Exploration

TGD Inspired Model for Freezing in Nano Scales

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Abstract

Freezing is a phase transition, which challenges the existing view of condensed matter in nanoscales. In the TGD framework, quantum coherence is possible in all scales and gravitational quantum coherence should characterize hydrodynamics in astrophysical and even shorter scales. The hydrodynamics at the surface of the planet such as Earth the mass of the planet and even that of the Sun should characterize gravitational Planck constant h_{gr} assignable to gravitational flux tubes mediating gravitational interactions. In this framework, quantum criticality involving $h_{eff} = nh_0 > h$ phases of ordinary matter located at the magnetic body (MB) and possibly controlling ordinary matter, could be behind the criticality of also ordinary phase transitions.

In this article, a model inspired by the finding that the water-air boundary involves an ice-like layer. The proposal is that also at criticality for the freezing a similar layer exists and makes possible fluctuations of the size and shape of the ice blob. At criticality the change of the Gibbs free energy for water would be opposite that for ice and the Gibbs free energy liberated in the formation of ice layer would transform to the energy of surface tension at water-ice layer.

This leads to a geometric model for the freezing phase transition involving only the surface energy proportional to the area of the water-ice boundary and the constraint term fixing the volume of water. The partial differential equations for the boundary surface are derived and discussed.

If $\Delta P = 0$ at the critical for the two phases at the boundary layer, the boundary consists of portions, which are minimal surfaces analogous to soap films and conformal invariance characterizing 2-D critical systems is obtained. Clearly, 3-D criticality reduces to rather well-understood 2-D criticality. For $\Delta P \neq 0$, conformal invariance is lost and analogs of soap bubbles are obtained.

In the TGD framework, the generalization of the model to describe freezing as a dynamical time evolution of the solid-liquid boundary is suggestive. An interesting question is whether this boundary could be a light-like 3-surface in $M^4 \times CP_2$ and thus have a vanishing 3-volume. A huge extension of ordinary conformal symmetries would emerge.

1 Introduction

Freezing is a phase transition, which challenges the existing view of condensed matter in nanoscales. For this reason it is interesting to look whether TGD could say something interesting about this phenomenon.

In the TGD framework, quantum coherence is possible in all scales and gravitational quantum coherence should characterize hydrodynamics in astrophysical and even in small scales [27, 31, 30]. The hydrodynamics at the surface of planet such as Earth should have mass of planet and even that of Sun should characterize gravitational Planck constant h_{gr} [3] [20, 21, 22] [24] assignable to gravitational flux tubes mediating gravitational interactions. In this framework, quantum criticality involving $h_{eff} = nh_0 > h$ phases of ordinary matter located at the MB and possibly controlling ordinary matter, could be behind the criticality of also ordinary phase transitions.

1.1 Freezing inside porous structures

The stimulus for considering freezing phenomenon came from a discussion of what happens in the freezing of water inside porous structure such as concrete. The freezing of conrete is of high interest for practical reasons. The ordinary freezing involves a reduction of temperature, which is above the criticality to the

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critical value T_{cr}. In the case of water, the temperatures slightly above T_{cr} lead to an increase of the volume that can can destructive effects on a porous material.

The porous structures like concrete have sizes in the size range of nanoparticles between 1 t 100 nm. The freezing is known to occur at considerably lower critical temperature which can be as low as -70 °C than ordinary freezing and to be a very slow process. Somehow this should relate to surface tension which carries a lot energy and to the small volume of pore implying that the large volume limit of thermodynamics does not apply as such.

Are the pores filled with water completely or only partially? From [9, 8] (`https://cutt.ly/cCVWKXx` and `https://cutt.ly/2C1GZHR`) one learns that the pores are partially filled with water so that there is also a gas phase present.

What motivates the interest on the physics of porous materials, is that the hydrodynamic phenomena in nano scales are hot topics of the recent condensed matter physics. Since TGD predicts all kinds of new quantum phenomena in these scales, it is interesting to see whether the TGD view could provide some new insights on the phenomenon of freezing.

1.2 The surface of water contains an ice-like layer

There is quite recent evidence that the surface of water bounded by air contains an ice-like layer consisting of 2-3 molecular layers [4] (`https://cutt.ly/DCVWM6C`). Second popular article telling that the boundary layer is neither water of ice and is conducting [5] (`https://cutt.ly/KC9Q2EA`).

Could the water at the surface freeze and liberate free energy as essentially thermal energy of motion, which is transformed to the energy of surface tension associated with the ice layer? This would explain what surface tension is at the fundamental level.

The ice layer at the surface would be analogous to a metal foil. Metal foils are unstable against warping, which means stretching without bending so that the induce metric remains flat ($z = z(x)$,$g_{xx} = 1 + (dz/dx)^2, g_{yy} = 1$ $\sqrt{g_2} = \sqrt{1 + (dz/dx)^2}$. Could the simplest surface waves of water be essentially warping waves in which the area increases and involves therefore phase transition creating more ice at the surface layer.

This would require that the surface of water is at criticality. In the TGD framework, this would correspond to quantum criticality and I have indeed proposed that at least some boundary layers involve membrane like structures at quantum criticality at the level of the MB of the system [28, 29]. Light-like boundaries of space-time surface define an analogous but not equivalent proposal. The quantum criticality would be essential for the ability of the water volume to change its shape while preserving its volume (volume preserving flow combined with a phase transitions occurring at the boundary layers.)

The temperature at the surface layer would be considerably higher than freezing temperature. Can one regard this phase as super-heated ice or some kind of quantum ice with long range correlations? Could one think that hydrogen bonds create long range order, which solidifies the boundary layer above the normal freezing temperatures. Here the notion of ordered water proposed to be associated with living systems such as DNA strand is suggestive.

The fourth phase of water, proposed by Pollack [11, 10, 13, 12], is a good candidate for this phase. This phase is formed in the presence of a gel phase and consists of hexagonal layers with an effective $H_{1.5}O$ stoichiometry.

TGD leads to a model of this phase in terms of the MB carrying dark protons transferred from so called exclusion zones (EZs), which are negatively charged and have properties suggesting time reversal at the level of the MB of the system [23] . For instance, EZs seem to dissipate in the reverse time direction [25, 26].

The fourth phase of water would differ from the ordinary water inside the nanopores. For instance, the freezing temperature would be much lower than for the ordinary water.

1.3 Immersed freezing and contact freezing

Freezing can occur as immersion freezing or contact freezing. In immersion freezing [7] (https://cutt.ly/bCVEPqT), which is the dominating mode for freezing, the growing region of ice is inside a possibly supercooled water volume. In the contact freezing [D2] (https://cutt.ly/BCVEHZj), the collision of a volume of possibly supercooled water with another object initiates the freezing. That is occurs could be understood from criticality.

Why contact freezing occurs much faster than immersion freezing is not well-understood [6] (https://cutt.ly/6CVEM9Q). Could the contact freezing start at the entire area of the outer surface of water blob and proceed to the interior and transform the liberated thermal kinetic energy to the energy of surface tension of the boundary layer between liquid and solid. If the boundary layer is and ice-like quantum coherent structure, the coherent freezing would be natural and occur coherently for the entire boundary layer. In the case of immersion freezing, temperature fluctuations imply that the ice nuclei can increase or decrease so that the process is less coherent and therefore slower.

Note that the freezing inside pores should be immersion freezing since collisions or other perturbations are not plausible inside pores. If this is the case, then the surface tension would be associated with the solid-liquid boundary inside pore.

1.4 TGD based model for freezing

In the sequal a model inspired by the finding that the water-air boundary involves an ice-like layer [4, 5] (see https://cutt.ly/DCVWM6C and https://cutt.ly/KC9Q2EA). The proposal is that also at criticality for the freezing a similar layer exists and makes possible fluctuations of the size and shape of the ice blob. At criticality the change of the Gibbs free energy for water would be opposite that for ice and the Gibbs free energy liberated in the formation of ice layer would transform to the energy of surface tension at water-ice layer.

This leads to a geometric model for the freezing phase transition involving only the surface energy proportional to the area of the water-ice boundary and the constraint term fixing the volume of water. This reduces freezing as a 3-D critical phenomenon to a 2-D critical phenomenon. The partial differential equations for the boundary surface are derived and discussed.

If $\Delta P = 0$ at the critical for the two phases at the boundary layer, the boundary consists of portions, which are minimal surfaces analogous to soap films and conformal invariance characterizing 2-D critical systems is obtained. For $\Delta P \neq 0$, conformal invariance is lost and analogs of soap bubbles are obtained.

In the TGD framework, the generalization of the model to describe freezing as a dynamical time evolution of the solid-liquid boundary is suggestive. An interesting question is whether this boundary could be a light-like 3-surface in $M^4 \times CP_2$ and thus have a vanishing 3-volume. A huge extension of ordinary conformal symmetries would emerge.

This leads to a proposal for the space-time description of the phase transition using basic TGD. If the light-like 3-surface corresponds to the condition $det(g_4) = 0$, the normal isometry currents appearing in the boundary conditions are finite. Kähler Chern-Simons term is necessary as a boundary part of the action. In consistency with the original belief given up later, it is possible to have space-time surfaces with boundaries.

2 TGD based model for freezing

In the following a TGD based model of freezing is developed by posing questions inspired by the empirical findings.

2.1 Can one assign a surface tension to the boundary between solid and liquid phase?

Can one assign a surface tension to the boundary layer between water and ice when the the boundary is at criticality?

If so, the increase of the size of the frozen volume inside (possibly supercooled) water near criticality increases the area of the boundary of the frozen volume between the two phases. If there is surface tension involved, energy is needed to increase the area.

The energy could come from the freezing proceeding by an addition of molecular thick boundary layers liberating the free energy as essentially thermal kinetic energy, which would transform to the energy assignable to the surface tension. Melting would be opposite to this process. If molecular layers are added, the liberated free energy (thermal energy) from the layer of water molecules is proportional to the area of the surface generated in this way as is also the energy of surface tension. Therefore one can have criticality and the area of of the surface between the two phases can fluctuate.

The boundary layer between the solid and liquid phases [4, 5], consisting of few molecule layers, would be critical. This motivates the proposal that at criticality the liberated Gibbs free energy (`https://cutt.ly/tC3flRn`), essentially thermal kinetic energy, identifiable as the heat of freezing equals the energy assignable to the surface tension.

The situation would be highly unstable due to temperature fluctuations. The volume could increase at some areas of the surface and decrease at other areas. The volume possessed by either phase would fluctuate as it indeed does at criticality. The situation is analogous to the interphase between water and air, which supports the view that the notion of water-ice surface tension indeed makes sense at criticality.

If the the increments of Gibbs free energy are the same apart at the critical temperature, temperature fluctuations make the situation highly critical. If the energy liberated in the freezing overcomes the energy of surface tension the additional heat tends to induce melting. If the energy is below that energy of surface tension, freezing is prevented.

2.2 Why the freezing temperature reduced with the decreasing pore size?

The freezing temperature inside nano pores (down to -70 °C) is considerably lower than the freezing temperature for a large volume of fluid. It would be tempting to assume that the freezing temperature at the ice-water boundary layer determines the freezing temperature T_{cr} and that for some reason T_{cr} is lower than at the limit of large volume.

1. The intuitive idea is that at criticality a layer of ice molecules can be generated as water molecules freeze [4, 5]. Also the reverse transition is possible. The freezing would proceed as new layers would be added. The liberated thermal kinetic energy of liquid molecules would go to the energy assignable to the layer.

   The recent finding that the surface of water is accompanied by few molecule layers behaving like ice suggests that the additional energy of the layer has an interpretation as surface tension. At critical temperature T_{cr} this layer is highly dynamic and its size can vary.

   At the criticality, the liberated energy per molecule equals the energy of surface tension per molecule. If the liberated energy is higher, it melts the molecular ice layer formed in this way. If the liberated energy is smaller, the formation of the layer is not possible. Small fluctuations of T around T_{cr} affect the shape and volume of the ice region in the ice phase, which are therefore highly sensitive to fluctuations of T.

2. T_cr should correlate with the pore size L. Gibbs free energy seems to be the correct thermodynamical function if only the change for number of ice molecules matters. Both ΔG_{water} and $\Delta G_{ice} = \Delta E_s$ could depend on T and pore size L. The naive first guess is that in nano scales the presence of

hydrogen bonded regions, analogous to seeds for the formation of ice, would increase the freezing temperature. However, just the opposite occurs.

One can argue that the increase of the number of hydrogen bonded structures already representing ice-like structures in the ice-water boundary layer reduces the size of ΔG_{water} liberated in the formation of a new boundary layer. Therefore ΔG_{water} decreases with the pore size L. Also the reduction of temperature reduces ΔG_{water}.

3. For instance, if $|\Delta G_{ice}| = \Delta E_s < |\Delta G_{water}|$ is true at the normal critical temperature and is not appreciably affected by the reduction of the temperature, the lowering of the temperature to T_{cr} cannot lead to to $\Delta E_s = \Delta G_{ice} = |\Delta G_{water}|$. Therefore no freezing would occur. On the other hand, the criticality at the infinite volume limit also gives $\Delta G_{ice} = \Delta G_{water}$ at the normal freezing temperature.

   A more realistic looking possibility is that $\Delta E_s = \Delta G_{ice}$ is of the form $a - b/L$ with $a = \Delta_{s,N}$ that is Δ_s at the normal freezing point at the infinite volume limit. Als a could have weak dependence on L As L is reduced below some critical value, ΔG_{ice} would decrease faster than $|\Delta G_{water}|$ and eventually one could have $\Delta G_{ice} = \Delta E_s = |\Delta G_{water}|$ at lowered T_{cr}.

   The core idea is that the freezing is not a 3-D phase transition but 2-D phase transition at the liquid-solid boundary which is critical. Critical temperature would characterize this boundary rather than the entire 3-D phase.

4. How could one understand the negative contribution b/L to Δ_s? Could one think that Coulomb energy is in question. Could a charge separation, analogous to that taking place in the Pollack effect, occur and give rise to a negative Coulomb interaction energy between dark protons at flux tubes and the negative charges in exclusion zone (EZ)? In short scales L this contribution would increase. This would mean that the new physics predicted by TGD would play a key role in freezing and presumably also in other phase transitions.

2.3 A more precise thermodynamical formulation using Gibbs free energy

Assume that one has $\Delta P = 0$ and $\Delta T = 0$ for the phase water-ice phase transition at criticality. If so, only the number of molecules in the two phases change and one has $\Delta N_{water} = -\Delta N_{ice}$.

1. Gibbs free energy $G = H - TS = F + pV$ is the proper thermodynamic function to describe the situation. One has $\Delta G = \Delta(H - TS) = S\Delta T + \sum \mu_i \Delta N_i + V\Delta P$. Let us assign index $i = 1 \rightarrow water$ to liquid phase and $i = 2 \rightarrow ice$ to ice so that one can define $G = G_{water} + G_{ice}$.

   For $\Delta P = 0$ and $\Delta T = 0$, one has $\Delta G = \mu_{water}\Delta N_{water} = \mu_{ice}\Delta N_{ice} = (\mu_{water} - \mu_{ice})\Delta N_{water}$. One cannot exclude the possibility $\Delta P \neq 0$.

2. At criticality, one has $\mu_{water} = \mu_{ice}$ if the new molecular layer assignable to the ice-water boundary is indeed analogous to that found to accompany the water-air boundary at criticality. This assumption is natural since at criticality the shape of ice regions is highly varying just like the shape of a water blob.

   If G_{ice} decreases with temperature (being analogous to the thermal energy of water molecules), the chemical potential μ_{water} decreases with the temperature. It is not obvious how the energy μ_{ice} per molecule assignable to surface tension depends on T and pore size L The first guess is that the dependence on T is weak. As proposed, the charge separation occurring in the Pollack effect could explain dependence on L as being caused by the attractive Coulomb potential.

3. Criticality at the interface means that water and ice molecules correspond to the same value of Gibbs free energy G so that their numbers N_i fluctuate but satisfy the condition $N_{water} + N_{ice} = constant$. This requires that $\Delta G_{water} < 0$ for water molecules identifiable as the thermal energy liberated in

the freezing is apart from sign equal to the increase ΔG_{ice} for ice molecules at criticality, and is identifiable as the increase of energy of the surface tension assignable to the additional area of the solid-liquid boundary layer. This requires $\mu_{water} = \mu_{ice}$ at criticality. Otherwise the phase transition cannot proceed.

If $\Delta G_{water} = \Delta N_{water}\mu_{water}$ and $\Delta E_s = \Delta G_{ice} = \Delta N_{ice}\mu_{ice}$, $\Delta N_{water} = -\Delta N_{ice}$, do not sum up to zero at the normal freezing temperature inside the pore, the transition does not occur until the temperature has been reduced to the critical temperature T_{cr} with $\Delta G_{water} = -\Delta G_{ice}$ so that one has $\mu_{water} = \mu_{ice}$.

If $\mu_{ice} > \mu_{water}$ is true at the normal critical temperature at infinite volume limit, the reduction of the temperature should reduce μ_{ice} faster than μ_{water} so that eventually $\mu_{water} = \mu_{ice}$ would hold true. This should be due to the finite pore size. This could be due to the $\mu_{ice} = a - b/L$ type dependence on pore size caused by the charge separation as part of protons of water molecules are transferred to the magnetic flux tubes in the Pollack effect.

3 A geometrical model for the ice-liquid system

In the following a simple purely geometric model for water-gas system and ice-liquid system in a finite volume such as pore is developed.

1. For a water-gas system, one assumes that the water phase has a fixed volume $V = V_0$ (incompressible flow) and there is a water-air boundary layer analogous to ice layer giving rise to surface tension by the proposed model. It is assumed that the boundary is critical in the sense that its area can increase or decrease without a change in the total free energy of the 3-D system. This is true if one has $\Delta G_{water} = \Delta G_{ice} = \Delta E_s$, where E_s is the energy associated to the surface tension assumed to be assignable to the water-air boundary at criticality. This assumption involves new physics.

   The interesting part of the free energy of the ice-water system is assumed to be associated with the surface tension at the boundary layer with a constant thickness measured as the number of water molecule layers. This part of energy is proportional the surface area of the layer in the case that the layer has constant thickness measures as number of water molecule layers.

   The relevant part of the Gibss free energy of the system in this case is given by

$$G_s = \sigma S + \Lambda(V - V_0) \ , \tag{3.1}$$

   where Λ isz Lagrange multiplier guaranteeing that the volume V of the entire system is fixed: $V = V_0$. The shape of the water blob can however vary. Without the volume constraing the variation would give as a solution minimal surface, which cannot be closed.

   Note that Λ does not depend on the coordinates of X^2 and its value is chosen in such a manner that the volume enclosed by X^2 equals to V_0.

2. The second model is for water-ice system inside a pore with volume V_{tot}. In this case, one can have several volumes V_i of ice phase and one can assume that the total volume of liquid is fixed $\sum V_i = V(water) = V(total) - V(gas)$. If only the boundary layers matter, one can treat each volume V_i separately and has

$$G_s = \sigma S + \Lambda(V - V_i) \ . \tag{3.2}$$

   Now the surface tension is assigned with the ice-water layer and it is assumed that it is at criticality also now so that one obtains large number of shapes for V_i.

The variation of G_s reduces to a variation of S and V to determine possible boundary solid-liquid boundary surfaces X^2.

1. The induced metric at X^2 is given by $g_{\alpha\beta} = g_{kl}\partial_\alpha x^k \partial_\beta x^l$. It is convenient to use Cartesian coordinates since in these coordinates one has $g_{kl} = \delta_{kl}$. One can always select the local representation of the surface in such a manner that two coordinate, say x, y serve as coordinates x^α ($x^1 = x, y^1 = y$) for the surface and the third coordinate z is given by $z = z(x, y)$. For a closed surface such as a sphere z is two-valued.

2. The use of Cartesian coordinates for 3-space implies that the formulas are not completely general: in particular, the expression of second fundamental form lacks terms coming from Riemann connection od 3-space E^3, which is non-vanishing for a general coordinate choice (such as spherical or cylindrical coordinates). The general formulas are obtained by replacing ordinary derivatives by covariant ones in appropriate places. The vector x^k appearing in the Gauss formula, is a vector field of E^3 and has a simple expression only in Cartesian coordinates. The index raising for x^k is performed using the flat metric $g_{kl} = \delta_{kl}$ of E^3.

   In Cartesian coordinates for E^3 index raising is a trivial operation $X^k = x_k$. A distinction is however made between these indices since this allows us to use Einstein's summation convention for repeated indices meaning that $A^k B_k$ therefore involves summation over k.

3.1 Derivation of the variational equations

The deduction of the equations for X^2 from the variational principle is rather straightforward but due to the non-linearity rather tedious.

3.1.1 Variation of the area

The variation of the area term S gives the following expression

$$
\begin{aligned}
&\delta S = \int_{X^2} \delta x^k Tr(H^k)\sqrt{g_2}dA \ \ , \\
&H^k \equiv g^{\alpha\beta} H^k_{\alpha\beta} \ \ , \\
&H^k_{\alpha\beta} = D_\beta(\partial_\alpha x^k) = \partial_\alpha \partial_\beta x^r P^k_r \ \ , \\
&P^k_r = g^k_r - g^{\mu\nu}\partial_\mu x^k \partial_\mu x_r \ \ .
\end{aligned}
\tag{3.3}
$$

Here $H^k_{\alpha\beta}$ is the second fundamental form defined as covariant derivatives of tangent vectors $\partial_\alpha x^k$. $H^k_{\alpha\beta}$ is orthogonal to the surface as the projection operator P^k_r projects to the normal space of the surface. dA is the coordinate area in the local coordinates of X^2, say $dA = dxdy$.

This gives the term $H^k = g^{\alpha\beta} H^k_{\alpha\beta}\sqrt{g_2}$ to the left hand side of equations for X^2. If the constraint term is absent, one obtains the equation of minimal surface:

$$
H^k = D_\beta(\partial_\alpha x^k) = g^{\alpha\beta} H^k_{\alpha\beta} = 0 \ \ . \tag{3.4}
$$

The equation states the conservation of momentum currents $j^{\alpha k} = \partial_\alpha x^k \sqrt{g_2}$.

The first interpretation is that the analog of acceleration for 2-D particle vanishes so that an analog of a geodesic line is in question. The second interpretation is as a non-linear geometrization of Laplace equation giving an analogy with electrostatics. The contribution from the volume consraint would give a non-line source term analogous to a density of electric charge.

As explained, closed minimal surfaces are not possible. It is however possible to have local regions which are minimal surfaces, say, the planar surface of a water blob. Physically the surface identifiable as pieces of crystal having planar faces and edges which meet at vertices are expected. These would correspond to surfaces, which possibly fail to be minimal surfaces at the edges serving as analogs of line charges. If the normal component of the conserverd current $j^{\alpha k}$ is continuous at the edge, one can say that the minimal surface equations hold true also at the edge.

3.1.2 Variation of the constraint term

The variation with respect to Λ (no dependence on the coordinates of X^2) gives rise to the constraint $V = V_0$. The variation of the volume V in the constraint term gives a source term to the right hand side of the minimal surface equation.

1. Gauss theorem allows to express the volume V as a surface integral

$$V = x^k n_k \sqrt{g_2} dA \ . \tag{3.5}$$

Here n_k is a unit normal vector for X^2 and expressible in terms of H^k

$$n^k = \frac{H^k}{\sqrt{H^r H_r}} \equiv \frac{H^k}{H} \ . \tag{3.6}$$

The unavoidable presence of the normal vector implies that the constraint term contains second derivatives. The naive expectation that the constraint terms give rise to third order partial differential equations. This expectation is in conflict with the intuitive expectations and is indeed wrong.

One can calculate n^k explicitly by taking two planar coordinates of E^3 as coordinates of X^2 so that one has $(x^1 = x, x^2 = y, x^3 = z(x,y))$. In these coordinates one has

$$H^k = (g^k_z - g^{\mu\nu}\partial_\mu x^k \partial_\nu z) g^{\alpha\beta}\partial_\alpha \partial_\beta z \ . \tag{3.7}$$

All components of H^k are proportional to $(\partial_\alpha \partial_\beta z)$, which completely disappears from the expression for the second fundamental form so that n^k reduces to the form

$$n^k = \frac{h^k}{\sqrt{h^r h_r}} \equiv \frac{h^k}{h} = \frac{h^k}{\sqrt{h^k h_k}} \ , \quad h^k = P^k_z \ . \tag{3.8}$$

This makes manifest the fact that only the equation for $m^3 = z$ is needed: this is implied by general coordinate invariance.

2. The variation δV can be written as

$$\begin{aligned}
\delta V &= \delta V_1 + \delta V_2 + \delta V_3, \\
\delta V_1 &= \int_{X^2} \delta x^k n_k \sqrt{g_2} dA \ , \\
\delta V_2 &= \int_{X^2} x^k \delta n_k \sqrt{g_2} dA \ , \\
\delta V_3 &= \int_{X^2} x^k n_k \delta(\sqrt{g_2}) dA \ .
\end{aligned} \tag{3.9}$$

3. δV_1 gives to the source term a contribution

$$X_1^k = n^k\sqrt{g_2} \quad . \tag{3.10}$$

having a direction normal to the surface.

4. The calculation of δV_2 requires the calculation of δn^k

$$\begin{aligned}
&\delta n^k = B_l^k \delta h^l \quad , \\
&B_l^k \equiv \frac{\partial (h^k/h)}{\partial h_l} = \frac{g_l^k}{h} - \frac{h^k h_l}{h^3} \quad , \\
&\delta h^l = \delta(P_s^l) = -\delta[g^{\mu\nu}\partial_\mu x^l \partial_\nu x_s] = 2g^{\mu\rho}g^{\nu\sigma}g_{mn}\partial_\sigma x^n \delta(\partial_\rho x^m)\partial_\mu x^l \partial_\nu x_s \\
&-g^{\mu\nu}[\delta(\partial_\mu x^l)\partial_\nu x_s + \partial_\mu x l \delta(\partial_\nu x_s)] \quad .
\end{aligned} \tag{3.11}$$

The outcome of partial integrations transforming variations of the partial derivatives of x^k to x^k can be expressed as operator action in which partial derivatives and from left to the part of the integrand multiplying the variation.

1. The δV_2 is proportional to δP_s^l. The contribution of a given term in the partial differential equations is written after $\rightarrow$:

$$\begin{aligned}
&\delta P_z^l = -\delta[g^{\mu\nu}\partial_\mu x^l \partial_\nu z] = 2g^{\mu\rho}[g^{\nu\sigma}g_{mn}\partial_\sigma x^n \delta(\partial_\rho x^m)\partial_\mu x^l \partial_\nu z] - g^{\mu\nu}[\delta(\partial_\mu x_l)\partial_\nu z + \partial_\mu x^l \delta(\partial_\nu z)] \\
&\rightarrow -2\Lambda \delta x^k D_\alpha [g^{\mu\alpha}g^{\nu\sigma}g_{kn}\partial_\sigma x^n \partial_\mu x^l \partial_\nu z O_l^z] + \delta x^k D_\alpha [g^{\alpha\nu}\partial_\nu z O_k^z] \quad , \\
&O^{zl} = x_k B^{kl}\sqrt{g_2} \quad .
\end{aligned} \tag{3.12}$$

2. The third term $\delta V_3 = \int_{X^2} x^k n_k \delta(\sqrt{g_2}) dA$ involves the variation of $\sqrt{g_2}$.

$$\begin{aligned}
&\delta\sqrt{g_2} = -\frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}\sqrt{g_2} = -g^{\alpha\beta}[g_{rs}\partial_\alpha x^r \delta(\partial_\beta x^s)]\sqrt{g_2} \\
&\rightarrow \Lambda \delta x^k D_\alpha [g^{\alpha\beta}g_{rs}\partial_\beta x^r B^{rl}h_l]\sqrt{g_2} \quad ,
\end{aligned} \tag{3.13}$$

Combining various terms one obtains the following equations for X^2.

$$\begin{aligned}
&D_\beta(g^{\alpha\beta}\partial_\beta x^k) = \frac{\Lambda}{\sigma}X^k \quad , \\
&X^k = X_1^k + X_2^k + X_3^k \quad , \\
&X_1^k = n^k \quad , \\
&X_i^k = D_\alpha(X_i^{k\alpha}), \text{for } i = 2,3 \quad , \\
&X_2^{k\alpha} = -2[g^{\mu\alpha}g^{\nu\sigma}g_{kn}\partial_\sigma x^n \partial_\mu x_l \partial_\nu z O^{zl} + g^{\alpha\nu}\partial_\nu z O^{zk}] \quad , \\
&X_3^{k\alpha} = g^{\alpha\beta}\partial_\beta x^k x^s B_{sl}h^l\sqrt{g_2} \quad , \\
&O^{zl} = x_k B^{kl}\sqrt{g_2} \quad .
\end{aligned} \tag{3.14}$$

3.1.3 Explicit form of equations

The equations can be written in two alternative forms

$$\begin{aligned} D_\alpha(g^{\alpha\beta}\partial_\beta x^k) &= \frac{\Lambda}{\sigma}(n^k + D_\alpha X^{k\alpha}) \ , \\ D_\alpha[g^{\alpha\beta}\partial_\beta x^k - \frac{\Lambda}{\sigma}X^{k\alpha}] &= \frac{\Lambda}{\sigma}n^k \ . \end{aligned} \tag{3.15}$$

One can argue that by the general coordinate invariance of equations at the level of X^2, only the equation for $x^3 = z$ is needed. The objection is that the nice form of equations is due to a choice of linear coordinates at both X^2 and E^3. Also the presence of $n_k m^k$ and n^k in the equations might mean that all 3 equations are necessary.

3.2 Various solution types

For $\Lambda = 0$ the equations reduce to minimal surface equations (`https://cutt.ly/eC3aQYM`). Note that in the TGD framework space-time surfaces in $H = M^4 \times CP_2$ as preferred extremals of action are conjectured to be minimal surfaces [28]. The fact that the second derivative terms vanish from the expression of n^k means that this limit involves no singular terms.

1. This option is favoured by the connection of the conformal invariance with 2-D criticality, which led to a very detailed understanding of the 2-D critical systems leading to the classification of critical systems in terms of criticality and conformal field theories [1]. Minimal surface equations for z indeed allow solutions as real or imaginary parts of analytic functions of complex coordinate z for X^2.

2. Minimal surface equations cannot hold everywhere since minimal surfaces are not closed. One can however consider gluing portions of minimal surfaces together along their boundaries serving as singularities. A natural condition is that the normal components of the currents $J^{\alpha k} = g^{\alpha\beta}D_\beta x^k$ are continuous at the discontinuity line. This could pose conditions to the angles between faces meeting at the edges, which could be somewhat analogous to the frames of soap films. These discontinuities would be analogous to cuts of analytic functions. The edges of planar faces of an ice crystal would provide an example of this kind of discontinuities.

3. The simplest solutions for the minimal surface equations would have interpretation as planar boundaries of crystals. More general crystals would be obtained by gluing together portions of curved minimal surfaces along edges. This could be perhaps tested experimentally.

4. One can criticize this picture. The value of Λ should determine the volume. For genuine minimal surfaces $\Lambda = 0$ means infinite volume. For piece-wise minimal surfaces the volume is finite and not determined by Λ but by the boundary conditions expressing the continuity of the normal component of $J^{\alpha k}$ along edges. A possible interpretation is that $\Lambda = 0$ corresponds to the thermodynamic infinite volume limit.

The second option is just the general equation and gives up the conformal invariance.

1. The equation $D_\alpha[g^{\alpha\beta}\partial_\beta x^k - (\Lambda/\sigma)X^{k\alpha}] = (\Lambda/\sigma)n^k$ contains right-hand side as a source term and is analogous to the equation $D_\alpha[g^{\alpha\beta}\partial_\beta x^k] = (\Delta P/\sigma)n^k$ for a soap bubble, where ΔP is pressure difference and σ is surface tension. Λ is analogous to a constant pressure difference ΔP.

   This kind of surface is highly analogous to a soap bubble and could for instance correspond to a spherical region of ice phase. Note that if soap bubbles and films involve an ice-like critical layer, they could be seen as solutions to the proposed equations.

2. For the conformally invariant option $\Lambda = \Delta P = 0$ would hold true at criticality since ΔP is a dimensional parameter and dimensional parameters should vanish at criticality and for conformal invariance. One might think that these solutions, in particular the surfaces X^2 consisting of planar faces, are some kind of limiting solutions when ΔP approaches zero.

3. The warped solutions were already mentioned. For them z depends on a single linear coordinate and they have a flat metric. The equations reduce to ordinary differential equations for z and Λ and are therefore easy to solve numerically. These solutions are not minimal surfaces. They could represent surface waves in water.

Concerning the model, it seems that the crucial thermodynamic question is whether $\Delta P = 0$ is true at criticality as the associated conformal invariance, requiring the absence of dimensional parameters, suggests.

4 What about phase transitions as dynamical phenomena describable using TGD proper?

Could one generalize the proposed thermodynamic model, which is actually a static model, to a genuinely thermodynamic model for the freezing or its reversal?

4.1 The first guess does not quite work

The first naive guess, inspired by TGD in which space-time is 4-D surface in $H = M^4xCP_2$, would be the replacement of X^2 with its orbit X^3 in 4-D Minkowski space M^4 and a generalization of free energy to what one might call a thermodynamic action.

1. The four-volume V_4 of the system would be fixed by a Lagrange multiplier term $\Lambda(V_4 - V_{4,0})$ and a generalization of 3-D Gauss law to 4-D situation would be used to express the 4-volume as integral $\int x^k n_k dV$ over X^3. The ends X_i^2 and X_f^2 of the 3-surface at times t_i and t_f would be fixed and not subject to variation whereas space-like boundaries would be varied.

   Also now the surface tension would appear as the coefficient of 3-volume of X^3 and the Lagrange multiplier Λ would have an interpretation as Δp. The thermodynamic action has dimensions of $\hbar$ as required. Stationary solutions would correspond to the extrema of Gibbs free energy. One would obtain the 2-D criticality and conformal invariance for these solutions.

2. What makes this so interesting is that for 3-surfaces X^3, which are light-like, and therefore have vanishing (and indeed minimal!) volume V_3, the induced metric is metrically 2-dimensional. This implies a huge extension of conformal symmetries and even the isometries form an infinite-dimensional group [33, 35]. Could these light-like 3-surfaces represent phase transitions as dynamical phenomena?

   This raises the question whether there any need for the 3-D volume action if light-likeness implies minimal surface property in the strongest possible form? This implies also that the mere light-likeness of X^3 might be enough. Could-likeness follow from some deeper principle. One can also ask, whether there any need for the 4-D volume constraint either.

3. What about extended conformal invariance? The effective 2-dimensionality plus the fact that 2-D surface X^2 as a spacelike section of X^3 always allows Kähler " structure with complex coordinate W suggests that extended conformal invariance is possible.

Consider now the objections against the naivest proposal.

1. There is a strong mathematical objection related to light-likeness. By a suitable choice of coordinates 3-D metric can be always made diagonal. Now this metric would be of the form $(g_{uu}, g_{W\overline{W}})$ since transversal degrees of freedom allow always Kähler metric. By light-likeness, one would have $g_{uu} = 0$ so that $g^{uu}H^{k_{uu}}\sqrt{g_3}$ would in general diverge and could be non-vanishing or ill-defined even if $H^k_{uu} = 0$ holds true. Therefore the 3-D volume term as thermodynamic action is not a promising idea.

2. The physical objection is that the light-like 3-surfaces X^3 of M^4 are unrealistic as evolutions of solid-liquid boundary: X^3 would represent an expanding light front.

3. The stability of light-like 3-surfaces is questionable. This suggests that they are such that their small variations cannot affect the light-likeness.

4.2 Could the phase transition have a space-time description at the level of basic TGD?

These objections suggest a modification of the naive proposal.

1. The basic observation, natural in the TGD framework, is that if one allows X^3 to be a surface of $H = M^4 \times CP_2$, the situation changes since light-likeness as 3-surface of H does not imply that the M^4 projection of X^3 expands with light-velocity. One could also have 3-D minimal surfaces in $M^4§^1 \subset M^4 \times CP_2$, where S^1 is rotating geodesic circle, with an E^3 projection, which is closed and has a finite size [28] so that the problem due to the infinite size of minimal surfaces might be solved.

   For instance, one might think that the light-like coordinate varies along a light-like geodesic in $M^4 \times S^1$ involving a rotation along the geodesic circle S^1. If the lightlike geodesic of $M^2 \times S^1$ has the form $(t, z, R\Phi) = K(w, \overline{w}) \times (\omega, k, \omega_1)U\ \omega^2 - k^2 - \omega_1^2) = 0$ its Minkowski projection corresponds to a sub-luminal velocity. The 2-surface in E^3 corresponds to $z = K(w, \overline{w})$ and can be closed if K is two-valued.

2. One should add to the action a 4-D part, say volume term or a more general term. If the description is a fundamental quantum description for the phase transition, one can ask whether one should give up the interpretation as a thermodynamic action and use the action defining classical TGD. This action contains a 4-D volume term and the Kähler action. This action would give rise to a boundary term representing a normal flow of isometry currents through the boundary. The boundary conditions would replace the minimal surface equations.

3. There could also be a 3-D part in the action but by light-likeness it cannot depend on induced metric. The Chern-Simons term for the Kähler action is the natural choice. Twistor lift suggests that it is present also in M^4 degrees of freedom. Topological field theories utilizing Chern-Simons type actions are standard in condensed matter physics, in particular in the description of anyonic systems, so that the proposal is not so radical as one might think. One might even argue that in anyonic systems, the fundamental dynamics of the space-time surface is not masked by the information loss caused by the approximations leading to the field theory limit of TGD.

   Boundary conditions would state that the normal components of the isometry currents are equal to the divergences of Chern-Simons currents and in this way guarantee conservation laws. In CP_2 degrees of freedom the conditions would be for color currents and in M^4 degrees of freedom for 4-momentum currents.

4. This picture would conform with the general view of TGD. In zero energy ontology (ZEO) [25, 26] phase transitions would be induced by macroscopic quantum jumps at the level of the magnetic body (MB) of the system. In ZEO, they would have as geometric correlates classical deterministic time evolutions of space-time surface leading from the initial to the final state [32]. The findings of Minev et al provide [2] lend support for this picture.

4.3 Light-like 3-surfaces from $det(g_4) = 0$ condition

How the light-like 3- surfaces could be realized?

1. A very general condition considered already earlier is the condition $det(g_4) = 0$ at the light-like 4-surface. This condition means that the tangent space of X^4 becomes metrically 3-D and the tangent space of X^3 becomes metrically 2-D. In the local light-like coordinates, $(u, v, W, \overline{W})$ $guv = g_{vu})$ would vanish (g_{uu} and g_{vv} vanish by definition.

   Could $det(g_4) = 0$ and $det(g_3) = 0$ condition implied by it allow a universal solution of the boundary conditions? Could the vanishing of these dimensional quantities be enough for the extended conformal invariance?

2. 3-surfaces with $det(g_4) = 0$ could represent boundaries between space-time regions with Minkowskian and Euclidean signatures or genuine boundaries of Minkowskian regions.

   A highly attractive option is that what we identify the boundaries of physical objects are indeed genuine space-time boundaries so that we would directly see the space-time topology. This was the original vision. Later I became cautious with this interpretation since it seemed difficult to realize, or rather to understand, the boundary conditions.

   The proposal that the outer boundaries of different phases and even molecules make sense and correspond to 3-D membrane like entities [28], served as a partial inspiration for this article but this proposal is not equivalent with the proposal that light-like boundaries defining genuine space-time boundaries can carry isometry charges and fermions.

3. How does this relate to $M^8 - H$ duality [36, 37]? At the level of rational polynomials P determined 4-surfaces at the level of M^8 as their "roots" and the roots are mass shells. The points of M^4 have interpretation as momenta and would have values, which are algebraic integers in the extension of rationals defined by P.

   Nothing prevents from posing the additional condition that the region of $H^3 \subset M^4 \subset M^8$ is finite and has a boundary. For instance, fundamental regions of tessellations defining hyperbolic manifolds (one of them appears in the model of the genetic code [38]) could be considered. $M^8 - H$ duality would give rise to holography associating to these 3-surfaces space-time surfaces in H as minimal surfaces with singularities as 4-D analogies to soap films with frames.

   The generalization of the Fermi torus and its boundary (usually called Fermi sphere) as the counterpart of unit cell for a condensed matter cubic lattice to a fundamental region of a tessellation of hyperbolic space H^3 acting is discussed is discussed in [34]. The number of tessellations is infinite and the properties of the hyperbolic manifolds of the "unit cells" are fascinating. For instance, their volumes define topological invariants and hyperbolic volumes for knot complements serve as knot invariants.

This picture resonates with an old guiding vision about TGD as an almost topological quantum field theory (QFT) [16, 18, 14], which I have even regarded as a third strand in the 3-braid formed by the basic ideas of TGD based on geometry-number theory-topology trinity.

1. Kähler Chern-Simons form, also identifiable as a boundary term to which the instanton density of Kähler form reduces, defines an analog of topological QFT.

2. In the recent case the metric is however present via boundary conditions and in the dynamics in the interior of the space-time surface. However, the preferred extremal property essential for geometry-number theory duality transforms geometric invariants to topological invariants. Minimal surface property means that the dynamics of volume and Kähler action decouple outside the singularities, where minimal surface property fails. Coupling constants are present in the dynamics only at these lower-D singularities defining the analogs of frames of a 4-D soap film.

Singularities also include string worlds sheets and partonic 2-surfaces. Partonic two-surfaces play the role of topological vertices and string world sheets couple partonic 2-orbits to a network. It is indeed known that the volume of a minimal surface can be regarded as a homological invariant.

3. If the 3-surfaces assignable to the mass shells H^3 define unit cells of hyperbolic tessellations and therefore hyperbolic manifolds, they also define topological invariants. Whether also string world sheets could define topological invariants is an interesting question.

4.4 Can one allow macroscopic Euclidean space-time regions

Euclidean space-time regions are not allowed in General Relativity. Can one allow them in TGD?

1. CP_2 extremals with a Euclidean induced metric and serving as correlates of elementary particles are basic pieces of TGD vision. The quantum numbers of fundamental fermions would reside at the light-like orbit of 2-D wormhole throat forming a boundary between Minkowskian space-time sheet and Euclidean wormhole contact- parton as I have called it. More precisely, fermionic quantum numbers would flow at the 1-D ends of 2-D string world sheets connecting the orbits of partonic 2-surfaces. The signature of the 4-metric would change at it.

2. It is difficult to invent any mathematical reason for excluding even macroscopic surfaces with Euclidean signature or even deformations of CP_2 type extremals with a macroscopic size. The simplest deformation of Minkowski space is to a flat Euclidean space as a warping of the canonical embedding $M^4 \subset M^4 \times S^1$ changing its signature.

3. I have wondered whether space-time sheets with an Euclidean signature could give rise to black-hole like entities. One possibility is that the TGD variants of blackhole-like objects have a space-time sheet which has, besides the counterpart of the ordinary horizon, an additional inner horizon at which the signature changes to the Euclidean one. This could take place already at Schwarzschild radius if g_{rr} component of the metric does not change its sign.

4.5 Conformal confinement at the level of H

The proposal of [39], inspired by p-adic thermodynamics, is that all states are massless in the sense that the sum of mass squared values vanishes for physical states. Conformal weight, as essentially mass squared value, is naturally additive and conformal confinement as a realization of conformal invariance would indeed mean that the sum of mass squared values vanishes. Since complex mass squared values with a negative real part are allowed as roots of polynomials, the condition is highly non-trivial.

M^8-H duality [36, 37] would make it natural to assign tachyonic masses with CP_2 type extremals and with the Euclidean regions of the space-time surface. What looks like a problem is that the regions with Euclidean signature look Minkowskian to the outsider since the M^4 projection is time-like. The resolution of the problem could be simple: the light-like momenta assignable with the light-like boundaries of the Euclidean regions would make them look like Minkowskian regions.

4.6 But are the normal components of isometry currents finite?

Whether this scenario works depends on whether the normal components for the isometry currents are finite.

1. $det(g_4) = 0$ condition gives boundaries of Euclidean and Minkowskian regions as 3-D light-like minimal surfaces. There would be no scales in accordance with generalized conformal invariance. g_{uv} in light-cone coordinates for M^2 vanishes and implies the vanishing of $det(g_4)$ and light-likeness of the 3-surface.

What is important is that the formation of these regions would be unavoidable and they would be stable against perturbations.

2. $g^{uv}\sqrt{|g_4|}$ is finite if $det(g_4) = 0$ condition is satisfied, otherwise it diverges. The terms $g^{ui}\partial_i h^k\sqrt{|g_4|}$ must be finite. $g^{ui} = cof(g_{iu})/det(g_4)$ is finite since $g_{uv}g_{vu}$ in the cofactor cancels it from the determinant in the expression of g^{ui}. The presence of $\sqrt{|g_4|}$ implies that the these contributions to the boundary conditions vanish. Therefore only the condition boundary condition for g^{uv} remains.

3. If also Kähler action is present, the conditions are modified by replacing $T^{uk} = g^{u\alpha}\partial_\alpha h^k\sqrt{|g_4|}$ with a more general expression containing also the contribution of Kähler action. I have discussed the details of the variational problem in [19, 18].

   The Kähler contribution involves the analogy of Maxwell's energy momentum tensor, which comes from the variation of the induced metric and involves sum of terms proportional to $J_{\alpha\mu}J_\mu^{beta}$ and $g^{\alpha\beta}J^{\mu\nu}J_{\mu\nu}$.

   In the first term, the dangerous index raisings by g^{uv} appear 3 times. The most dangerous term is given by $J^{uv}J_v^v\sqrt{|g|} = g^{u\mu}g^{v\nu}J_{\alpha\beta}g^{vu}J_{vu}\sqrt{|g|}$. The divergent part is $g^{uv}g^{vu}J_{uv}g^{vu}J_{vu}\sqrt{|g|}$. The diverging g^{uv} appears 3 times and $J_{uv} = 0$ condition eliminates two of these. $g^{vu}\sqrt{|g|}$ is finite by $\sqrt{|g|} = 0$ condition. $J_{uv} = 0$ guarantees also the finiteness of the most dangerous part in $g^{\alpha\beta}J^{\mu\nu}J_{\mu\nu}\sqrt{|g|}$.

   There is also an additional term coming from the variation of the induced Kähler form. This to the normal component of the isometry current is proportional to the quantity $J^{n\alpha}J_l^k\partial_\beta h^l\sqrt{|g|}$. Also now, the most singular term in $J^{u\beta} = g^{u\mu}g^{\beta\nu}J_{\mu\nu}$ corresponds to J^{uv} giving $g^{uv}g^{vu}J^{uv}\sqrt{|g|}$. This term is finite by $J_{uv} = 0$ condition.

   Therefore the boundary conditions are well-defined but only because $det(g_4) = 0$ condition is assumed.

4. Twistor lift strongly suggests that the assignment of the analogy of Kähler action also to M^4 and also this would contribute. All terms are finite if $det(g_4) = 0$ condition is satisfied.

5. The isometry currents in the normal direction must be equal to the divergences of the corresponding currents assignable to the Chern-Simons action at the boundary so that the flow of isometry charges to the boundary would go to the Chern-Simons isometry charges at the boundary.

   If the Chern-Simons term is absent, one expects that the boundary condition reduces to $\partial_v h^k = 0$. This would make X^3 2-dimensional so that Chern-Simons term is necessary. Note that light-likeness does not force the M^4 projection to be light-like so that the expansion of X^2 need not take with light-velocity. If CP_2 complex coordinates are holomorphic functions of W depending also on $U = v$ as a parameter, extended conformal invariance is obtained.

4.7 $det(g_4) = 0$ condition as a realization of quantum criticality

Quantum criticality is the basic dynamical principle of quantum TGD. What led to its discovery was the question "How to make TGD unique?". TGD has a single coupling constant, Kähler couplings strength, which is analogous to a critical temperature. The idea was obvious: require quantum criticality. This predicts a spectrum of critical values for the Kähler coupling strength. Quantum criticality would make the TGD Universe maximally complex. Concerning living matter, quantum critical dynamics is ideal since it makes the system maximally sensitive and maximallt reactive.

Concerning the realization of quantum criticality, it became gradually clear that the conformal invariance accompanying 2-D criticality, must be generalized. This led to the proposal that super symplectic symmetries, extended isometries and conformal symmetries of the metrically 2-D boundary of lightcone of M^4, and the extension of the Kac-Moody symmetries associated with the light-like boundaries of deformed CP_2 type extremals should act as symmetries of TGD extending the conformal symmetries of 2-D conformal symmetries. These huge infinite-D symmetries are also required by the existence of the Kähler geometry of WCW [16, 15, 17] [33, 35].

However, the question whether light-like boundaries of 3-surfaces with scale larger than CP_2 are possible, remained an open question. On the basis of preceding arguments, the answer seems to be affirmative and one can ask for the implications.

1. At M^8 level, the concrete realization of holography would involve two ingredients. The intersections of the space-time surface with the mass shells H^3 with mass squared value determined as the roots of polynomials P and the tlight-like 3-surfaces as $det(g_4) = 0$ surfaces as boundaries (genuine or between Minkowskian and Euclidean regions) associated by $M^8 - H$ duality to 4-surface of M^8 having associative normal space, which contains commutative 2-D subspace at each point. This would make possible both holography and $M^8 - H$ duality.

   Note that the identification of the algebraic geometric characteristics of the counterpart of $det(g_4) = 0$ surface at the level of H remains still open.

   Since holography determines the dynamics in the interior of the space-time surface from the boundary conditions, the classical dynamics can be said to be critical also in the interior.

2. Quantum criticality means ability to self-organize. Number theoretical evolution allows us to identify evolution as an increase of the algebraic complexity. The increase of the degree n of polynomial P serves as a measure for this. $n = h_{eff}/h_0$ also serves as a measure for the scale of quantum coherence, and dark matter as phases of matter would be characterized by the value of n.

3. The 3-D boundaries would be places where quantum criticality prevails. Therefore they would be ideal seats for the development of life. The proposal that the phase boundaries between water and ice serve as seats for the evolution of prebiotic life, is discussed from the point of TGD based view of quantum gravitation involving huge value of gravitational Planck constant $\hbar_{eff} = \hbar_{gr} = GMm/v_0$ making possible quantum coherence in astrophysical scales [30]. Density fluctuations would play an essential role, and this would mean that the volume enclosed by the 2-D M^4 projection of the space-time boundary would fluctuate. Note that these fluctuations are possible also at the level of the field body and magnetic body.

4. It has been said that boundaries, where the nervous system is located, distinguishes living systems from inanimate ones. One might even say that holography based on $det(g_4) = 0$ condition realizes nervous systems in a universal manner.

5. I have considered several variants for the holography in the TGD framework, in particular strong form of holography (SH). SH would mean that either the light-like 3-surfaces or the 3-surfaces at the ends of the causal diamond (CD) determine the space-time surface so that the 2-D intersections of the 3-D ends of the space-time surface with its light-like boundaries would determine the physics.

   This condition is perhaps too strong but a fascinating, weaker, possibility is that the internal consistency requires that the intersections of the 3-surface with the mass shells H^3 are identifiable as fundamental domains for the coset spaces $SO(1,3)/\Gamma$ defining tessellations of H^3 and hyperbolic manifolds. This would conform nicely with the TGD inspired model of genetic code [38].

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