Article

Quantum information, Complexity & Spacetime

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Abstract

Connections between quantum entanglement and complexity are examined in the context of the vacuum and black holes. The growth in quantum complexity is argued to approximate the distribution of prime number according to the Riemann ζ-function.

Keywords: Quantum information, complexity, spacetime, quantum entanglement, black hole, prime number, Riemann zeta function.

1 Spacetime from squeezed vacuum states

Spacetime from Jacobson [1]to Van Raamsdonk [2] is argued to be built from quantum states and entanglement. Similarly the singularity of a black hole is a region where spacetime appears to end and geodesics end. It is then reasonable consider the possibility that spacetime may emerge from quantum states or fields that have no explicit reference to some embedding spacetime. The geometry of quantum entanglement itself may be fabric that constructs spacetime.

Entanglement between two EPR pairs may be conserved in spacetime. This could even be with a black hole. Alice may take her EPR pair into the black hole and may be able to receive a teleportation of states from Bob outside. The converse is not so apparent, for Bob loses sight of Alice and her EPR pair due to red-shift. Bob only finds an entanglement with the black hole. A volume in space has a boundary with constant area, which the Weyl curvature may distend. In phase space this is a measure of the distention of phase space volume of a system. If entanglement is conserved, then this phase space volume is conserved. This is a squeezed state.

Mutual information of two states X and Y is $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$[3]. It is symmetric with $I(X;Y) = I(Y;X)$, which is similar to a metric. If we assume mutual information of a closed quantum system is conserved under differentiation with proper time $\frac{dI(X;Y)}{ds} = 0$, formally the same as $\frac{dg(U,V)}{ds} = \nabla g(U,V)\frac{dU}{ds} = 0$, the constancy of the metric $\nabla g(U,V) = 0$ is equal to the constancy of mutual information. Mutual information of a quantum system is a measure of entanglement, and the constancy of quantum entanglements is equivalent to the covariant constancy of the metric.

Mutual information may be expressed as $I(X;Y) = H(X) + H(Y) - H(X,Y)$ for the joint entropy $H(X,Y) = H(X|Y) + H(Y|X)$. This then means $I(X;Y) = H(X,Y) - H(X|Y) - H(Y|X)$[3]. We can see this with respect to the Shannon-von Neumann information theory $S = -\sum_n p_n\ log(p_n)$ or $S = -Tr[\rho\ log(\rho)]$. The trace operation Tr_x here is a summation of $x \in X$ so mutual information is

$$I(X;Y) = Tr_xTr_y[p(x,y)\ log\frac{p(x,y)}{p(x)p(y)}$$

$$= Tr_xTr_y\left[p(x,y)\ log\frac{p(x,y)}{p(x)}\right] - Tr_xTr_y\left[p(x,y)\ log\ p(y)\right],$$

where we may continue to see Shannon information for joint probabilities as

$$I(X;Y) = Tr_xTr_y\left[p(x,y)\ log\ p(x,y)\right] - Tr_xTr_y\left[p(x,y)log\ p(x)\right]$$

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$$+\ Tr_x Tr_y\left[p(x,y)(log\ p(x)\ +\ log\ p(y))\right],$$

Which is $H(X,Y)\ -\ H(X)\ \ -\ H(Y)$. Ignoring this last step mutual information is then

$$Tr_x\left[p(x)H(Y|x)\right]\ -\ Tr_y\left[p(y)\ log\ p(y)\right]\ =\ -H(Y|X)\ +\ H(Y).$$

Mutual information emerges from relative entropy.

The mutual information between two states X and Y is

$$S(X;Y)\ =\ S(\rho_{XY}||\rho_X\otimes\rho_Y)\ =\ Tr[\rho_{XY}\ log\rho_{XY}]\ -\ Tr[\rho_X\otimes\rho_Y\ log\rho_X\otimes\rho_Y],$$

which by virtual of the logarithm $log\rho_{XY}\simeq\ log\rho_X\otimes\rho_Y$ the mutual information is approximately

$$S(X;Y)\ \simeq\ Tr[(\rho_{XY}\ -\ \rho_X\otimes\rho_Y)log\rho_{XY}]\ \geq\ \frac{1}{2}|\rho_{XY}\ -\ \rho_X\otimes\rho_Y|^2.$$

$$\geq\ \frac{1}{2}\frac{(\langle XY\rangle\ -\ \langle X\rangle\langle Y\rangle)^2}{(\langle X\rangle\langle Y\rangle)^2},$$

which is the Fubini-Study metric [4] and the uncertainty principle. The mutual information is the metric information content of quantum entanglement.

Entanglement of states is a tensor product $\mathcal{H}_X\otimes\mathcal{H}_Y$ of dimensions N_X and N_Y. We have local unitary transformation $U_X\ \in\ SU(N_X)$ and $U_Y\ \in\ SU(N_Y)$ so $\psi'\ =\ U_X\otimes U_Y\psi$. We have that $SU(N_X)\times SU(N_Y)$, with dimension $N_X^2+N_Y^2\ -\ 2$ is a subset of $SU(N_X\ +\ N_Y)$ with dimension $(N_X+N_Y)^2\ -\ 1$ and $SU(N_XN_Y)$ with dimension $(N_XN_Y)^2\ -\ 1$. Given $N\ \leq\ min(N_X,N_Y)$, the pure state in $\mathcal{H}_X\otimes\mathcal{H}_Y$ is expressed as

$$|\psi\rangle\ =\ \sum_{i=1}^{N}\sqrt{\chi_i}|\alpha_i\rangle\otimes|\beta_i\rangle,$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are elements of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. A pure state can be represented as the double sum over a related bases on N_X and N_Y.

$$|\psi\rangle\ =\ \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}C_{ij}|\alpha_i\rangle\otimes|\beta_j\rangle.$$

The two representations are Schmidt decompositions of a state[5].

The general Schmidt decomposition [6] with a Schmidt vector $\vec{\sigma}\ =\ (0^1,\dots 0^{m_0},\ v_1^1,v_1^2,\dots,v_1^{m_1},v_2^1,\dots,v_2^{m_2},\dots,v_n^1,\dots,v_n^{m_n})$ such that the base and fibre of a bundle are respectively

$$\frac{U(N)}{U(m_0)}\times U(m_1)\times\dots U(m_n)$$

and

$$\frac{U(N)}{U(m_0)\times U(1)}\ =\ \mathbb{C}\mathbf{P}^{m_0}.$$

The base manifold is a flag manifold $F(m_0,m_1,\dots,m_n,\mathbb{C})$ and the local orbit space is defined by the product so $\mathcal{O}_{loc}\ =F(m_0,m_1,\dots,m_n,\mathbb{C})\times\mathbb{C}\mathbf{P}^{m_0}$.

Now consider the flag manifold $F_{1,2}(\mathbb{C}^4)$ of interest over $\mathbb{C}^4$. This is a total space for 4-fold entanglement. This is the sort of entanglement considered in [7] In this entanglement we have the possibility of 2 sets of 2 states entangled with each other, or 1 state entangled with the other 3. We then consider the flag manifold $F_{1,2}(\mathbb{C}^4)$. This then leads to

$$F_{1,2}(\mathbb{C}^4)\ =\ \frac{U(4)}{U(3)\times U(1)}$$

$$\simeq \frac{U(4)}{U(3)\times U(1)}\times\frac{U(3)}{U(2)} \simeq \mathbb{C}P^3\times\mathbb{C}^2\times U(1).$$

Since $F_1(\mathbb{C}^4) = \mathbb{C}^3$ and $F_2(\mathbb{C}^4) = G_2(\mathbb{C}^4)$, where $\mathbb{C}P^3$ and $G_2(\mathbb{C}^4)$ embed into $F_{1,2}(\mathbb{C}^4)$ as the double fibration of twister geometry,[8]

$$G_2(\mathbb{C}^4) \leftarrow^{\phi} F_{1,2}(\mathbb{C}^4)\ {}^{\chi}\rightarrow \mathbb{C}^3.$$

For $U(1)$ the orbit given by $ds = r\,U^{-1}dU$ gives the "volume" of the circle $Vol[U(1)] = 2\pi\sqrt{r}$. Now set $r = \frac{1}{2}$, the volume of $U(N)$ the follows as

$$Vol[U(N)] = Vol\left(\frac{U(N)}{U(1)^N}\right)Vol[U(1)]^N = 2^{N/2}\frac{\pi^{N(N+1)/2}}{\prod_{k=1}^N\Gamma(k)}$$

The volume of the Grassmannian $G_2(\mathbb{C}^4)$ is then $\frac{\pi^4}{72} \simeq 4.059$ and the volume of $\mathbb{C}P^3$ is $\frac{\pi^3}{6} \simeq 5.177$. This double fibration of twistor theory is a correspondence between lines in 3-dimensional space with 2-dimensional subspaces in 4-dimensions. This is an embedding of $Gr_2(\mathbb{C}^4)$ into projectivized wedge 2-products $\mathbf{P}(\Lambda^2\mathbb{C}^4)$ of 5 real dimensions[9].

The lines of $\mathbb{C}P^3$ are points in the 5-dimensional Klein quadric that is the projectivized $\mathbf{P}(\Lambda^2\mathbb{C}^4)$ of 5 real dimensions. This line, Plucker coordinates, is determined by a span of two vectors $p_i,\ p_j\ i < j$, so that $f_{ij} = [p_i,\ p_j]$ and that satisfy the quadratic relationship,

$$f_{12}f_{34} - f_{13}f_{24} + f_{14}f_{23} = 0.$$

The 6 dimensional $\mathbf{P}(\Lambda^2\mathbb{C}^4)$ define the set of 2-chains that above is for $p_i \rightarrow p_i + A_i$ means the Lagrangian term $F^{ab} * F_{ab} = 0$. This means there are no relevant edge states on the boundary of this 6 dimensional space. This is in an Euclideanized form equivalent as well to $SU(4) \simeq SO(6)$, where this loss of edge or boundary states means the relevant space of interest is $SO(4,2)/SO(4,1)$ which is the anti-de Sitter spacetime. The AdS_4 has correspondences with black hole physics, and is for this discussion important for $AdS \sim BH$ correspondence.

These are Schmidt states, and the entanglement entropy of the is determined by the equivalency of lines in $\mathbb{C}^4$ and points in the quadric and further the planes of $\mathbb{C}^4$ are spacetime[8]. These are the Schmidt states with the entanglement entropy determined by the von Neumann entropy of the reduce states, equivalently the Shannon information of the Schmidt states

$$S(\rho_X) = -\sum_{n=1}^N \rho_{Xi}\, ln\rho_{Xi},$$

with $S(\rho_X) \leq k\, ln(N)$. In this manner for the reduced state with 2 qubits the entropy is $\simeq 3.366\times 10^{-23}J/K$. Given this occurs at the Planck temperature $T_p = 1.417\times10^{32}K$ gives $E = ST = 4.769\times 10^9J = 1.972m_p$ or about 2 Planck units of mass-energy. For a quNit with N units or N qubits entropy is larger. A solar mass black hole has $S = 4\times10^{77}k$. However, if this is due to an entanglement between two states with a large N superposition of states, say a Schwarzschild black hole that is a perfectly entangled wormhole in a Penrose diagram, the entropy $\simeq k\, ln(N)$ is only $S \simeq 178k$. In this case the black hole would not only be completely entangled as a non-traversable wormhole, but analogous to two entangled hydrogen atoms with a large set of superposed Rydberg states. The entropy for an astrophysical black hole is largely due to the inability of an external observer to localize quantum bits and is large and bounded by the Bekenstein bound $S \leq A/4\ell_p^2$, $\ell_p = \ = \sqrt{G\hbar/c^3}$.

The emergence of spacetime from a Hilbert space of states requires an additional component. Clearly, the entropy is far larger than expected from the entanglement of superposed Rydberg-like states. For black hole physics this low entropy would correspond to a high temperature. This black hole would explode catastrophically.

There is the matter of quantum complexity. Two entangled black holes contain a shared interior that grows. The volume of this interior expands enormously, and it corresponds to the increase in quantum complexity. Given N qubit states the maximal entropy is $N\ ln2$, and for n quNit state with N quantum levels this is $Nk\ ln(n)$. However, maximal complexity is $\mathcal{C}\ =\ e^S$, which corresponds to the time of maximal complexity[10]. The recurrence time is the exponential of that. A black hole interior expands according to an increase in quantum complexity. This quantum complexity is from the perspective on an exterior observer a mixing of quantum states. The corresponding exponentiation of $S\ =\ 178k$ is then 2×10^{77} which in this numerical correspondence means the Bekenstein bound for exterior entropy is determined by the interior complexity. Then entropy is in effect the logarithm of complexity, where complexity may be enormously large. The time for a black hole to achieve maximum entropy is $\simeq\ log(N)$. For a solar mass black hole that is $\simeq\ 10^{-6}sec$. A black hole placed in an AdS_4 spacetime at equilibrium would be "eternal," and its quantum recurrence time would be on the order of $10^{10^{10}}$ years. Of course, a black hole in the observable universe will quantum evaporate long before this time.

Quantum complexity is vast, yet this does not contribute in a summation to the number of states or the energy of the black hole. Quantum complexity in fact can double with the introduction of a single qubit. There are then a vast number of state configurations that must have the same eigenvalues, say with the Hamiltonian, in a huge degeneracy. This would hold for the vacuum state. Quantum complexity obeys the logistic equation, which has a sigmoid growth curve. It has analogues to the saturation of a population by a pandemic. The growth of complexity is a logistic curve that approximates the log integral function $Li(x)\ = \int_0^x dx/ln(x)$ for $x\ >>\ 1$. An explicit calculation may be found here [11]. Because $li(x)$ approximates the distribution of primes numbers this leads to the prospect that the Riemann ζ-function is a representation of the supersymmetric Hamiltonian that annuls the vacuum state.

This will fit into conformal supersymmetry. The dilatation operator $D\ =\ x\frac{d}{dx}$ can be the generator (derivation) of an operator such as $\sum_n(-1)^{n+1}ne^D$ or

$$\mathcal{O}\ =\ \sum_{n=1}^{\infty}(-1)^{n+1}e^{ln(x)\ D}$$

where the inverse operator must then be

$$\mathcal{O}^{-1}\ =\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{ln(x)\ D}$$

It is not hard to see this operator acts on x^{-s} to give $\mathcal{O}x^{-s}\ = (1\ -\ 2^{1-s})\zeta(s)x^{-s}$ and its inverse $\mathcal{O}^{-1}x^{-s}\ = (1\ -\ 2^s)\zeta(1-s)x^{-s}$. Further, the lowering and raising operators can be shown to be

$$a_\omega\ =\ |x|^{-i\omega/2}\mathcal{O}|x|^{-i\omega/2},\ a^\dagger_\omega\ =\ |x|^{i\omega/2}\mathcal{O}^{-1}|x|^{i\omega/2},$$

where upon the supersymmetric Hamiltonians $H_-\ =\ a^\dagger_\omega a_\omega$ and $H_+\ =\ a_\omega a^\dagger_\omega$ or

$$H_-\ =\ |x|^{i\omega/2}\mathcal{O}^\dagger\mathcal{O}|x|^{-i\omega/2}$$
$$H_+\ =\ |x|^{-i\omega/2}\mathcal{O}\mathcal{O}^\dagger|x|^{i\omega/2}.$$

The commutator of these is zero, which is evident from the fact they are constructed from the same dilatation operator. A matrix of these on the vacuum state is zero, in line with supersymmetric annulment of the vacuum state[12]. In this way quantum complexity is a form of conformal supersymmetry, where the vast number of vacuum state configurations are annulled by a supersymmetry Hamiltonian with a Riemann ζ-function representation.

H_+ and H_- define a form of px and xp according to their difference and summation. The commutator of these is zero, since they are constructed from the dilatation operator. In this way quantum complexity is a form of conformal supersymmetry, where the vast number of vacuum state configurations are annulled by a supersymmetry Hamiltonian with a Riemann ζ-function representation.

Received November 4, 2023; Accepted December 19, 2023

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