Article

$F_qR_1R_2$ Skew Constacyclic Codes & Their Gray Images

Gamze Avcı*[1], Abdullah Dertli[2] & Yasemin Çengellenmiş[1]

[1]Dept. of Math., Trakya University, Edirne, Turkey
[2]Dept. of Math, Ondokuz Mayıs Univ., Samsun, Turkey

Abstract

In this study, we investigate the structure of linear codes over a mixed alphabet $F_qR_1R_2$, where $R_1 = F_q + u_1F_q$, $u_1^2 = \beta_1u_1$ and $R_2 = F_q + u_1F_q + u_2F_q$, $u_1^2 = \beta_1u_1$, $u_2^2 = \beta_2u_2$, $u_1u_2 = u_2u_1 = 0$, $\beta_1, \beta_2 \in F_q^*$ are a family of finite rings and $R_0 = F_q$ is a finite field with $q = p^m$ elements for odd prime p, positive integer m. We mainly studied skew constacyclic codes over $F_qR_1R_2$. The duals of them are determined. The concept of separable code is introduced. A Gray map is introduced, and the Gray images of skew constacyclic codes over $F_qR_1R_2$ are obtained.

Keywords: Skew constacyclic codes, mixed alphabet, Gray map.

1. Introduction

Getting optimal error-correcting codes is very important in coding theory. There are a lot of methods in literature. One of them is establishing a relationship between certain types of codes over the finite rings and codes over the finite fields. Initially, researchers used an alphabet that was either a finite ring or a family of finite rings. In 1997, Rifa first defined the codes using a mixed alphabet [6]. Since the last two decades, they have studied about the codes over the mixed alphabet. Especially by defining cyclic, constacyclic and skew cyclic codes over some mixed alphabet, they investigated their properties and obtained optimal codes over the finite fields from them. Moreover, they used them to get optimal quantum error-correcting codes. The skew constacyclic codes form a larger class than cyclic, constacyclic and skew cyclic codes. They produce better codes over the finite fields.

The skew constacyclic codes over finite fields were introduced by H. Dinh et al. first in [1]. In [3], skew constacyclic codes over finite chain rings were studied. Later, these codes over finite non-chain rings are extensively studied. In [4], Bo Kong et al. studied skew constacyclic code over $R_k = F_q[u_1, u_2, \ldots, u_k]/\langle u_i^2 = \beta_iu_i, u_iu_j = u_ju_i = 0\rangle$, where F_q is a finite field and $1 \leq i, j \leq k$, $i \neq j$.

This paper is organized as follows: In section 2, some definitions are given. In section 3, the skew constacyclic codes over $F_qR_1R_2$ are defined, and the concept of separable code is

*Correspondence Author: Gamze Avcı, Department. of Mathematics, Trakya University, Edirne, Turkey.
Email: gmzavci99@gmail.com

introduced. In section 4, a Gray map is introduced. The Gray images of skew constacyclic codes over $F_qR_1R_2$ are obtained.

2. Preliminaries

The set of all ordered α_0-tuples $F_q^{\alpha_0}$ is a vector space over F_q with the usual component-wise addition and multiplication by scalars, where F_q is a finite field with q elements, where $q = p^m$ for odd prime p and positive integer m. A code C of length α_0 over F_q is a non-empty subset of $F_q^{\alpha_0}$, and a code C is a linear code over F_q if it is a subspace of $F_q^{\alpha_0}$. Let $\mathbf{c} = (c_0, c_1, ..., c_{\alpha_0-1}) \in C$, then the Hamming weight of the element $\boldsymbol{c}$, $w_H(\boldsymbol{c})$ is defined as the number of non-zero components of $\mathbf{c}$. The Hamming distance between two codewords $\boldsymbol{c}, \boldsymbol{c}' \in C$ is given by $d_H(\boldsymbol{c}, \boldsymbol{c}') = w_H(\boldsymbol{c} - \boldsymbol{c}')$. The minimum distance of C is defined as $d_H(C) = min\{d_H(\boldsymbol{c}, \boldsymbol{c}') | \boldsymbol{c} \neq \boldsymbol{c}', \forall \boldsymbol{c}, \boldsymbol{c}' \in C\}$.

From [5], we know that the distinct automorphisms of F_{p^m} over F_p are exactly the mapping $\sigma_0, \sigma_1, ..., \sigma_{m-1}$ defined by $\sigma_j(\gamma) = \gamma^{p^j}$ for $\gamma \in F_{p^m}^*$ and $0 \leq j \leq m-1$. These automorphisms of F_{p^m} over F_p form a cyclic group of order m generated by σ_1.

The skew constacyclic codes over finite fields were introduced by H. Dinh et al. first in [1]. For more information about skew constacyclic codes over finite fields, see [1].

Definition 2.1 [1] Given a non-trivial automorphism σ_t of F_{p^m} and a unit λ_0 in F_{p^m}, a code C is said to be a skew σ_t-λ_0 constacyclic code of length α_0 if it is closed under the skew σ_t-λ_0 constacyclic shift $\tau_{\lambda_0}^{\sigma_t}: F_{p^m}^{\alpha_0} \longrightarrow F_{p^m}^{\alpha_0}$ which is defined by $\tau_{\lambda_0}^{\sigma_t}(c_0, c_1, \ldots, c_{\alpha_0-1}) = (\sigma_t(\lambda_0 c_{\alpha_0-1}), \sigma_t(c_0), \ldots, \sigma_t(c_{\alpha_0-2}))$.

Theorem 2.2 [1] Let σ_t be a non-trivial automorphism of F_{p^m}, length α_0 an integer divisible by the order of σ_t and λ_0 a unit in F_{p^m} which is a fixed by σ_t. Then the code C is a skew σ_t-λ_0 constacyclic code if and only if C is a left ideal of $F_{p^m}[x, \sigma_t]/\langle x^{\alpha_0} - \lambda_0 \rangle$, where $C = \langle g(x) \rangle$ and $g(x)$ is a right divisor of $x^{\alpha_0} - \lambda_0$.

In [4], the skew constacyclic codes over $R_k = F_q[u_1, u_2, \ldots, u_k]/\langle u_i^2 = \beta_i u_i, u_i u_j = u_j u_i = 0 \rangle$ were studied by Bo Kong et al., where $q = p^m$, p is an odd prime, α_i is a unit over F_q and $1 \leq i, j \leq k$, $i \neq j$.

According to that, the family of rings R_k is semilocal and has q^{k+1} elements. For example, for $k = 1, R_1 = F_q + u_1 F_q, u_1^2 = \beta_1 u_1$, for $k = 2$, $R_2 = F_q + u_1 F_q + u_2 F_q, u_1^2 = \beta_1 u_1$, $u_2^2 = \beta_2 u_2$, $u_2 u_1 = u_1 u_2 = 0$ for $k = 3$, $R_3 = F_q + u_1 F_q + u_2 F_q + u_3 F_q, u_1^2 = \beta_1 u_1, u_2^2 = \beta_2 u_2$, $u_3^2 = \beta_3 u_3, u_2 u_1 = u_1 u_2 = 0$, $u_1 u_3 = u_3 u_1 = 0$, $u_2 u_3 = u_3 u_2 = 0$. It has the orthogonal idempotents, $\xi_1 = \frac{u_1}{\beta_1}, \xi_2 = \frac{u_2}{\beta_2}, \ldots, \xi_k = \frac{u_k}{\beta_k}, \xi_{k+1} = 1 - \frac{u_1}{\beta_1} - \frac{u_2}{\beta_2} - \ldots - \frac{u_k}{\beta_k}$. They satisfy the following conditions;

1. $\xi_1 + \xi_2 + \ldots + \xi_{k+1} = 1$,
2. $\xi_s . \xi_f = 0$ for $s, f = 1,2, \ldots, k+1$, $s \neq f$,

3. $\xi_s^2 = \xi_s$, for $s = 1,2,\dots,k+1$.
   So $R_k \cong \xi_1 F_q \oplus \xi_2 F_q \oplus \dots \oplus \xi_{k+1} F_q$.

Any element $r_k \in R_k$, there exist $r_1, r_2, \dots, r_{k+1} \in F_q$ such that r_k can be uniquely written as $r_k = r_{k,1}\xi_1 + \dots + r_{k,k+1}\xi_{k+1}$.

In [4], it was stated that an element $\lambda_k = \lambda_{k,1}\xi_1 + \dots + \lambda_{k,k+1}\xi_{k+1}$ is a unit in R_k if and only if $\lambda_{k,j}$ is a unit in F_q for $j = 1,2,\dots,k+1$.

If C_k is a code of length α_k over R_k, then C_k is a subset of $R_k^{\alpha_k}$. C_k is a linear code of length n over R_k if and only if C_k is an R_k-submodule of $R_k^{\alpha_k}$.

In [4], for any $r_k = r_{k,1}\xi_1 + \dots + r_{k,k+1}\xi_{k+1} \in R_k$, the Gray map ϕ_k was defined as

$$\phi_k : R_k \longrightarrow F_q^{k+1}$$

$$r_k \longmapsto (r_{k,1}, r_{k,2}, \dots, r_{k,k+1}).$$

The authors extended ϕ_k as follows

$$\phi_k : R_k^{\alpha_k} \longrightarrow F_q^{(k+1)\alpha_k}$$

$$(r_k^0, r_k^1, \dots, r_k^{\alpha_k - 1}) \longmapsto (r_{k,1}^0, \dots, r_{k,k+1}^0, r_{k,1}^1, \dots, r_{k,k+1}^1, \dots, r_{k,1}^{\alpha_k - 1}, \dots, r_{k,k+1}^{\alpha_k - 1})$$

where $r_k^z = r_{k,1}^z \xi_1 + r_{k,2}^z \xi_2 + \dots + r_{k,k+1}^z \xi_{k+1} \in R_k^{\alpha_k}$, where $z = 0,1,\dots,\alpha_k - 1$.

The Gray weight of an element $r_k = r_{k,1}\xi_1 + \dots + r_{k,k+1}\xi_{k+1} \in R_k$ is defined as follows

$$w_G(r_k) = w_H(\phi_k(r_k))$$

where w_H denotes the usual Hamming weight on F_q.

The Gray weight of the element $\boldsymbol{u} = (u_0, u_1, \dots, u_{\alpha_k - 1}) \in R_k^{\alpha_k}$ is defined by $w_G(\boldsymbol{u}) = \sum_{i=0}^{\alpha_k - 1} w_G(u_i)$.

For any elements $\boldsymbol{u}, \boldsymbol{v} \in R_k^{\alpha_k}$, the Gray distance is given by $d_G(\boldsymbol{u}, \boldsymbol{v}) = w_G(\boldsymbol{u} - \boldsymbol{v})$. The minimum Gray distance of a code C_k is the smallest non-zero Gray distance between all pairs of distinct codewords.

The Gray map ϕ_k is an F_q-linear and distance-preserving map from $R_k^{\alpha_k}$ (Gray distance) to $F_q^{(k+1)\alpha_k}$(Hamming distance).

Moreover, in [4], it was stated that the linear code C_k of length α_k over R_k can be uniquely expressed as

$$C_k = \oplus_{j=1}^{k+1} \xi_j C_{k,j}$$

where $C_{k,j} = \left\{ \boldsymbol{x_j} \in F_q^{\alpha_k} \middle| \sum_{i=1}^{k+1} \boldsymbol{x_j}\xi_j \in C_k,\ \exists \boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_{j-1}}, \boldsymbol{x_{j+1}}, \dots, \boldsymbol{x_{k+1}} \in F_q^{\alpha_k} \right\}$.

In [4], they defined the automorphism of R_k as follows

$$\delta_k : R_k \longrightarrow R_k$$

$$a_0 + a_1u_1 + a_2u_2 + \dots + a_ku_k \longmapsto \sigma_t(a_0) + \sigma_t(a_1)u_1 + \dots + \sigma_t(a_k)u_k$$

where σ_t is Frobenius automorphism $\sigma_t : F_q \longrightarrow F_q$ by $\sigma_t(a) = a^{p^t}$ to define skew constacyclic codes over R_k.

Definition 2.3 [4] Let δ_k be a non-trivial automorphism of R_k and λ_k be a unit of R_k. Then a nonempty subset C_k of $R_k^{\alpha_k}$ is called a skew δ_k-λ_k constacyclic code of length α_k over R_k is

- C_k is an R_k-submodule of $R_k^{\alpha_k}$
- If $\boldsymbol{c} = (c_0, \dots, c_{\alpha_k-1}) \in C_k$, then $\tau_{\lambda_k}^{\delta_k}(\boldsymbol{c}) = (\lambda_k\delta_k(\mathrm{c}_{\alpha_k-1}), \delta_k(\mathrm{c}_0), \dots, \delta_k(c_{\alpha_k-2})) \in C_k$.

For the polynomial representation,

A map was defined as

$$\psi_k : R_k^{\alpha_k} \longrightarrow R_k[x, \delta_k]/\langle x^{\alpha_k} - \lambda_k \rangle$$

$$(a_0, a_1, \dots, a_{\alpha_k-1}) \longmapsto a_0 + a_1x + \dots + a_{\alpha_k-1}x^{\alpha_k-1}.$$

et l be the order of δ_k. That is, $l = ord(\delta_k)$ if $l \mid \alpha_k$, they defined a skew δ_k-λ_k constacyclic code of length α_k over R_k as a left ideal of $R_k[x, \delta_k]/\langle x^{\alpha_k} - \lambda_k \rangle$.

Theorem 2.4 [4] Let $C_k = \oplus_{j=1}^{k+1} \xi_j C_{k,j}$ be a linear code of length α_k over R_k and $\lambda_k = \lambda_{k,1}\xi_1 + \dots + \lambda_{k,k+1}\xi_{k+1}$ is a unit in R_k, $ord(\delta_k) \mid \alpha_k$, $\delta_k(\lambda_k) = \lambda_k$. Then, C_k is a skew δ_k-λ_k constacyclic code of length α_k over R_k if and only if $C_{k,j}$ is a skew σ_t-$\lambda_{k,j}$ constacyclic code of length α_k over F_q for $j = 1,2, \dots, k+1$.

Theorem 2.5 [4] Let $C_k = \oplus_{j=1}^{k+1} \xi_j C_{k,j}$ be a skew δ_k-λ_k constacyclic code of length α_k over R_k, $\lambda_k = \lambda_{k,1}\xi_1 + \dots + \lambda_{k,k+1}\xi_{k+1}$ is a unit in R_k, $ord(\delta_k) \mid \alpha_k$, $\delta_k(\lambda_k) = \lambda_k$. Then, $C_k^\perp = \oplus_{j=1}^{k+1} \xi_j C_{k,j}^\perp$ is a skew δ_k-λ_k^{-1} constacyclic code of length α_k over R_k, where $j = 1, \dots, k+1$ and $C_{k,j}^\perp$ is a skew σ_t-$\lambda_{k,j}^{-1}$ constacyclic code of length α_k over F_q for $j = 1,2, \dots, k+1$, where $\lambda_k^{-1} = \lambda_{k,1}^{-1}\xi_1 + \dots + \lambda_{k,k+1}^{-1}\xi_{k+1}$.

Theorem 2.6 [4] Let $C_k = \oplus_{j=1}^{k+1} \xi_j C_{k,j}$ be a skew δ_k-λ_k constacyclic code of length α_k over R_k, $\lambda_k = \lambda_{k,1}\xi_1 + \dots + \lambda_{k,k+1}\xi_{k+1}$ is a unit in R_k, $ord(\delta_k) \mid \alpha_k$, $\delta_k(\lambda_k) = \lambda_k$. Then there exists a polynomial $\xi_1 g_{k,1}(x) + \dots + \xi_{k+1} g_{k,k+1}(x) \in R_k[x, \delta_k]$ subject to $\psi_k(C_k) = \langle \xi_1 g_{k,1}(x) + \dots + \xi_{k+1} g_{k,k+1}(x) \rangle$, where the right divisor of $x^{\alpha_k} - \lambda_k$ is

$\xi_1 g_{k,1}(x)+\ldots+\xi_{k+1} g_{k,k+1}(x)$, the generator polynomial of skew σ_t-$\lambda_{k,j}$ constacyclic $C_{k,j}$ is $g_{k,j}(x) \in F_q[x,\sigma_t]$ and $g_{k,j}(x) \mid x^{\alpha_k} - \lambda_{k,j}$ on the right for $j = 1,2,\ldots k+1$.

3. Skew Constacyclic Codes

In this section, we study the structure of $F_qR_1R_2$ linear codes where $R_1 = F_q + u_1F_q$, $u_1^2 = \beta_1 u_1$ and $R_2 = F_q + u_1F_q + u_2F_q$, $u_1^2 = \beta_1 u_1$, $u_2^2 = \beta_2 u_2$, $u_1u_2 = u_2u_1 = 0$, $\beta_1, \beta_2 \in F_q^*$ and investigate the skew constacyclic codes over $F_qR_1R_2$.

Proposition 3.1 The set $F_qR_1R_2 = \{(a,b,c) \mid a \in F_q, b \in R_1, c \in R_2\}$ forms an R_2-module under usual addition and scalar multiplication defined as

$$r.(a,b,c) = (\phi_0(r)a, \phi_1(r)b, rc),$$

where $\phi_0: R_2 \longrightarrow F_q$ is a linear transformation such that $\phi_0(x+yu_1+zu_2) = x$ and $\phi_1: R_2 \longrightarrow R_1$ is a linear transformation such that $\phi_1(x+yu_1+zu_2) = x+yu_1$ for $r = x+yu_1+zu_2 \in R_2$.

This '.' can be extended componentwise to $F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$ by $r.(a_0,\ldots,a_{\alpha_0-1},b_0,\ldots,b_{\alpha_1-1},c_0,\ldots,c_{\alpha_2-1}) = (\phi_0(r)a_0,\ldots,\phi_0(r)a_{\alpha_0-1},\phi_1(r)b_0,\ldots,$

$\phi_1(r)b_{\alpha_1-1}, rc_0,\ldots,rc_{\alpha_2-1})$, where $(a_0,\ldots,a_{\alpha_0-1},b_0,\ldots,b_{\alpha_1-1},c_0,\ldots,c_{\alpha_2-1}) \in F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$. It can be easily seen that $F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$ is an R_2-module.

Definition 3.2 A non-empty subset $\boldsymbol{C}$ of $F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$ is called an $F_qR_1R_2$-linear code of length $(\alpha_0,\alpha_1,\alpha_2)$ if it is R_2-submodule of $F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$.

Definition 3.3 Let $\sigma_t \in Aut(F_q)$ such that $ord(\sigma_t) \mid \alpha_0$ and a unit $\lambda_0 \in F_q$ is fixed by σ_t.

$\delta_u \in Aut(R_u)$ such that $ord(\delta_u) \mid \alpha_u$ and a unit $\lambda_u \in R_u$ is fixed by δ_u, for $u = 1,2$. An $F_qR_1R_2$-linear code $\boldsymbol{C}$ of length $(\alpha_0,\alpha_1,\alpha_2)$ is called a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code, if $(a_0,\ldots,a_{\alpha_0-1},b_0,\ldots,b_{\alpha_1-1},c_0,\ldots,c_{\alpha_2-1}) \in \boldsymbol{C}$ implies $\tau_{\lambda_0,\lambda_1,\lambda_2}^{\sigma_t,\delta_1,\delta_2}(a_0,\ldots,a_{\alpha_0-1},b_0,\ldots,b_{\alpha_1-1},c_0,\ldots,c_{\alpha_2-1}) = (\lambda_0\sigma_t(a_{\alpha_0-1}),\sigma_t(a_0),\ldots,\sigma_t(a_{\alpha_0-2}),$

$\lambda_1\delta_1(b_{\alpha_1-1}),\delta_1(b_0),\ldots,\delta_1(b_{\alpha_1-2}),\lambda_2\delta_2(c_{\alpha_2-1}),\delta_2(c_0),\ldots,\delta_2(c_{\alpha_2-2})) \in \boldsymbol{C}$.

There is a one to one correspondence between $F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$ and $\Lambda = F_q[x,\sigma_t]/\langle x^{\alpha_0}-\lambda_0\rangle \times R_1[x,\delta_1]/\langle x^{\alpha_1}-\lambda_1\rangle \times R_2[x,\delta_2]/\langle x^{\alpha_2}-\lambda_2\rangle$ as follows;

$$\varphi: F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2} \longrightarrow \Lambda$$

$$(a_0,\ldots,a_{\alpha_0-1},b_0,\ldots,b_{\alpha_1-1},c_0,\ldots,c_{\alpha_2-1})\longmapsto(a(x),b(x),c(x))$$

where $a(x)=a_0+a_1x+\ldots+a_{\alpha_0-1}x^{\alpha_0-1}$, $b(x)=b_0+b_1x+\ldots+b_{\alpha_1-1}x^{\alpha_1-1}$, $c(x)=c_0+c_1x+\ldots+c_{\alpha_2-1}x^{\alpha_2-1}$.

Theorem 3.4 The set Λ is a left $R_2[x,\delta_2]$-module under the usual addition, and the left scalar multiplication defined as

$$r(x).(a(x),b(x),c(x))=(\phi_0(r(x))a(x),\phi_1(r(x))b(x),r(x)c(x)),$$

where $r(x)=r_0+r_1x+\ldots+r_tx^t\in R_2[x,\delta_2]$, $(a(x),b(x),c(x))\in\Lambda$ and $\phi_v(r(x))=\phi_v(r_0)+\phi_v(r_1)x+\ldots+\phi_v(r_t)x^t\in R_v[x,\delta_v]$ for $v=0,1$ and $\sigma_t=\delta_0$.

According to that, we can give the polynomial representation of the skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code as follows;

Theorem 3.5 An $F_qR_1R_2$-linear code $\boldsymbol{C}$ is a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code of length $(\alpha_0,\alpha_1,\alpha_2)$ if and only if $\varphi(\boldsymbol{C})$ is a left $R_2[x,\delta_2]$-submodule of Λ.

Proof Let $\boldsymbol{C}$ be a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code of length $(\alpha_0,\alpha_1,\alpha_2)$. Then by definition $x.\big(a(x),b(x),c(x)\big)\in\varphi(\boldsymbol{C})$, we have $x.(a(x),b(x),c(x))=(\sigma_t(a_0)x+\sigma_t(a_1)x^2+\cdots+\lambda_0\sigma_t(a_{\alpha_0-1}),\delta_1(b_0)x+\delta_1(b_1)x^2+\cdots+\lambda_1\delta_1(b_{\alpha_1-1}),\delta_2(c_0)x+\delta_2(c_1)x^2+\ldots+\lambda_2\delta_2(c_{\alpha_2-1}))\in\varphi(\boldsymbol{C})$ and by using linearity of $\boldsymbol{C}$, we get $r(x).(a(x),b(x),c(x))\in\varphi(\boldsymbol{C})$ for some $r(x)\in R_2[x,\delta_2]$. Conversely, $\varphi(\boldsymbol{C})$ is a left $R_2[x,\delta_2]$-submodule over Λ, then we have $x.(a(x),b(x),c(x))\in\varphi(\boldsymbol{C})$ implies $\boldsymbol{C}$ is a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code. □

Definition 3.6 Let $\boldsymbol{C}$ be an $F_qR_1R_2$-linear code of length $(\alpha_0,\alpha_1,\alpha_2)$. The dual code of $\boldsymbol{C}$ is defined as,

$$\boldsymbol{C}^{\perp}=\left\{\boldsymbol{c}'\in F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}\,\middle|\,\boldsymbol{cc}'=0,\forall\boldsymbol{c}\in\boldsymbol{C}\right\}$$

where the inner product is defined,

$$\boldsymbol{cc}'=\sum_{i=0}^{\alpha_0-1}\boldsymbol{a}_i\boldsymbol{a}_i'+\sum_{j=0}^{\alpha_1-1}\boldsymbol{b}_j\boldsymbol{b}_j'+\sum_{e=0}^{\alpha_2-1}\boldsymbol{c}_e\boldsymbol{c}_e'$$

for any $\boldsymbol{c}=(a_0,\ldots,a_{\alpha_0-1},b_0,\ldots,b_{\alpha_1-1},c_0,\ldots,c_{\alpha_2-1})$ and $\boldsymbol{c}'=(a_0',\ldots,a_{\alpha_0-1}',b_0',\ldots,b_{\alpha_1-1}',c_0',\ldots,c_{\alpha_2-1}')\in F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$.

Proposition 3.7 Let $ord(\sigma_t)\,|\,\alpha_0$, $ord(\delta_1)\,|\,\alpha_1$, $ord(\delta_2)\,|\,\alpha_2$ and $\sigma_t(\lambda_0)=\lambda_0$, $\delta_1(\lambda_1)=\lambda_1$, $\delta_2(\lambda_2)=\lambda_2$. If $\boldsymbol{C}$ is a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code of length $(\alpha_0,\alpha_1,\alpha_2)$ over $F_qR_1R_2$, then $\boldsymbol{C}^{\perp}$ is a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0^{-1},\lambda_1^{-1},\lambda_2^{-1})$ constacyclic code of length $(\alpha_0,\alpha_1,\alpha_2)$ over $F_qR_1R_2$.

Proof Let $\boldsymbol{c} = (a_0, a_1, \ldots, a_{\alpha_0-1}, b_0, b_1, \ldots, b_{\alpha_1-1}, c_0, c_1, \ldots, c_{\alpha_2-1}) \in \boldsymbol{C}$ and $\boldsymbol{c}' = (\mathrm{a}'_0, \ldots, \mathrm{a}'_{\alpha_0-1}, \mathrm{b}'_0, \ldots, \mathrm{b}'_{\alpha_1-1}, \mathrm{c}'_0, \ldots, \mathrm{c}'_{\alpha_2-1}) \in \boldsymbol{C}^{\perp}$. We show that $\tau^{\sigma_t,\delta_1,\delta_2}_{\lambda_0^{-1},\lambda_1^{-1},\lambda_2^{-1}}(\mathrm{a}'_0, \ldots, \mathrm{a}'_{\alpha_0-1}, \mathrm{b}'_0,$
$\ldots, \mathrm{b}'_{\alpha_1-1}, \mathrm{c}'_0, \ldots, \mathrm{c}'_{\alpha_2-1}) = (\lambda_0^{-1}\sigma_t(\mathrm{a}'_{\alpha_0-1}), \sigma_t(\mathrm{a}'_0), \ldots, \sigma_t(\mathrm{a}'_{\alpha_0-2}), \lambda_1^{-1}\delta_1(\mathrm{b}'_{\alpha_1-1}), \delta_1(\mathrm{b}'_0), \ldots,$
$\delta_1(\mathrm{b}'_{\alpha_1-2}), \lambda_2^{-1}\delta_2(\mathrm{c}'_{\alpha_2-1}), \delta_2(\mathrm{c}'_0), \ldots, \delta_2(\mathrm{c}'_{\alpha_2-2})) \in \mathbf{C}^{\perp}$.

Hence it is suffices to show that $\boldsymbol{c}\tau^{\sigma_t,\delta_1,\delta_2}_{\lambda_0^{-1},\lambda_1^{-1},\lambda_2^{-1}}(\boldsymbol{c}') = 0$. i.e.
$\left[\lambda_0^{-1} a_0 \sigma_t\left(\mathrm{a}'_{\alpha_0-1}\right) + a_1\sigma_t(\mathrm{a}'_0) + \ldots + a_{\alpha_0-1}\sigma_t\left(\mathrm{a}'_{\alpha_0-2}\right)\right] +$
$\left[\lambda_1^{-1} b_0 \delta_1\left(\mathrm{b}'_{\alpha_1-1}\right) + b_1\delta_1(\mathrm{b}'_0) + \ldots + b_{\alpha_1-1}\delta_1\left(\mathrm{b}'_{\alpha_1-2}\right)\right] +$
$\left[\lambda_2^{-1} c_0 \delta_2\left(\mathrm{c}'_{\alpha_2-1}\right) + c_1\delta_2(\mathrm{c}'_0) + \ldots + c_{\alpha_2-1}\delta_2\left(\mathrm{c}'_{\alpha_2-2}\right)\right] = 0.$

Let h_{λ_0} be the order λ_0 in F_q, h_{λ_1} be the order of λ_1 in R_1 and h_{λ_2} be the order of λ_2 in R_2. Then $\lambda_0^{h_{\lambda_0}} = 1$ in F_q, $\lambda_1^{h_{\lambda_1}} = 1$ in R_1, $\lambda_2^{h_{\lambda_2}} = 1$ in R_2. Let $l = h_{\lambda_0}h_{\lambda_1}h_{\lambda_2}\alpha_0\alpha_1\alpha_2$ and denote $K = h_{\lambda_0}h_{\lambda_1}h_{\lambda_2}\alpha_1\alpha_2$, $N = h_{\lambda_0}h_{\lambda_1}h_{\lambda_2}\alpha_0\alpha_2$, $M = h_{\lambda_0}h_{\lambda_1}h_{\lambda_2}\alpha_0\alpha_1$. Since $\boldsymbol{c} \in \boldsymbol{C}$ and $\boldsymbol{C}$ is a skew $(\sigma_t, \delta_1, \delta_2)$-$(\lambda_0, \lambda_1, \lambda_2)$ constacyclic code, we have $\left(\tau^{\sigma_t,\delta_1,\delta_2}_{\lambda_0,\lambda_1,\lambda_2}\right)^{l-1}(\boldsymbol{c}) \in \boldsymbol{C}$, where $(\tau^{\sigma_t,\delta_1,\delta_2}_{\lambda_0,\lambda_1,\lambda_2})^{l-1}(\boldsymbol{c}) =$
$(\lambda_0^{K}\sigma_t^{l-1}(a_1), \ldots, \lambda_0^{K}\sigma_t^{l-1}(a_{\alpha_0-1}), \lambda_0^{K-1}\sigma_t^{l-1}(a_0), \lambda_1^{N}\delta_1^{l-1}(b_1), \ldots, \lambda_1^{N}\delta_1^{l-1}(b_{\alpha_1-1}),$
$\lambda_1^{N-1}\delta_1^{l-1}(b_0), \lambda_2^{M}\delta_2^{l-1}(c_1), \ldots, \lambda_2^{M}\delta_2^{l-1}(c_{\alpha_2-1}), \lambda_2^{M-1}\delta_2^{l-1}(c_0))$.

The inner product of $(\tau^{\sigma_t,\delta_1,\delta_2}_{\lambda_0,\lambda_1,\lambda_2})^{l-1}(\boldsymbol{c})$ and $\boldsymbol{c}'$ is equal to zero. That is,

$$[\sigma_t^{l-1}(a_1)\mathrm{a}'_0 + \sigma_t^{l-1}(a_2)\mathrm{a}'_1 + \ldots + \sigma_t^{l-1}\left(a_{\alpha_0-1}\right)\mathrm{a}'_{\alpha_0-2} + \lambda_0^{-1}\sigma_t^{l-1}(a_0)\mathrm{a}'_{\alpha_0-1}] +$$
$$[\delta_1^{l-1}(b_1)\mathrm{b}'_0 + \delta_1^{l-1}(b_2)\mathrm{b}'_1 + \ldots + \delta_1^{l-1}(b_{\alpha_1-1})\mathrm{b}'_{\alpha_1-2} + \lambda_1^{-1}\delta_1^{l-1}(b_0)\mathrm{b}'_{\alpha_1-1}] +$$
$$[\delta_2^{l-1}(c_1)\mathrm{c}'_0 + \delta_2^{l-1}(c_2)\mathrm{c}'_1 + \ldots + \delta_2^{l-1}\left(c_{\alpha_2-1}\right)\mathrm{c}'_{\alpha_2-2} + \lambda_2^{-1}\delta_2^{l-1}(c_0)\mathrm{c}'_{\alpha_2-1}] = 0$$

which means,

$$\left[\lambda_0^{-1}\sigma_t^{l-1}(a_0)\mathrm{a}'_{\alpha_0-1} + \sigma_t^{l-1}(a_1)\mathrm{a}'_0 + \sigma_t^{l-1}(a_2)\mathrm{a}'_1 + \ldots + \sigma_t^{l-1}\left(a_{\alpha_0-1}\right)\mathrm{a}'_{\alpha_0-2}\right] +$$
$$\left[\lambda_1^{-1}\delta_1^{l-1}(b_0)\mathrm{b}'_{\alpha_1-1} + \delta_1^{l-1}(b_1)\mathrm{b}'_0 + \delta_1^{l-1}(b_2)\mathrm{b}'_1 + \ldots + \delta_1^{l-1}\left(b_{\alpha_1-1}\right)\mathrm{b}'_{\alpha_1-2}\right] +$$
$$[\lambda_2^{-1}\delta_2^{l-1}(c_0)\mathrm{c}'_{\alpha_2-1} + \delta_2^{l-1}(c_1)\mathrm{c}'_0 + \delta_2^{l-1}(c_2)\mathrm{c}'_1 + \ldots + \delta_2^{l-1}(c_{\alpha_2-1})\mathrm{c}'_{\alpha_2-2}] = 0.$$

Since $ord(\sigma_t) \,\big|\, \alpha_0$, $ord(\delta_1) \,\big|\, \alpha_1$, $ord(\delta_2) \,\big|\, \alpha_2$, we have $\sigma_t^l = identity$, $\delta_1^l = identity$ and $\delta_2^l = identity$. Note that $\sigma_t(\lambda_0) = \lambda_0$, $\delta_1(\lambda_1) = \lambda_1$, $\delta_2(\lambda_2) = \lambda_2$ and $\delta_2\big|_{F_q} = \sigma_t$, $\delta_2\big|_{R_1} = \delta_1$. Applying δ_2 on both sides of the above equation, we get
$\left[\lambda_0^{-1} a_0 \sigma_t\left(\mathrm{a}'_{\alpha_0-1}\right) + a_1\sigma_t(\mathrm{a}'_0) + \ldots + a_{\alpha_0-1}\sigma_t\left(\mathrm{a}'_{\alpha_0-2}\right)\right] +$
$\left[\lambda_1^{-1} b_0 \delta_1\left(\mathrm{b}'_{\alpha_1-1}\right) + b_1\delta_1(\mathrm{b}'_0) + \ldots + b_{\alpha_1-1}\delta_1\left(\mathrm{b}'_{\alpha_1-2}\right)\right] + [\lambda_2^{-1} c_0 \delta_2(\mathrm{c}'_{\alpha_2-1}) +$
$c_1\delta_2(\mathrm{c}'_0) + \ldots + c_{\alpha_2-1}\delta_2(\mathrm{c}'_{\alpha_2-2})] = 0.$

This shows that $\tau^{\sigma_t,\delta_1,\delta_2}_{\lambda_0^{-1},\lambda_1^{-1},\lambda_2^{-1}}(\boldsymbol{c}')\boldsymbol{c} = 0$. So $\tau^{\sigma_t,\delta_1,\delta_2}_{\lambda_0^{-1},\lambda_1^{-1},\lambda_2^{-1}}(\boldsymbol{c}') \in \mathbf{C}^{\perp}$. Thus $\mathbf{C}^{\perp}$ is a skew $(\sigma_t, \delta_1, \delta_2)$-$(\lambda_0^{-1}, \lambda_1^{-1}, \lambda_2^{-1})$ constacyclic code of length $(\alpha_0, \alpha_1, \alpha_2)$ over $F_qR_1R_2$. □

An $F_qR_1R_2$-linear code C of length $(\alpha_0,\alpha_1,\alpha_2)$ is called separable code if $C=B_0\otimes B_1\otimes B_2$, where B_0, B_1 and B_2 are punctured codes of C. They are obtained by deleting the coordinates outside the α_0, α_1 and α_2 components respectively.

Proposition 3.8 Let $\boldsymbol{C}=B_0\otimes B_1\otimes B_2$ be an $F_qR_1R_2$ linear code of length $(\alpha_0,\alpha_1,\alpha_2)$, where B_ϱ is a linear code of length α_ϱ over F_q, R_1, R_2, for $\varrho=0,1,2$, respectively. Then $\boldsymbol{C}$ is a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code of length $(\alpha_0,\alpha_1,\alpha_2)$ if and only if B_0 is a skew σ_t-λ_0 constacyclic code of length α_0 over F_q, B_u is a skew δ_u-λ_u constacyclic code of length α_u over R_u for $u=1,2$, respectively.

Proof Let $\boldsymbol{C}=B_0\otimes B_1\otimes B_2$ be a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code of length $(\alpha_0,\alpha_1,\alpha_2)$ over $F_qR_1R_2$ and $(a_0,a_1,\dots,a_{\alpha_0-1},b_0,b_1,\dots,b_{\alpha_1-1},c_0,c_1,\dots,c_{\alpha_2-1})\in\boldsymbol{C}$, where $(a_0,a_1,\dots,a_{\alpha_0-1})\in B_0$, $(b_0,b_1,\dots,b_{\alpha_1-1})\in B_1$, $(c_0,c_1,\dots,c_{\alpha_2-1})\in B_2$. Since $\boldsymbol{C}$ is a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code, we have $(\lambda_0\sigma_t(a_{\alpha_0-1}),\sigma_t(a_0),\dots,\sigma_t(a_{\alpha_0-2}),$

$\lambda_1\delta_1(b_{\alpha_1-1})\ \delta_1(b_0),\delta_1(b_1),\dots,\delta_1(b_{\alpha_1-2}),\lambda_2\delta_2(c_{\alpha_2-1}),\delta_2(c_0),\delta_2(c_1),\dots,\delta_2(c_{\alpha_2-2}))\in\boldsymbol{C}$

which implies $(\lambda_0\sigma_t(a_{\alpha_0-1}),\sigma_t(a_0),\dots,\sigma_t(a_{\alpha_0-2}))\in B_0$, $(\lambda_1\delta_1(b_{\alpha_1-1}),\delta_1(b_0),\delta_1(b_1),\dots,\delta_1(b_{\alpha_1-2}))\in B_1$, $(\lambda_2\delta_2(c_{\alpha_2-1}),\delta_2(c_0),\delta_2(c_1),\dots,\delta_2(c_{\alpha_2-2}))\in B_2$. Therefore, B_0 is a skew σ_t-λ_0 constacyclic code of length α_0 over F_q, B_u is a skew δ_u-λ_u constacyclic code of length α_u over R_u for $u=1,2$, respectively.

Conversely, let $(a_0,a_1,\dots,a_{\alpha_0-1},b_0,b_1,\dots,b_{\alpha_1-1},c_0,c_1,\dots,c_{\alpha_2-1})\in\boldsymbol{C}$, where $(a_0,a_1,\dots,a_{\alpha_0-1})\in B_0$, $(b_0,b_1,\dots,b_{\alpha_1-1})\in B_1$, $(c_0,c_1,\dots,c_{\alpha_2-1})\in B_2$. Suppose that B_0 is a skew σ_t-λ_0 constacyclic code of length α_0 over F_q, B_u is a skew δ_u-λ_u constacyclic code of length α_u over R_u for $u=1,2$, respectively. This means, $(\lambda_0\sigma_t(a_{\alpha_0-1}),\sigma_t(a_0),\dots,\sigma_t(a_{\alpha_0-2}))\in B_0$,

$(\lambda_1\delta_1(b_{\alpha_1-1}),\delta_1(b_0),\delta_1(b_1),\dots,\delta_1(b_{\alpha_1-2}))\in B_1$, $(\lambda_2\delta_2(c_{\alpha_2-1}),\delta_2(c_0),\delta_2(c_1),\dots,$

$\delta_2(c_{\alpha_2-2}))\in B_2$.

Therefore $(\lambda_0\sigma_t(a_{\alpha_0-1}),\sigma_t(a_0),\dots,\sigma_t(a_{\alpha_0-2}),\lambda_1\delta_1(b_{\alpha_1-1}),\delta_1(b_0),\delta_1(b_1),\dots,\delta_1(b_{\alpha_1-2}),$ $\lambda_2\delta_2(c_{\alpha_2-1}),\delta_2(c_0),\delta_2(c_1),\dots,\delta_2(c_{\alpha_2-2}))\in\boldsymbol{C}$, then $\boldsymbol{C}$ be a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code of length $(\alpha_0,\alpha_1,\alpha_2)$ over $F_qR_1R_2$. □

Corollary 3.9 Let $\boldsymbol{C}=B_0\otimes B_1\otimes B_2$ be an $F_qR_1R_2$-linear code of length $(\alpha_0,\alpha_1,\alpha_2)$, where B_ϱ is a linear code of length α_ϱ over R_ϱ for $\varrho=0,1,2$, respectively. Then $\boldsymbol{C}$ is a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code of length $(\alpha_0,\alpha_1,\alpha_2)$ if and only if B_0 is a skew σ_t-λ_0 constacyclic code of length α_0 over F_q, $B_{1,u}$ are skew σ_t-$\lambda_{1,u}$ constacyclic codes of length α_1 over F_q for $u=1,2$ and $B_{1,s}$ are skew σ_t-$\lambda_{2,s}$ constacyclic codes of length α_2 over F_q for $s=1,2,3$.

4. The Gray Image of Skew Constacyclic Code

In this section, we investigate the Gray image of skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code over $F_qR_1R_2$.

We know that from [4]

$$\phi_1: R_1 \longrightarrow F_q^2$$

$$a_{1,1}\xi_1 + a_{1,2}\xi_2 \longmapsto (a_{1,1}, a_{1,2}) \quad \text{and}$$

$$\phi_2: R_2 \longrightarrow F_q^3$$

$$a_{2,1}\xi_1' + a_{2,2}\xi_2' + a_{2,3}\xi_3' \longmapsto (a_{2,1}, a_{2,2}, a_{2,3})$$

where $\xi_1 = \frac{u_1}{\beta_1}$, $\xi_2 = 1 - \frac{u_1}{\beta_1}$ and $\xi_1' = \frac{u_1}{\beta_1}$, $\xi_2' = \frac{u_2}{\beta_2}$, $\xi_3' = 1 - \frac{u_1}{\beta_1} - \frac{u_2}{\beta_2}$.

By using them, we can define the Gray map on $F_qR_1R_2$ as follows;

$$\Phi: F_qR_1R_2 \longrightarrow F_q^6$$

$$(a_0, a_1, a_2) = (a_0, a_{1,1}\xi_1 + a_{1,2}\xi_2, a_{2,1}\xi_1' + a_{2,2}\xi_2' + a_{2,3}\xi_3') \longmapsto (a_0, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}, a_{2,3})$$

We can extend this map on $F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$ as follows;

$$\Phi: F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2} \longrightarrow F_q^{\alpha_0+2\alpha_1+3\alpha_2}$$

$$(\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_2}) = (a_{0,0}, a_{0,1}, \dots, a_{0,\alpha_0-1}, a_{1,0}, a_{1,1}, \dots, a_{1,\alpha_1-1}, a_{2,0}, a_{2,1}, \dots, a_{2,\alpha_2-1}) \longmapsto$$
$$(a_{0,0}, a_{0,1}, \dots, a_{0,\alpha_0-1}, \phi_1(a_{1,0}), \dots, \phi_1(a_{1,\alpha_1-1}), \phi_2(a_{2,0}), \dots, \phi_2(a_{2,\alpha_2-1})).$$

The Gray weight of $(\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_2}) \in F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$ is defined as $w_G(\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_2}) = w_H(\boldsymbol{a_0}) + w_G(\boldsymbol{a_1}) + w_G(\boldsymbol{a_2})$, where w_H and w_G denote Hamming weight and Gray weight over F_q, R_1, R_2. The Gray distance between $\boldsymbol{a'}, \boldsymbol{b'} \in F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$ is defined as $d_G(\boldsymbol{a'}, \boldsymbol{b'}) = w_G(\boldsymbol{a'} - \boldsymbol{b'}) = w_H(\Phi(\boldsymbol{a'} - \boldsymbol{b'}))$.

Theorem 4.1 The Gray map Φ is an F_q-linear map that preserves distance from $F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$ (Gray distance) to $F_q^{\alpha_0+2\alpha_1+3\alpha_2}$ (Hamming distance).

Proof Let $\boldsymbol{y} = (\boldsymbol{a_0'}, \boldsymbol{a_1'}, \boldsymbol{a_2'})$, $\boldsymbol{r} = (\boldsymbol{a_0''}, \boldsymbol{a_1''}, \boldsymbol{a_2''}) \in F_q^{\alpha_0}R_1^{\alpha_1}R_2^{\alpha_2}$, where $\boldsymbol{a_1'} = \boldsymbol{a'_{1,1}}\xi_1 + \boldsymbol{a'_{1,2}}\xi_2$, $\boldsymbol{a_1''} = \boldsymbol{a''_{1,1}}\xi_1 + \boldsymbol{a''_{1,2}}\xi_2 \in R_1$, $\boldsymbol{a_2'} = \boldsymbol{a'_{2,1}}\xi_1' + \boldsymbol{a'_{2,2}}\xi_2' + \boldsymbol{a'_{2,3}}\xi_3'$, $\boldsymbol{a_2''} = \boldsymbol{a''_{2,1}}\xi_1' + \boldsymbol{a''_{2,2}}\xi_2' + \boldsymbol{a''_{2,3}}\xi_3' \in R_2$. Then,

$$\Phi(\boldsymbol{y} + \boldsymbol{r}) = (\boldsymbol{a_0'} + \boldsymbol{a_0''}, \boldsymbol{a'_{1,1}} + \boldsymbol{a''_{1,1}}, \boldsymbol{a'_{1,2}} + \boldsymbol{a''_{1,2}}, \boldsymbol{a'_{2,1}} + \boldsymbol{a''_{2,1}}, \boldsymbol{a'_{2,2}} + \boldsymbol{a''_{2,2}}, \boldsymbol{a'_{2,3}} + \boldsymbol{a''_{2,3}})$$

$$= (\boldsymbol{a_0'}, \boldsymbol{a'_{1,1}}, \boldsymbol{a'_{1,2}}, \boldsymbol{a'_{2,1}}, \boldsymbol{a'_{2,2}}, \boldsymbol{a'_{2,3}}) + (\boldsymbol{a_0''}, \boldsymbol{a''_{1,1}}, \boldsymbol{a''_{1,2}}, \boldsymbol{a''_{2,1}}, \boldsymbol{a''_{2,2}}, \boldsymbol{a''_{2,3}})$$

$$= \Phi(\boldsymbol{y}) + \Phi(\boldsymbol{r}), \text{ and}$$

$\Phi(\rho \boldsymbol{y}) = (\rho \boldsymbol{a'_0}, \rho \boldsymbol{a'_{1,1}}, \rho \boldsymbol{a'_{1,2}}, \rho \boldsymbol{a'_{2,1}}, \rho \boldsymbol{a'_{2,2}}, \rho \boldsymbol{a'_{2,3}}) = \rho \Phi(\boldsymbol{y})$, where $\rho \in F_q$. Then Φ is an F_q-linear map.

Since Φ is an F_q-linear map, we have $d_G(\boldsymbol{y}, \boldsymbol{r}) = w_G(\boldsymbol{y} - \boldsymbol{r}) = w_H\big(\Phi(\boldsymbol{y} - \boldsymbol{r})\big) = d_H(\Phi(\boldsymbol{y}), \Phi(\boldsymbol{r}))$. Therefore, Φ is an F_q-linear distance-preserving map. □

Theorem 4.2 Let $\boldsymbol{C}$ be an $F_qR_1R_2$-linear code of length $(\alpha_0, \alpha_1, \alpha_2)$ with d_G. Then $\Phi(\mathbf{C})$ is an $[\alpha_0 + 2\alpha_1 + 3\alpha_2, k, d_H]$ linear code over F_q, where $d_G = d_H$ and k is the dimension of $\Phi(\mathbf{C})$.

Definition 4.3 Let $\omega = \left(a_1, a_2, \ldots, a_{k_1}, b_1, b_2, \ldots, b_{k_2}, c_1, c_2, \ldots, c_{k_3}\right) \in F_q^{\alpha_0 k_1} F_q^{\alpha_1 k_2} F_q^{\alpha_2 k_3} = \Gamma$, where $a_i \in F_q^{\alpha_0}$, for $i = 1,2,\ldots,k_1$, $b_j \in F_q^{\alpha_1}$ for $j = 1,2,\ldots,k_2$, $c_s \in F_q^{\alpha_2}$ for $s = 1,2,\ldots,k_3$. Let

$$\eta : \Gamma \longrightarrow \Gamma$$

$$\omega \longmapsto \mu$$

where $\mu = (\tau_{\lambda_0}^{\sigma_t}(a_1), \tau_{\lambda_0}^{\sigma_t}(a_2), \ldots, \tau_{\lambda_0}^{\sigma_t}(a_{k_1}), \tau_{\lambda_{1,1}}^{\sigma_t}(b_1), \tau_{\lambda_{1,2}}^{\sigma_t}(b_2), \ldots, \tau_{\lambda_{1,k_2}}^{\sigma_t}(b_{k_2}), \tau_{\lambda_{2,1}}^{\sigma_t}(c_1), \tau_{\lambda_{2,2}}^{\sigma_t}(c_2), \ldots, \tau_{\lambda_{2,k_3}}^{\sigma_t}(c_{k_3}))$, $\tau_{\lambda_0}^{\sigma_t}$ is a skew σ_t-λ_0 constacyclic shift, $\tau_{\lambda_{1,j}}^{\sigma_t}$ is a σ_t-$\lambda_{1,j}$ constacyclic shift for $j = 1,2,\ldots,k_2$, $\tau_{\lambda_{2,w}}^{\sigma_t}$ is a σ_t-$\lambda_{2,w}$ constacyclic shift for $w = 1,2,\ldots,k_3$.

Then a code E of length $(\alpha_0 k_1, \alpha_1 k_2, \alpha_2 k_3)$ in Γ is called a mixed generalized skew quasi $(\sigma_t, \sigma_t, \sigma_t)$-$(\lambda_0, \lambda_{1,j}, \lambda_{2,w})$ constacyclic code of index (k_1, k_2, k_3) if $\eta(E) = E$.

Proposition 4.4 Let Φ be the Gray map, $\tau_{\lambda_0,\lambda_1,\lambda_2}^{\sigma_t,\delta_1,\delta_2}$ be $(\sigma_t, \delta_1, \delta_2)$-$(\lambda_0, \lambda_1, \lambda_2)$ constacyclic shift over $F_q^{\alpha_0} R_1^{\alpha_1} R_2^{\alpha_2}$. Then $\Phi \tau_{\lambda_0,\lambda_1,\lambda_2}^{\sigma_t,\delta_1,\delta_2} = \eta \Phi$.

Proof Let $(a_0, a_1, \ldots, a_{\alpha_0 - 1}, b_0, b_1, \ldots, b_{\alpha_1 - 1}, c_0, c_1, \ldots, c_{\alpha_2 - 1}) \in F_q^{\alpha_0} R_1^{\alpha_1} R_2^{\alpha_2}$, where each $b_i = r_{1,1}^i \xi_1 + r_{1,2}^i \xi_2$ for $i = 0,1,\ldots,\alpha_1 - 1$, $c_j = r_{2,1}^j \xi'_1 + r_{2,2}^j \xi'_2 + r_{2,3}^j \xi'_3$ for $j = 0,1,\ldots,\alpha_2 - 1$, $\lambda_1 = \lambda_{1,1}\xi_1 + \lambda_{1,2}\xi_2$ and $\lambda_2 = \lambda_{2,1}\xi'_1 + \lambda_{2,2}\xi'_2 + \lambda_{2,3}\xi'_3$.

Then
$\Phi(\tau_{\lambda_0,\lambda_1,\lambda_2}^{\sigma_t,\delta_1,\delta_2}(a_0, a_1, \ldots, a_{\alpha_0 - 1}, b_0, b_1, \ldots, b_{\alpha_1 - 1}, c_0, c_1, \ldots, c_{\alpha_2 - 1})) = \Phi((\lambda_0 \sigma_t(a_{\alpha_0 - 1}), \sigma_t(a_0),$

$\ldots, \sigma_t(a_{\alpha_0 - 2}), \lambda_1 \delta_1(b_{\alpha_1 - 1}), \delta_1(b_0), \ldots, \delta_1(b_{\alpha_1 - 2}), \lambda_2 \delta_2(c_{\alpha_2 - 1}), \delta_2(c_0), \ldots, \delta_2(c_{\alpha_2 - 2})) =$

$(\lambda_0 \sigma_t(a_{\alpha_0 - 1}), \sigma_t(a_0), \ldots, \sigma_t(a_{\alpha_0 - 2}), \lambda_{1,1} \sigma_t(r_{1,1}^{\alpha_1 - 1}), \sigma_t(r_{1,1}^0), \ldots, \sigma_t(r_{1,1}^{\alpha_1 - 2}), \lambda_{1,2} \sigma_t(r_{1,2}^{\alpha_1 - 1}),$

$\sigma_t(r_{1,2}^0), \ldots, \sigma_t(r_{1,2}^{\alpha_1 - 2}), \lambda_{2,1} \sigma_t(r_{2,1}^{\alpha_2 - 1}), \sigma_t\left(r_{2,1}^0\right), \ldots, \sigma_t\left(r_{2,1}^{\alpha_2 - 2}\right), \lambda_{2,2} \sigma_t\left(r_{2,2}^{\alpha_2 - 1}\right), \sigma_t\left(r_{2,2}^0\right), \ldots,$

$\sigma_t\left(r_{2,2}^{\alpha_2 - 2}\right), \lambda_{2,3} \sigma_t\left(r_{2,3}^{\alpha_2 - 1}\right), \sigma_t\left(r_{2,3}^0\right), \ldots, \sigma_t\left(r_{2,3}^{\alpha_2 - 2}\right)) \ldots (1)$

On the other hand,

$$\eta\big(\Phi(a_0,a_1,\ldots,a_{\alpha_0-1},b_0,b_1,\ldots,b_{\alpha_1-1},c_0,c_1,\ldots,c_{\alpha_2-1}\big)$$
$$=\eta\big(a_0,a_1,\ldots,a_{\alpha_0-1},r_{1,1}^0,r_{1,1}^1,\ldots,r_{1,1}^{\alpha_1-1},r_{1,2}^0,\ldots,r_{1,2}^{\alpha_1-1},r_{2,1}^0,\ldots,r_{2,1}^{\alpha_2-1},r_{2,2}^0,\ldots,r_{2,2}^{\alpha_2-1},r_{2,3}^0,\ldots,r_{2,3}^{\alpha_2-1}\big)$$

$$(\lambda_0\sigma_t\big(a_{\alpha_0-1}\big),\sigma_t(a_0),\ldots,\sigma_t\big(a_{\alpha_0-2}\big),\lambda_{1,1}\sigma_t\big(r_{1,1}^{\alpha_1-1}\big),\sigma_t\big(r_{1,1}^0\big),\ldots,\sigma_t\big(r_{1,1}^{\alpha_1-2}\big),\lambda_{1,2}\sigma_t\big(r_{1,2}^{\alpha_1-1}\big),$$

$$\sigma_t\big(r_{1,2}^0\big),\ldots,\sigma_t\big(r_{1,2}^{\alpha_1-2}\big),\lambda_{2,1}\sigma_t\big(r_{2,1}^{\alpha_2-1}\big),\sigma_t\big(r_{2,1}^0\big),\ldots,\sigma_t\big(r_{2,1}^{\alpha_2-2}\big),\lambda_{2,2}\sigma_t\big(r_{2,2}^{\alpha_2-1}\big),\sigma_t\big(r_{2,2}^0\big),\ldots,$$

$$\sigma_t\big(\mathrm{r}_{2,2}^{\alpha_2-2}\big),\lambda_{2,3}\sigma_t\big(r_{2,3}^{\alpha_2-1}\big),\sigma_t\big(r_{2,3}^0\big),\ldots,\sigma_t\big(r_{2,3}^{\alpha_2-2}\big)) \ \ldots \ (2)$$

Since $(1)=(2)$, we have $\Phi\tau_{\lambda_0,\lambda_1,\lambda_2}^{\sigma_t,\delta_1,\delta_2}=\eta\Phi$. □

Theorem 4.5 Let $\boldsymbol{C}$ be a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code of length $(\alpha_0,\alpha_1,\alpha_2)$ over $F_qR_1R_2$. Then the Gray image of $\boldsymbol{C}$ is a mixed generalized skew quasi $(\sigma_t,\sigma_t,\sigma_t)$-$(\lambda_0,\lambda_{1,1},\lambda_{1,2},\lambda_{2,1},\lambda_{2,2}\lambda_{2,3})$ constacyclic code of index (1,2,3).

Proof Let $\boldsymbol{C}$ be a skew $(\sigma_t,\delta_1,\delta_2)$-$(\lambda_0,\lambda_1,\lambda_2)$ constacyclic code of length $(\alpha_0,\alpha_1,\alpha_2)$ over $F_qR_1R_2$. So $\tau_{\lambda_0,\lambda_1,\lambda_2}^{\sigma_t,\delta_1,\delta_2}(\boldsymbol{C})=\boldsymbol{C}$. We have $\Phi(\tau_{\lambda_0,\lambda_1,\lambda_2}^{\sigma_t,\delta_1,\delta_2}(\boldsymbol{C}))=\Phi(\boldsymbol{C})$. By using Proposition 4.4, we have $\Phi(\tau_{\lambda_0,\lambda_1,\lambda_2}^{\sigma_t,\delta_1,\delta_2}(\boldsymbol{C}))=\eta(\Phi(\boldsymbol{C}))=\Phi(\boldsymbol{C})$. Therefore $\Phi(\boldsymbol{C})$ is a mixed generalized skew quasi $(\sigma_t,\sigma_t,\sigma_t)$-$(\lambda_0,\lambda_{1,1},\lambda_{1,2},\lambda_{2,1},\lambda_{2,2}\lambda_{2,3})$ constacyclic code of index (1,2,3). □

5. Conclusion

This work searches the skew constacyclic codes over a mixed alphabet. The duals of them determined. By introducing a Gray map, the Gray images of them are obtained. Thanks to this, optimal codes over F_q can be obtained by using MAGMA.

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