Article

On the Associated Curves of a Null & Pseudo Null Curve in R_1^4

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Abstract

In this study on curves, we defined principal direction and binormal direction curves of a granted null curve and pseudo null curve by using integral curves for 4-dimensional Minkowski space-time. Also, we obtain some characterizations of such curves.

Keywords: Minkowski spacetime, principal direction curve, binormal direction curve.

1. Introduction

Curves theory has a important place in differential geometry. Helices, slant helices (slant) curves, which have an prominent place in the theory of curves, have been processed on the Euclidean space [6,7]. Bertrand curve, manheim partner curve, spherical indicatrices and rectifying curve are the most employed in curves so far [4,8].

Gawell show that Non-Euclidean geometry was harnessed for architecture built from past to present [2]. Non-Euclidean geometries have an important place in the study area. In many area of science, we can come across types of non-Euclidean geometry. Null curves were first studied E. Cartan. Also, these curves were profoundly endeavored by W.B. Bonnor in Minkowski space-time [11,12,13]. For the uniform space, it is expressed by J.Walrave in ancient times for the curves we will employ in our study. If we add another, Minkowski space may discrepant from the characters of curves. New studies have been commentated to the literature by many researchers with the alms of frenet equations defined on null and pseudo null curves[1,3,7,9]. Over and above these curves have been investigationed by Ilarslan and Nesovic [5]. İn addition, perspectives have been upgraded within the admissible frenet curves[10].

In this paper, we have given some relations with the curvatures of the curves given in Minkowski space-time. Also, principal and binormal direction curves are defined for null and psedo null curves. For pseudo and null curves we research connections between slant helix and B_2 −slant helix.

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2. Matherial and Methods

Let R_1^4 be 4-dimensional vector space endowed with the scalar product $<,>$

which is defined by

$$\langle,\rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

R_1^4 is 4-dimensional vector space equipped with the scalar product $<,>$

then R_1^4 is called Lorentzian 4-space or 4-dimensional Minkowski space.. There are three separate cases for the w vector in Minkowski space-time:

$$\begin{cases} <w,w> \ >0 \quad or \quad w=0 & ,spacelike \\ <w,w> \ <0 & ,timelike \\ <w,w>=0 \quad and \quad w\neq 0 & ,null(lightlike) \end{cases}$$

If all of velocity vector $\alpha'(s)$ are spacelike, timelike or lightlike respectively. The norm of a vector $v \in {R_1}^4$ is given by $\|v\| = \sqrt{(|<v,v>|)}$. Therefore, v is a unit vector $<v,v> \pm 1$.A (spacelike, or timelike) curve is parametrized by the arc length if $\alpha'(s)$ is unit vector for any s. Also we say that the vectors $v, w \in R_1^4$ are orthogonal if $<v,w>=0$

when $a(s)$ is a null curve, Frenet equations are stated as

$$\begin{vmatrix} T' \\ N' \\ B_1' \\ B_2' \end{vmatrix} = \begin{vmatrix} 0 & K_1 & 0 & 0 \\ K_2 & 0 & -K_1 & 0 \\ 0 & -K_2 & 0 & K_3 \\ -K_3 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} T \\ N \\ B_1 \\ B_2 \end{vmatrix}$$

For T, N, B_1, B_2 vectors, there are the following circumstances.

$$\langle T,T\rangle = \langle B_1,B_1\rangle = 0, \langle N,N\rangle = \langle B_2,B_2\rangle = 1$$

$$\langle T,N\rangle = \langle T,B_2\rangle = \langle N,B_1\rangle = \langle N,B_2\rangle = \langle B_1,B_2\rangle = 0, \langle T,B_1\rangle = 1$$

when $a(s)$ is a pseudo null curve, Frenet equations are followed as

$$\begin{vmatrix} T' \\ N' \\ B_1' \\ B_2' \end{vmatrix} = \begin{vmatrix} 0 & K_1 & 0 & 0 \\ 0 & 0 & K_2 & 0 \\ 0 & K_3 & 0 & -K_2 \\ -K_1 & 0 & -K_3 & 0 \end{vmatrix} \begin{vmatrix} T \\ N \\ B_1 \\ B_2 \end{vmatrix}$$

For T, N, B_1, B_2 vectors, there are the following condition.

$$\langle T,T\rangle = \langle B_1,B_1\rangle = 1, \langle N,N\rangle = \langle B_2,B_2\rangle = 0$$

$$\langle T,N\rangle = \langle T,B_2\rangle = \langle T,B_1\rangle = \langle N,B_1\rangle = \langle B_1,B_2\rangle = 0, \langle N,B_2\rangle = 1$$

[5].

3. Results

First of all, we introduce associated curves of a null curve and then we express associated curves of a pseudo null curve. For these two section, we express a new relationship for our curves curves in Minkowski space-time.

Definition 1. Let us consider a null curve α in the Minkowski space-time known as the Frenet frame $\{T, N, B_1, B_2\}$. The integral curve of the principal normal, B_1 binormal and B_2 binormal vector fields of the α, respectively, it is called principal direction, B_1-direction and B_2-direction curves of α.

Theorem 1. Let us consider a null curve with α curvatures signified as K_1, K_2, K_3, the principal direction curve of α by $\tilde{\alpha}$. The curvatures of $\tilde{\alpha}$ are as follows

$$\widetilde{K}_1(s) = \frac{-2K_1K_2}{\sqrt{-2K_1K_2}}, \qquad K_1, K_2 \text{ opposed sign}$$

$$\widetilde{K}_2(s) = 0$$

Proof. Let $\{\widetilde{T}, \widetilde{N}, \widetilde{B}_1, \widetilde{B}_2, \widetilde{K}_1, \widetilde{K}_2, \widetilde{K}_3\}$ take the Frenet apparatus of $\tilde{\alpha}$. We have the following equality because of the principal direction curve.

$$N(s)\,\Big|_{\tilde{\alpha}(s)} = \tilde{\alpha}'(s) = \widetilde{T}(s)$$

We have,

$$\widetilde{N}(s) = \frac{\tilde{\alpha}''(s)}{\|\tilde{\alpha}''(s)\|} = \frac{K_2T - K_1B_1}{\sqrt{-2K_1K_2}}$$

$$\widetilde{B}_2(s) = \frac{\tilde{\alpha}'(s) \times \tilde{\alpha}''(s) \times \tilde{\alpha}'''(s)}{\|\tilde{\alpha}'(s) \times \tilde{\alpha}''(s) \times \tilde{\alpha}'''(s)\|} = \frac{K_2K_1B_1 - K_1^2T}{K_2\sqrt{-2K_1K_2}}$$

Finally,

$$\tilde{B}_1(s) = \tilde{B}_2 \times \tilde{T} \times \tilde{N} = \frac{-(K_1^2 + K_2^2)}{K_2} B_2$$

The $\widetilde{K}_1(s)$ curvature of $\tilde{\alpha}$ is given as:

$$\widetilde{K}_1(s) = \langle \widetilde{T}', \widetilde{N} \rangle = \langle N', \frac{K_2 T - K_1 B_1}{\sqrt{-2K_1 K_2}} \rangle = \frac{-2K_1 K_2}{\sqrt{-2K_1 K_2}}$$

and the $\widetilde{K}_2(s)$ curvature of $\tilde{\alpha}$ is given as :

$$\widetilde{K}_2(s) = \langle \widetilde{N}', \tilde{B}_1 \rangle = \langle \frac{K_2 T - K_1 B_1}{\sqrt{-2K_1 K_2}}, \frac{-(K_1^2 + K_2^2)}{K_2} B_2 \rangle = 0$$

Theorem 2. Let us consider a null curve α whose curvatures signified as K_1, K_2, K_3 demonstrate the B_1-direction curve of α by $\tilde{\alpha}$. The curvatures of $\tilde{\alpha}$ are given

$$\widetilde{K}_1(s) = \sqrt{K_2^2 + K_3^2}$$

$$\widetilde{K}_2(s) = \frac{-K_1 K_2}{\sqrt{K_2^2 + K_3^2}}$$

Proof. Let $\{\widetilde{T}, \widetilde{N}, \tilde{B}_1, \tilde{B}_2, \widetilde{K}_1, \widetilde{K}_2, \widetilde{K}_3\}$ take the Frenet apparatus of $\tilde{\alpha}$. We have the following equality because of the B_1-direction curve.

$$B_1(s)\,\Big|_{\tilde{\alpha}(s)} = \tilde{\alpha}'(s) = \tilde{T}(s)$$

And so ,

$$\widetilde{N}(s) = \frac{\tilde{B}_1'(s)}{\|\tilde{B}_1'(s)\|} = \frac{-K_2 N - K_3 B_2}{\sqrt{K_2^2 + K_3^2}}$$

$$\tilde{B}_2(s) = \frac{K_2 B_2 - K_3 N}{\sqrt{K_2^2 + K_3^2}}$$

And finally ,

$$\tilde{B}_1(s) = \tilde{B}_2 \times \tilde{T} \times \tilde{N} = -T$$

The $\tilde{K}_1(s)$ curvature of $\tilde{\alpha}$ is given as

$$\tilde{K}_1(s) = \langle \tilde{T}', \tilde{N} \rangle = \sqrt{K_2^2 + K_3^2}$$

and the $\tilde{K}_2(s)$ curvature of $\tilde{\alpha}$ is given as

$$\tilde{K}_2(s) = \langle \tilde{N}', \tilde{B}_1 \rangle = \frac{-K_1 K_2}{\sqrt{K_2^2 + K_3^2}}$$

Theorem 3. Let us consider a null curve a whose curvatures signified as K_1, K_2, K_3. the B_2-direction curve of a by $\tilde{\alpha}$. The curvetures of $\tilde{\alpha}$ are obtained

$$\tilde{K}_1(s) = 0$$

$$\tilde{K}_2(s) = K_1$$

Proof. Let $\{\tilde{T}, \tilde{N}, \tilde{B}_1, \tilde{B}_2, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3\}$ take the Frenet apparatus of $\tilde{\alpha}$. We have following equality because of the B_2-direction curve.

$$B_2(s)\big|_{\tilde{\alpha}(s)} = \tilde{\alpha}'(s) = \tilde{T}(s)$$

Using frenet vector fields,

$$\tilde{N}(s) = \frac{\tilde{B}_2{}'(s)}{\|\tilde{B}_2{}'(s)\|} = \frac{-K_3 T}{|K_3|} = -sgn(K_3)T$$

$$\tilde{B}_2(s) = \frac{K_3^2 K_1 B_1}{|K_3^2 K_1|} = B_1$$

And finally

$$\tilde{B}_1(s) = \tilde{T} \times \tilde{N} \times \tilde{B}_2 = B_2 \times -sgn(K_3)T \times B_1 = -sgn(K_3)N$$

The $\tilde{K}_1(s)$ curvature of $\tilde{\alpha}$ is given as

$$\tilde{K}_1(s) = \langle \tilde{T}', \tilde{N} \rangle = \langle \tilde{B}_2{}'(s), -sgn(K_3)T \rangle = \langle -K_3 T, -sgn(K_3)T \rangle = 0$$

and the $\tilde{K}_2(s)$ curvature of $\tilde{\alpha}$ is given as

$$\tilde{K}_2(s) = \langle \tilde{N}', \tilde{B}_1 \rangle = \langle -sgn(K_3)K_1 N, -sgn(K_3)N \rangle = K_1.$$

Theorem 4. Let α be a null curve in Minkowski space-time and the principal direction curve of α by $\tilde{\alpha}$. α helix if and only if $\tilde{\alpha}$ is a general helix.

Proof. We know that $\{\widetilde{T}, \widetilde{N}, \tilde{B}_1, \tilde{B}_2\}$ is the Frenet frame of α. We have the following equality because of the principal direction curve.

$$N(s) = \tilde{\alpha}'(s) = \tilde{T}(s)$$

So,

$$\begin{aligned} \alpha \text{ is a slant helix } &\Leftrightarrow \langle N, w\rangle = a \text{ , } a = \text{const.} \\ &\Leftrightarrow \langle \widetilde{T}, w\rangle = a \\ &\Leftrightarrow \tilde{\alpha} \text{ is a general helix} \end{aligned}$$

Theorem 5. Let α be a null curve in Minkowski space-time and the B_2 direction curve of α by $\tilde{\alpha}$. α is a B_2 slant helix if and only if $\tilde{\alpha}$ is a general helix.

Proof. We know that $\{\widetilde{T}, \widetilde{N}, \tilde{B}_1, \tilde{B}_2\}$ is the frenet frame of α. We have the following equality because of the B_2-direction curve.

$$B_2(s) = \tilde{\alpha}'(s) = \tilde{T}(s)$$

So,

$$\begin{aligned} \alpha \text{ is a } B_2 \text{ slant helix } &\Leftrightarrow \langle B_2, v\rangle = a \text{ , } a = \text{const.} \\ &\Leftrightarrow \langle \widetilde{T}, v\rangle = a \\ &\Leftrightarrow \tilde{\alpha} \text{ is a general helix} \end{aligned}$$

Definition 2. Let be α a pseudo null curve in the Minkowski space-time known as the frenet frame $\{T, N, B_1, B_2\}$.The integral curve of the principal normal, B_1 bi-normal and B_2 bi-normal vector fields of the α, respectively, it is added principal direction, B_1-direction and B_2-direction curves of α.

Theorem 6. Let be α pseudo null curve with α curvatures signified as K_1, K_2, K_3 , and the principal direction curve of α by $\tilde{\alpha}$.The curvetures of α are obtained

$$\widetilde{K}_1(s) = K_2$$

$$\widetilde{K}_2(s) = -K_3 sgn(K_2^3)$$

Proof. Let $\{\widetilde{T}, \widetilde{N}, \tilde{B}_1, \tilde{B}_2, \widetilde{K}_1, \widetilde{K}_2, \widetilde{K}_3\}$ take the Frenet apparatus of $\tilde{\alpha}$. We have the following equality because of the principal direction curve.

$$N(s)\,|_{\tilde{\alpha}(s)} = \tilde{\alpha}'(s) = \tilde{T}(s)$$

We can write,

$$\widetilde{N}(s) = \frac{N'(s)}{\|N'(s)\|} = \frac{K_2 B_1}{|K_2|} = B_1$$

$$\tilde{B}_2(s) = -sgn(K_2^3)T$$

After all

$$\tilde{B}_1(s) = \tilde{B}_2 \times \widetilde{T} \times \widetilde{N} = -sgn(K_2^3)B_2$$

The $\widetilde{K}_1(s)$ curvature of $\tilde{\alpha}$ is given as

$$\widetilde{K}_1(s) = \langle \widetilde{T}\,', \widetilde{N} \rangle = \langle N', B_1 \rangle = K_2$$

and the $\widetilde{K}_2(s)$ curvature of $\tilde{\alpha}$ is given as

$$\widetilde{K}_2(s) = \langle \widetilde{N}\,', \tilde{B}_1 \rangle = \langle B'_1, -sgn(K_2^3)B_2 \rangle = -K_3 sgn(K_2^3)$$

Theorem 7. Let be a a pseudo null curve with a whose curvatures signified as K_1, K_2, K_3 . B_1-direction curve of a by $\tilde{\alpha}$. The curvetures of $\tilde{\alpha}$ are obtained

$$\widetilde{K}_1(s) = \frac{-2K_2K_3}{\sqrt{-2K_2K_3}}, \qquad K_2, K_3 \; opposed\; sign$$

$$\widetilde{K}_2(s) = \frac{K_1K_2}{\sqrt{-2K_2K_3}}\left(\frac{K_3}{2K_2} + \frac{K_3^2}{2K_2^2}\right) \quad K_2, K_3 \; opposed\; sign$$

Proof. Let $\{\widetilde{T}, \widetilde{N}, \tilde{B}_1, \tilde{B}_2, \widetilde{K}_1, \widetilde{K}_2, \widetilde{K}_3\}$ take the Frenet apparatus of $\tilde{\alpha}$. We have the following equality because of the B_1- direction curve.

$$B_1(s)\,|_{\tilde{\alpha}(s)} = \tilde{\alpha}'(s) = \tilde{T}(s)$$

And so ,

$$\widetilde{N}(s) = \frac{\tilde{B}_1{}'(s)}{\|\tilde{B}_1{}'(s)\|} = \frac{K_3N - K_2B_2}{\sqrt{-2K_2K_3}}$$

$$\tilde{B}_2(s) = \frac{K_1K_2K_3B_2 - K_2^2K_1N}{\sqrt{-2K_1^2K_2^3K_3}}$$

And finally,

$$\tilde{B}_1(s) = \tilde{B}_2 \times \tilde{T} \times \tilde{N} = \frac{K_3T}{2K_2} + \frac{K_3^2T}{2K_2^2}$$

The $\tilde{K}_1(s)$ curvature of $\tilde{\alpha}$ is given as

$$\tilde{K}_1(s) = \langle \tilde{T}', \tilde{N} \rangle = \langle B_1', \frac{K_3N - K_2B_2}{\sqrt{-2K_2K_3}} \rangle = \frac{-2K_2K_3}{\sqrt{-2K_2K_3}}$$

and the $\tilde{K}_2(s)$ curvature of $\tilde{\alpha}$ is given as

$$\tilde{K}_2(s) = \langle \tilde{N}', \tilde{B}_1 \rangle = \frac{K_1K_2}{\sqrt{-2K_2K_3}}\left(\frac{K_3}{2K_2} + \frac{K_3^2}{2K_2^2}\right)$$

Theorem 8. Let be α a pseudo null curve with α curvatures signified as K_1, K_2, K_3 ,demonstrate the B_2-direction curve of α by $\tilde{\alpha}$. The curvatures of $\tilde{\alpha}$ are obtained

$$\tilde{K}_1(s) = \frac{K_1K_2 + K_3^2}{\sqrt{K_2^2 + K_3^2}}$$

$$\tilde{K}_2(s) = \frac{K_3K_2(-K_1K_2 - K_3^2)}{(K_2^2 + K_3^2)\sqrt{K_1^2 + K_3^2}}$$

Proof. Let $\{\tilde{T}, \tilde{N}, \tilde{B}_1, \tilde{B}_2, \tilde{K}_1, \tilde{K}_2, \tilde{K}_3\}$ be the Frenet apparatus of $\tilde{\alpha}$. We have the following equality because of the B_2-direction curve.

$$B_2(s)\,\big|_{\tilde{\alpha}(s)} = \tilde{\alpha}'(s) = \tilde{T}(s)$$

Using frenet vector fields,

$$\tilde{N}(s) = \frac{\tilde{B}_2'(s)}{\|\tilde{B}_2'(s)\|} = \frac{-K_2T - K_3B_1}{\sqrt{K_2^2 + K_3^2}}$$

$$\tilde{B}_2(s) = \frac{-K_3T + K_1B_1}{\sqrt{K_2^2 + K_3^2}} = B_1$$

And finally,

$$\tilde{B}_1(s) = \tilde{T} \times \tilde{N} \times \tilde{B}_2 = \frac{-K_1K_2N - K_3^2N}{\sqrt{K_1^2 + K_3^2}\sqrt{K_2^2 + K_3^2}}$$

The $\tilde{K}_1(s)$ curvature of $\tilde{\alpha}$ is given as

$$\tilde{K}_1(s) = \langle \tilde{T}', \tilde{N} \rangle = \langle \tilde{B}_2'(s), \frac{-K_2T - K_3B_1}{\sqrt{K_2^2 + K_3^2}} \rangle$$

$$= \langle -K_1T - K_3B_1, \frac{-K_2T - K_3B_1}{\sqrt{K_2^2 + K_3^2}} \rangle = \frac{K_1K_2 + K_3^2}{\sqrt{K_2^2 + K_3^2}}$$

and the $\tilde{K}_2(s)$ curvature of $\tilde{\alpha}$ is given as

$$\tilde{K}_2(s) = \langle \tilde{N}', \tilde{B}_1 \rangle = \frac{K_3K_2(-K_1K_2 - K_3^2)}{(K_2^2 + K_3^2)\sqrt{K_1^2 + K_3^2}}$$

Theorem 9. Let α be an pseudo null curve in Minkowski space-time and let's demonstrate the principal direction curve of α by $\tilde{\alpha}$. Here, α is a slant helix if and only if $\tilde{\alpha}$ is a general helix.

Proof. We know that $\{\tilde{T}, \tilde{N}, \tilde{B}_1, \tilde{B}_2\}$ is the frenet frame of α. We have following equality because of the principal direction curve.

$$N(s) = \tilde{\alpha}'(s) = \tilde{T}(s)$$

So, w is constant vector,

α is a slant helix $\Leftrightarrow \langle N, w \rangle = a$, $a =$const.
$\Leftrightarrow \langle \tilde{T}, w \rangle = a$
$\Leftrightarrow \tilde{\alpha}$ is a general helix

Theorem 10. Let α be a pseudo null curve in Minkowski space-time and Let's demonstrate the B_2-direction curve of α by $\tilde{\alpha}$. α is a B_2 slant helix if and only if $\tilde{\alpha}$ is a general helix.

Proof. We know that $\{\tilde{T}, \tilde{N}, \tilde{B}_1, \tilde{B}_2\}$ is the frenet frame of α .We can receive the following equality because of the B_2-direction curve.

$$B_2(s) = \tilde{\alpha}'(s) = \tilde{T}(s)$$

So,

α is a $B_2 -$ slant helix $\Leftrightarrow \langle B_2, v \rangle = a$, $a =$const.
$\Leftrightarrow \langle \tilde{T}, v \rangle = a$
$\Leftrightarrow \tilde{\alpha}$ is a general helix.

4. Discussion and Conclusion

A new kind of curve called the associated curves of null and pseudo null curves are introduced and studied. We give some characterizations of such curves.It is aimed that these studies will continue in this way in the future.

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