t^2 - \nabla^2$ is the scalar component of a quaternion "Laplacian": $(\partial_t, \nabla)(\partial_t, \nabla) = (\partial_t^2 - \nabla^2, \partial_t \nabla + \nabla \partial_t + \nabla \times \nabla)$.

Example 2 – Dirac's Gamma Matrices as Quaternion Formalism

Another example of quaternion algebra is Dirac's Gamma Matrices. These matrices naturally arise from the solution of Dirac's Equation [16]:

$$(-i\gamma^\mu \partial_\mu + m) \psi = 0, \quad \text{where} \quad (7)$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix} \& \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} \quad (8)$$

are 4×4 matrices constructed of the 2×2 Pauli $(\sigma_x, \sigma_y, \sigma_z)$, Identity I and Null 0 matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \& 0_{2 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (9)$$

These Dirac matrices obey a quaternion $Cl_{1,3}(\mathbf{H}(2))$ algebra with the anti-commutator relation:

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Separately, the Pauli matrices obey a complex $Cl_{1,2}(\mathbf{C}(2))$ algebra. The fifth Gamma matrix:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (10)$$

behaves like a $Cl_{0,4}(\mathbf{H}(2))$ pseudoscalar – a scalar quaternion component that changes signs under inversion.

Example 3 – The Einstein Field Equations as Quaternion or Octonion Formalism

The Einstein Field Equations (EFE) may be written as [17, 18]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (11)$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature, Λ is the cosmological constant, and $T_{\mu\nu}$ is the stress-energy tensor. The EFE is a set of ten independent symmetric tensor equations. This particular formulation looks like a $Cl_{2,3}(\mathbf{C}(4)) \sim SO(3,2)$ algebra with (3,7) symmetries. This set of four complex algebras may be intertwined via a Cayley-Dickson construction into the equivalent of an Octonion algebra [19, 20].

However, the freedom of choice of the four spacetime coordinates allow us to reduce the EFE down to a set of six independent tensor equations that resembles a $Cl_{2,2}(\mathbf{R}(4)) \sim SO(2,2)$ algebra with (1,5) symmetries. This set of four real algebras may be intertwined into the equivalent of a Quaternion algebra.

Do the Einstein Field Equations represent the equivalent of a Quaternion or an Octonion algebra? That appears to be a matter of perspective.

Example 4 – A GUT of the Gravitational Sector Involving Quaternions and Octonions

Example 3 implies that a $Cl_{2,2}(\mathbf{R}(4)) \sim SO(2,2)$ algebra with (1,5) symmetries is the minimal representation for Einstein's Gravity. The "1" symmetry implies a spin-2 Graviton particle, but the "5" symmetry implies something else of spin-2. If the Gravitational particle spectrum is more complex than a simple Graviton, then we need to explain these extra degrees-of-freedom, and we may need to explain why the Graviton is massless.

In Electro-Weak Theory, a $U(1)_B$ mixes quantum numbers with an $SU(2)_W$ via a rotation:

$$\begin{pmatrix} \gamma \\ Z^0 \end{pmatrix} = \begin{pmatrix} \cos \theta_W & \sin \theta_W \\ -\sin \theta_W & \cos \theta_W \end{pmatrix} \begin{pmatrix} B^0 \\ W^0 \end{pmatrix} \quad (12)$$

where θ_W is the Weinberg (or weak) mixing angle [21] to yield a massless photon γ , a massive Z^0 boson, massive W^\pm bosons, and a unified Electro-Weak sector.

Suppose that a $Cl_{2,2}(\mathbf{R}(4)) \sim SO(2,2)$ six-plet of tensors (looks like the simplified EFE) mixes quantum numbers with a $Cl_{2,3}(\mathbf{C}(4)) \sim SO(3,2)$ ten-plet of tensors (looks like the general EFE) to yield a massless $U(1)_G$ of Graviton and a $Cl_{2,4}(\mathbf{H}(4)) \sim SO(4,2) \sim SU(2,2)$ fifteen-plet of massive WIMP-Gravitons [22] that resembles a Twistor Algebra [23]. These WIMP-Gravitons are “Weakly Interacting Massive Particle” Gravitons with five spin projections: $(\pm 2, \pm 1, 0)\hbar$, where the ± 2 spin projections are purely transverse, the 0 spin projection is purely longitudinal, and the ± 1 spin projections are a mixture of transverse and longitudinal components. Thus, these WIMP-Gravitons morph into an $SO(3,3) \sim SU(3,1)$ symmetry that implies three different WIMP-Gravitons, each with five different spin projections.

$U(1)_G \times SO(4,2)_{WG}$ is a 4-dimensional rotation of tensors – somewhat like Electro-Weak Theory, but more complex. Perhaps this may be represented as left- and right-isoclinic component equations:

$$\begin{pmatrix} g_L \\ F_{1L} \\ F_{2L} \\ F_{3L} \end{pmatrix} = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} G_{1HL} \\ G_{5HL} \\ G_{3OL} \\ G_{7OL} \end{pmatrix} \quad \& \quad \begin{pmatrix} g_R \\ F_{1R} \\ F_{2R} \\ F_{3R} \end{pmatrix} = \begin{pmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{pmatrix} \begin{pmatrix} G_{1HR} \\ G_{5HR} \\ G_{3OR} \\ G_{7OR} \end{pmatrix} \quad (13)$$

where a left isoclinic rotation is represented by left-multiplication on a unit quaternion $\hat{q}_L = a + b\hat{i} + c\hat{j} + d\hat{k}$, a right isoclinic rotation is represented by right-multiplication on a unit quaternion $\hat{q}_R = p + q\hat{i} + r\hat{j} + s\hat{k}$, and our initial massive Quaternion Graviton G_H and massive Octonion Graviton G_O tensors are represented as sets of vector components with L and R denoting left- and right-helicity – as are our final massless graviton g and massive WIMP-Graviton F tensors.

Note that all capitalized symbols are tensors with effective mass, and longitudinal components are not considered in this “simplified” formalism that permits up to the six unique mixing angles of $SO(4,2)$ capable of describing mixing of quantum numbers between the Graviton and WIMP-Gravitons. The $(1H,5H)$ and $(3O,7O)$ represent component Quaternion-like $Cl_{2,2}(\mathbf{R}(4)) \sim SO(2,2)$ and Octonion-like $Cl_{2,3}(\mathbf{C}(4)) \sim SO(3,2)$ symmetries, respectively.

This $Cl_{2,4}(\mathbf{H}(4))$ algebra is a set of four quaternion algebras that may be intertwined into the equivalent of two (7+1)-D Octonions via a Cayley-Dickson construction, and may thus be related to $E_8 \times E_8^* \sim Spin(32)/Z_2$ Heterotic String Theory [24] with dual 8-dimensional Gosset lattices. Thus, $U(1)_G \times SO(4,2)_{WG}$ may yield a potential GUT of the gravitational sector that also connects with Octonions, Gosset lattices, and Twistor and Heterotic String Theory.

Example 5 – An Octonion Inspired Seven-Color Mapping of a Torus

A recent FQXi Essay contest raised the question of whether Reality is fundamentally Discrete or Continuous. Both authors [25] presented arguments and evidence showing why we think Reality is fundamentally *both* Discrete *and* Continuous. If Octonion geometry is in effect near the Planck scale, this suggests that reality is actually seven ways analog before it manifests as digital, or discrete. If Spacetime has a discrete nature at certain Scales, though, this may explain the AdS – CFT Correspondence by using a Graphene analogy for Spacetime structure [26]. If a graphene analogy might represent Spacetime’s large-scaled structure, then another Carbon allotrope, the C-60 Buckyball [27] (or truncated icosahedron) might explain Spacetime’s small-scaled structure and the apparent stability of the Black Hole singularity.

It is well-known that – because of the approximate equality of the Compton wavelength and the Schwarzschild radius at the Planck scale – a Planck particle (if it emerged from the ‘Quantum soup,’) would almost immediately be swallowed up by a Black Hole [28], however some geometries might survive a collapse. The buckyball is a relatively-rigid, nearly-spherical geometry, but spinning buckyballs are unstable due to the Hairy Ball Theorem [29]. Thus a pair of nested, spinning buckyballs with 120 total vertices tend to morph into their homotopic cousin, a lattice-like torus. This initial toroidal lattice consisting of 24 pentagons and 40 hexagons has the wrong symmetries for long-term stability, and thus further decomposes into a seven-fold symmetry, which seems to be the most

stable of these geometries because of seven-color toroidal mapping [30] (see Figure 2 – the toroidal extension of four-color spherical surface mapping). At least two cases deserve consideration:

Case A – The toroidal lattice decomposes into a “donut” with $3 \times 7 = 21$ pentagons and $5 \times 7 = 35$ hexagons via the underlying $120 \sim SO(16) \sim SO(15,1) \sim 105+15$ symmetries within a minimal rank-7 representation. We obtain 7-D out of an apparently 2-D toroidal surface (a 2-brane) within a 3-D space (a 3-brane) because this torus is allowed to roll in on itself as a 2-D vortex effect (another 2-brane). These extra degrees-of-freedom may be the root cause of the concept of intrinsic spin.

The 15-plet becomes a “donut hole” with rank-3 $SO(3,3) \sim SU(3,1) \sim 12+3$ symmetries (a morphed Twistor Algebra like Example 4) shaped like an icosahedron within a minimum 3-D representation. Coincidentally, the original toroidal lattice of order 120 has the identical order as the Icosahedral Point group, I_h [31] : two singlets: I and i ; four 12-plets: C_5, C_5^2, S_{10} and S_{10}^3 ; two 15-plets: C_2 and σ ; and two 20-plets: C_3 and S_6 . This donut hole may represent the 15 degrees-of-freedom of the $C_2 = C_4^2$ icosahedral symmetries, and the WIMP-Gravitons from the previous example.



Figure 2 – Seven-Color Toroidal Mapping [32] – Top and Bottom Views – Note that each color shares a boundary with the other six colors. Such a mapping implies a minimum of a rank-7 representation.

How does this example relate to Quaternion and Octonion symmetries? It is interesting that this toroidal lattice might have 7-dimensional $SO(15)$ space-like symmetries plus one time-like symmetry (“Octonionic time” is the dynamic variable that separates these toroidal vertices and inflates this toroidal lattice – the negative time-like metric provides the “degree of fixity” required for the outward tension against gravity) while the icosahedron has 3-dimensional space-like symmetries plus another time-like symmetry (“Quaternionic time” is the dynamic variable that separates these icosahedral vertices and inflates this icosahedron). These dimensional symmetries are exactly what we might expect from the union of the (7+1)-D Octonion and the (3+1)-D Quaternion from the previous example into a 12-D F-Theory.

Case B – Another possible decomposition of the initial toroidal lattice would be a $120 \sim SO(16) \sim SO(14,2) \sim 91+29 \sim 91+14+12+3$, which produces a final donut with $3 \times 7 = 21$ pentagons and $4 \times 7 = 28$ hexagons, but now the donut hole is the 3-plet and icosahedron of the prior case plus a G_2 14-plet. A rank-2 G_2 contains the three-fold symmetries of $SU(3)$, and may thus represent three generations of matter (compare with the Strong Force with an $SU(3)$ of three colors). As such, this G_2 is fundamental to understanding both the CKM and PMNS Matrices [33, 34]. These symmetries of Octonion (7+1)-D, Quaternion (3+1)-D and minimum 2-D G_2 add up to an effective 14-D theory as we might expect from the rank-14 $Spin(14)$ sub-component of a $Spin(16) \sim Spin(14,2)$ Theory. Furthermore, a rank-(14+2) $Spin(14,2) \rightarrow Spin(10) \times Spin(4,2)$ could easily represent the union of an effectively 10-D $Spin(10)$ GUT (not including Gravity) [35] and an effectively (4+2)-D $Spin(4,2)$ Gravitational GUT from Example 4 while simultaneously mirroring the minimum (14+2)-D properties of a pair of (7+1)-D Octonions.

These lattices build an impenetrable structure around the Black Hole singularity, and thus prevent the singularity from consuming the entire Black Hole. More exotic allotropes with seven-fold symmetries might involve a pair of C-70 fullerenes – each with 12 pentagons and 25 hexagons, but these options seem to be less stable than pairs of C-60, and lack the beautiful symmetries of $SO(16)$ (although a final donut with 21 pentagons and 42 hexagons has 119 vertices – close to the 120 order of $SO(16)$).

Conclusions

There are numerous texts [36] which focus on the utility of the Quaternions and Octonions in Physics, and there have been quite a few papers by various authors [37] which explicitly utilize the Octonions as a basis for a GUT or TOE, or for explaining numerous aspects of known Physics. The Quaternions have come to be used quite extensively for computations in Video Games and in both Spacecraft and Aircraft Navigation [38], as they greatly simplify trajectory calculations. The Octonions, on the other hand, are still poorly regarded and seldom used. John Baez described them as ‘the crazy old uncle’ of the number family [39] which the rest of the family would rather keep in the attic, but perhaps this analogy is too harsh, and they should be thought of as the wealthy and eccentric old uncle who does not associate with others, but has a lot to share for those who take the time. What octonions offer lies mainly in the area of possibilities, as their seven imaginary components are the freedom to vary, and represent seven different kinds of variation that may be present at once.

Several interesting geometric structures and algebras were presented in this paper, and these deserve extra emphasis and a relational summary. A pair of nested C-60 Buckyballs is homotopic to a torus, and contains 120 vertices – equivalent to the order of an $SU(11) \sim SO(16)$ algebra, and the Icosahedral Point Group symmetries. The Gosset lattice contains 240 vertices within an 8-D space, which is equivalent to the order of a $Spin(16)$ algebra, or the number of E_8 roots. The Octonion has 480 different possible multiplication tables, which is equivalent to the number of $E_8 \times E_8^*$ roots. And finally, $SO(32) \sim E_8 \times E_8^*$ each have an order of 496, and $SO(32)$ has a complex representation capable of mathematically describing CP Symmetry violation.

We have provided some examples of the general usefulness of octonions and quaternions in Physics. We have also suggested a beautiful top-down example (Example 4) of the potential need for octonions in a Quantum Gravity Theory. And we give a bottom-up example (Example 5) suggesting the potential need for octonions in the stability of Black Hole singularities. Interestingly, the seven space-like dimensions of the octonion may correspond to the minimum 7-dimensions of Hyperspace that are required for 10-D String Theory, 11-D M-Theory and / or 12-D F-Theory. Thus it is seen that the degrees of freedom provided by the Octonions allow for the emergence of the full range of observed form, and may provide insights into additional particles to be found, and other aspects of reality we

have not observed yet. It is our hope that more researchers will give serious consideration to the potential usefulness of the Octonions in Quantum Gravity and String Theories.

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