Article

Some New Characterizations of F- Rectifying Curves Respect to Type-2 Quaternionic Frame in $\mathbb{R}^4$

Esra Erdem*

Abstract

Quaternions which are used in both theoriticial and applied sciences, were defined by Hamilton in 1843. Due to the wide application area for quaternions, there are numerous studies on the special defined quaternionic curves. In four dimensional spaces, rectifying curves are named as a curve whose position vector is completely lies in $\{T, N_2, N_3\}$. In this study, we present the notion of an f-rectifying curve in $\mathbb{R}^4$ as a curve β in $\mathbb{R}^4$ parametrized by its arc length s such that its f-position vector $\beta_f(s) = f(s)d_\beta$ for all s, every time lies in its rectifying space in $\mathbb{R}^4$, where f is a nonzero integrable function in parameter s of the curve β. With the help of this information, we obtain some characterizations for such curves ın $\mathbb{R}^4$.

Keywords: Quaternion, rectifying curve, f-rectifying curve.

1.Introduction

The quaternions were firstly introduced by W. R. Hamilton who explored appropriate generalization in which the real axis is left unchanged whereas the vector(imaginary) axis is supplemented by adding two further vector axis in 1843 [1]. The practical use of quaternions was minimal compared to other methods until the mid-20th century, that has now changed. Quaternion theory has developed rapidly in recent times and many mathematicians have focused on this field from different perspectives. Among these studies Baharathi and Nagaraj's study [2] on quaternion valued functions of the real variable Frenet-Serret equations is a touchstone. Getting inspired from this work, a new quaternionic framework begins with Aksoyak in $\mathbb{R}^4$[3].

In [4] notion of the rectifying curve is obtained as a space curve whose position vector always lies in its rectifying plane. Chen and Dillen [5] achieved a relationship between the rectifying curves and the centrodes given by the endpoints of the Darboux vector of a space curve which playing an significant role in mechanics. Also Güngör and Tosun determined the spatial quaternionic rectifying curves in $\mathbb{R}^3$. and obtained some charectarizations for these curves. In addition they explored quaternionic rectifying curves in $\mathbb{R}^4$ [6]. Another is Igbal and Sengupta's study. They obtained the notion of an f-rectifying curve in $\mathbb{R}^4$ [7]. In addition, non-null and null f-rectifying curves were studied in Minkowski 3-space [8,9] and null f-rectifying curves were researched in E_1^4 [10].

*Correspondence: Esra Erdem, Turkey. E-mail: esra.6666.23@gmail.com

In the first part, we give information about the articles made, and then we give requisite preliminaries. Thereafter, define the f- rectifying curves and we obtain some characterization of rectifying curves using type-2 quaternion framework.

2. Preliminaries

In this part briefly mentioned the quaternion theory in Euclidean space. Detailed information can be found in [3] and [11].

The set of quaternions Q is determined by

$$\{q = q_0 + q_1 i + q_2 j + q_3 k; \quad q_1 \in \mathbb{R}, \ 0 \leq i \leq 3\}$$

where i,j,k are orthogonal unit spatial vectors in three dimensional space so that

$$i^2 = j^2 = k^2 = -1,$$
$$ij = -\text{ji} = \text{k},$$
$$\text{jk} = -\text{kj} = \text{i},$$
$$\text{ki} = -\text{ik} = \text{j}.$$

If we indicate by $S_q = q_0$ and $\vec{V_q} = q_0 + q_1 i + q_2 j + q_3 k$, (where respectively,the symbols indicate scaler and vectoral part q) we can write quaternion as $q = S_q + \vec{V_q}$. The product of quaternions can be attained as

$$p \times q = S_p S_q - < V_p, V_q > + S_p V_q + S_q V_P + V_p \wedge V_q, \ \ \forall p, q \in \text{Q}.$$

We mean $\times$ and $\langle , \rangle$ cross and inner product in Euclidean space R³ respectively [12]. Then conjugate of q denoted by αq and determined as follows:

$$\alpha q = S_q - V_q = q_3 k - q_0 - q_1 i - q_2 j.$$

Taking into account of Hamiltonian conjugation α is an antiautomorphism of Q it satisfies the following equation:

$$\alpha(p \times q) = \alpha q \times \alpha p \quad for\ all\ p, q \in \text{Q}.$$

One may define the symmetric real-valued, non-degenerate, bilinear form h as follows

$$h: \text{Q} \times \text{Q} \to \mathbb{R}$$
$$(p, q) \to h(p, q) = (1/2)[(p \times \alpha q) + (q \times \alpha p)].$$

and named as the quaternion inner product. The norm of a real quaternion is.defined as:

$$\|q^2\| = h(q, q) = q \times \alpha q = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2.$$

If $\|q\| = 1$, then it is determined as a unit quaternion. A spatial quaternion q is determined when is $q + \alpha q = 0[2]$ and a temporal quaternion is determined when $q - \alpha q = 0$. Any q can be written as $q = (1/2)(q + \alpha q) + (1/2)(q - \alpha q)$ [12]. As discussed introduction part a new quaternionic framework obtained by Aksoyak in $\mathbb{R}^4$ as follows:

Theorem 2.1: Let $J = [0,1]$ indicate the unit interval in the real line $\mathbb{R}$ and

$$\delta : J \subset \mathbb{R} \to H$$

$$s \ \to \ \delta(s) = \delta_0(s) + \delta_1(s)i + \delta_2(s)j + \delta_3(s)k$$

be a an arc-length curve in $\mathbb{R}^4$.Then, Serret-Frenet formulas of δ are obtained by

$$\begin{bmatrix} T' \\ {N_1}' \\ {N_2}' \\ {N_3}' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & -\tau & 0 \\ 0 & \tau & 0 & (K-k) \\ 0 & 0 & -(K-k) & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} \tag{1}$$

where $T = \delta'$the is unit tangent and N_1, N_2, N_3 are the unit normal vectors of the curve δ and $K = \|T'\|$ is the principal curvature, $-\tau$ is the torsion and $(K - k)$ is the bitorsion of the curve δ [3].

In this study, our main purpose is to attain some characterizations of f-rectifying curve in $\mathbb{R}^4$ by using type-2 quaternionic frame.

3. Characterization of Quaternionic f-Rectifying Curves in $\mathbb{R}^4$ for Type-2 Quaternionic Frame

A unit-speed curve $\beta: I \to \mathbb{R}^4$ is a rectifying curve if and only if its position vector fully lies in its rectifying space can be phrased as

$$\beta(s) = \lambda(s)T_\beta(s) + \mu(s)N_{\beta 2}(s) + \gamma N_{\beta 3}(s)$$

for differentiable functions $\lambda, \mu, \gamma: I \to \mathbb{R}$ for each s$\in I$. Furthermore let $f: I \to \mathbb{R}$ in parameter s, the f-position vector of β in $\mathbb{R}^4$ is indicated and described by

$$\beta_f(s) = \int f(s) d_\beta$$

for s$\in I$.

Definition 3.1: Let $\beta: I \to \mathbb{R}^4$ be a unit-speed curve (parametrized by arc length function s) with Frenet apparatus $\{T_\beta, N_{\beta 1}, N_{\beta 2}, N_{\beta 3}, K_\beta, \tau_\beta, (K-k)_\beta$. Furthermore, let $f: I \to \mathbb{R}$ be a nonzero integrable function in parameter s with at least twice differentiable primitive function F. In that case β is determined an f-rectifying curve in $\mathbb{R}^4$ if its f-position vector β_f fully lies in its rectifying space in $\mathbb{R}^4$, i.e., if its f-position vector β_f in $\mathbb{R}^4$ can be phrased as

$$\beta(s) = \lambda(s)T_\beta(s) + \mu(s)N_{\beta 2}(s) + \gamma N_{\beta 3}(s) \tag{2}$$

for all $s \in I$. [7]

Theorem 3.1: Let $\beta: I \to \mathbb{R}^4$ be a unit-speed curve (parametrized by arc length function s), having nonzero curvatures $K_\beta, \tau_\beta, (K-k)_\beta$. Furthermore, let $f: I \to \mathbb{R}$ be a nonzero integrable function in parameter s with at least twice differentiable primitive function F. In that case ,until isometries of $\mathbb{R}^4$, β is congruent to an f-rectifying curve in $\mathbb{R}^4$ if and only if the following equation is provided:

$$\frac{d}{ds}\left(\frac{\frac{d}{ds}\left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right)}{(K-k)_\beta(s)}\right) + \left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right)(K-k)_\beta(s) = 0 \tag{3}$$

for all s$\in I$.

Proof: Let us first suppose that $\beta: I \to \mathbb{R}^4$ be an f-rectifying curve having nonzero curvatures $K_\beta, \tau_\beta, (K-k)_\beta$. At the time for ome differentiable $\lambda,\mu,\gamma: I \to \mathbb{R}$ in parameter s, its f-position vector β_f provides the equation (2). Differentiating (2) and at the time applying (1), we have

$$\begin{aligned} f(s)T_\beta(s) = \lambda' T_\beta(s) + \left(\lambda(s)K_\beta + \mu\tau(s)\right)N_{\beta 1}(s) + \left(\mu' - \gamma(K-k)_\beta\right)N_{\beta 2}(s) \\ + \left(\mu(K-k)_\beta + \gamma'\right)N_{\beta 3}(s) \end{aligned} \tag{4}$$

$\forall$ s$\in I$. From here we obtain

$$\lambda'(s) = f(s),$$

$$\lambda(s)K_\beta + \mu\tau(s) = 0,$$

$$\mu' - \gamma(K-k)_\beta = 0, \tag{5}$$

$$\mu(K-k)_\beta + \gamma' = 0$$

From first three equations (5), we get

$$\lambda(s) = F(s),$$

$$\mu(s) = \frac{-\tau_\beta(s)}{K_\beta F(s)}, \tag{6}$$

$$\gamma(s) = \frac{1}{(K-k)_\beta}\frac{d}{ds}\left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right)$$

If all of the equation (6) is substituted for in the fourth one of (5) we find,

$$\frac{d}{ds}\left(\frac{\frac{d}{ds}\left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right)}{(K-k)_\beta(s)}\right)+\left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right)(K-k)_\beta(s)=0$$

for s$\in I$. On the contrary, we suppose that $\beta: I \to \mathbb{R}^4$ is a unit speed curve having nonzero curvatures K_β, τ_β and $(K-k)_\beta$. Furthermore, let $f: I \to \mathbb{R}$ be a nonzero integrable function in parameter s with at least twice differentiable primitive function F. Thus, equation (3) is provided.

Determining a vector field α throughout β by

$$\alpha(s)=\beta_{f(s)}-F(s)T_\beta(s)+\left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right)N_{\beta 2}(s)$$
$$-\frac{1}{(K-k)_\beta(s)}\frac{d}{ds}\left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right)N_{\beta 3}(s) \quad (7)$$

for all s$\in I$. Differentiating (7) and substituting (1) and (3), we attain that $\alpha'(s)=0$. This means that α is constant throughout β. Therefore until isometries of $\mathbb{R}^4$, β is congruent to an f-rectifying curve in $\mathbb{R}^4$ according to type-2 quaternionic frame.

Remark 3.1: For an f-rectifying curve in $\mathbb{R}^4$, suppose that be $K_\beta=b_1\neq 0$, $\tau_\beta=b_2\neq 0$ and $(K-k)_\beta=b_3\neq 0$ for s$\in I$, in this case from (3), we find

$$-F''F^2+2F(F')^2+{b_3}^2F^3=0 \quad (8)$$

Assume that f is nonzero constant either linear, in this case from (8) we get $b_3=0$ which is a contradiction.Conversely, suppose that f is non-linear, then from (8) we obtain b_3 is non-constant which is a contradiction too.

Respect to the above remark, we have the following theorem:

Theorem 3.2: Let $\beta: I \to \mathbb{R}^4$ be a unit- speed curve having nonzero curvatures K_β, τ_β and $(K-k)_\beta$. In this case β is not congruent to an f-rectifying curve for any choice of f if and only if all its curvatures K_β, τ_β and $(K-k)_\beta$ are constants.

Theorem 3.3: Let $\beta: I \to \mathbb{R}^4$ be a unit-speed curve (parametrized by arc length function s), having nonzero curvatures $K_\beta, \tau_\beta, (K-k)_\beta$. Furthermore, let $f: I \to \mathbb{R}$ be a nonzero integrable function in parameter s with at least twice differentiable primitive function F. We obtain the following:

i-) If theK_β and τ_β are constant, then β is congruent to an f- rectifying curve in $\mathbb{R}^4$ according to type-2 quaternionic frame if and only if the $(K-k)_\beta$ satisfies the following differential equation:

$$\frac{F''F^2(K-k)_\beta-(K-k)'_\beta\, F'F^2-2F(F')^2(K-k)_\beta}{{(K-k)_\beta}^2F^4}-\frac{(K-k)_\beta}{F}=0$$

ii-) If the 1st curvature K_β and the 3rd curvature $(K-k)_\beta=b$ are constants, then β is congruent to an f- rectifying curve in $\mathbb{R}^4$ according to type-2 quaternionic frame if and only if 2nd curvature τ_β satisfies the following differential equation:

$$\frac{d^2}{ds^2}\left(\frac{\tau_\beta}{F}\right)+\frac{\tau_\beta b}{F}=0$$

iii-) If the 2nd τ_β and the 3rd curvature $(K-k)_\beta=b$ are constants, then β is congruent to an f-rectifying curve in $\mathbb{R}^4$ according to type-2 quaternionic frame if and only if 1th K_β satisfies the following differential equation:

$$\frac{d^2}{ds^2}\Big(K_\beta F(s)\Big)-\frac{b^2}{K_\beta F(s)}=0$$

Similar characterizations can be derived as a consequences of Theorem 3.1 when any one of K_β,τ_β or $(K-k)_\beta$ is a constant.

Theorem 3.4: Let $\beta\colon I\to\mathbb{R}^4$ be a unit-speed curve (parametrized by arc length function s), having nonzero curvatures $K_\beta,\tau_\beta,(K-k)_\beta$. Furthermore, let $f\colon I\to\mathbb{R}$ be a nonzero integrable function in parameter s with at least twice differentiable primitive function F. If β is an f-rectifying curve in $\mathbb{R}^4$ according to type-2 quaternionic frame, in this case, the following statements are valid.

i-) The norm function $q(s)=\left\|\beta_{f(s)}\right\|$ is grant by

$\forall$s$\in I$, c is a non-zero constant.

ii-) The tangential component $\langle\beta_f,T_\beta\rangle$ of the f-position vector β_f of β is given by

$$\left\langle\beta_{f(s)},T_\beta\,(s)\right\rangle=F(s)$$

$\forall$s$\in I$.

iii-) The normal component ${\beta_f}^{N_{\beta 1}}$ of the f-position vector β_f of β has constant length and the normal function q is non-constant.

iv-) The 1st binormal component $\langle\beta_f,N_{\beta 2}\rangle$ and 2nd binormal component $\langle\beta_f,N_{\beta 3}\rangle$ of the f-position vector β_f of β are given by

$$\left\langle\beta_{f(s)},N_{\beta 2}(s)\right\rangle=\frac{-\tau_\beta(s)}{K_\beta F(s)}$$

$$\left\langle\beta_{f(s)},N_{\beta 3}(s)\right\rangle=\frac{1}{(K-k)_\beta(s)}\frac{d}{ds}\left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right)$$

∀s∈ I.

On the contrary , suppose that $\beta: I \to \mathbb{R}^4$ is a unit-speed curve (parametrized by arc length function s), having nonzero curvatures $K_\beta, \tau_\beta, (K-k)_\beta$. Furthermore, let $f: I \to \mathbb{R}$ be a nonzero integrable function in parameter s with at least twice differentiable primitive function F so that any one of the expressions (i),(ii),(iii) or (iv) is provided, at the time β is an f-rectifying curve in $\mathbb{R}^4$ according to type-2 quaternionic frame .

Proof: Let'suppose first $\beta: I \to \mathbb{R}^4$ is an f-rectifying curve having nonzero curvatures $K_\beta, \tau_\beta, (K-k)_\beta$. At the time for some differentiable functions $\lambda, \mu, \gamma: I \to \mathbb{R}$ parameter s, the f-position vector β_f of the curve β in $\mathbb{R}^4$ provides equation (2) and from the proof of Theorem 3.1, we obtain equations (5) and (6). Now, ıf necessary operations are done on equation (5), we get

$$\mu(s)\mu'(s) + \gamma(s)\gamma'(s) = 0$$

∀s∈ I. if integrating the last equation, we find

$$\mu^2 + \gamma^2 = c^2 \tag{9}$$

∀s∈ I, c is a non-zero constant.

i-) Using equations (2),(6) and equation (9), the norm function $q(s) = \|\beta_{f(s)}\|$ is given by

$$q^2(s) = \|\beta_{f(s)}\|^2 = \langle \beta_{f(s)}, \beta_{f(s)} \rangle = F^2 + c^2$$

i.e.,

$$q(s) = \sqrt{F^2 + c^2}$$

∀s∈ I, c is a non-zero constant.

ii-) Using equations (2) and (6), the tangential compotent $\langle \beta_{f(s)}, T_\beta(s) \rangle$ of , $\beta_{f(s)}$ is given by

$$\langle \beta_{f(s)}, T_\beta(s) \rangle = \lambda(s) = F(s)$$

∀s∈ I.

iii-) An f-position vector x_f of an arbitrary curve $x: J \to \mathbb{R}^4$ can be separated as

$$x_f(t) = y(t)T_\beta(t) + x_f^{N_{\beta 1}(t)}, t \in J,$$

For some differentiable function $y: I \to \mathbb{R}$, indicates the normal component of x_f. Here $\beta: I \to \mathbb{R}^4$ is an f-rectifying curve and thus from equations (2), it is see that the normal component $\beta_f^{N_{\beta 1}}$ of x_f is given by

$$\beta_f^{N_{\beta 1}}(s) = \mu(s)N_{\beta 2}(s) + \gamma(s)\, N_{\beta 3}(s$$

∀s∈ I. For this reason, we get

$$\left\|{\beta_f}^{N_{\beta 1}}(s)\right\| = \sqrt{\langle {\beta_f}^{N_{\beta 1}}(s), {\beta_f}^{N_{\beta 1}}(s)\rangle} = \sqrt{\mu^2+\gamma^2}$$

∀s∈ I. Now by using equations (9), we evident that $\left\|{\beta_f}^{N_{\beta 1}}(s)\right\| = c$. This means that ${\beta_f}^{N_{\beta 1}}$ has constant length. Also, from expression (i), it follows that the norm function q=β_f is non-constant.

iv-) Using (2) and (6), the first binormal compotent $\langle \beta_{f(s)}, N_{\beta 2}(s)\rangle$ of β_f is given by

$\langle \beta_{f(s)}, N_{\beta 2}(s)\rangle = \mu(s) = \frac{-\tau_\beta(s)}{K_\beta F(s)}$

∀s∈ I and the second binormal component $\langle \beta_{f(s)}, N_{\beta 3}(s)\rangle$ of β_f is given by

$$\langle \beta_{f(s)}, N_{\beta 3}(s)\rangle = \gamma(s) = \frac{1}{(K-k)_\beta(s)}\frac{d}{ds}\left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right)$$

∀s∈ I .

Inversely, we suppose that $\beta: I \to \mathbb{R}^4$ be a unit-speed curve (parametrized by arc length function s), having nonzero curvatures $K_\beta, \tau_\beta, (K-k)_\beta$. Furthermore, let $f: I \to \mathbb{R}$ be a nonzero integrable function in parameter s with at least twice differentiable primitive function F so that any one of the expressions (i),(ii),(iii) or (iv) is valid. For statement (i), we get

$$\langle \beta_{f(s)}, \beta_{f(s)}\rangle = F^2 + c^2$$

∀s∈ I, where c is a non-zero constant. On differentiation of previous equation, we find

$$\langle \beta_{f(s)}, T_\beta(s)\rangle = \mathrm{F(s)} \tag{10}$$

∀s∈ I. Thus in both cases we get equation (10). Differentiating (10) and by using (1), we attain

$$\langle \beta_{f(s)}, N_{\beta 1}(s)\rangle = 0$$

∀s∈ I. It claims to us β is an f-rectifying curve in $\mathbb{R}^4$ according to type-2 quaternionic frame.

Next, we suppose that expression (iii) is valid. At the time $\left\|{\beta_f}^{N_{\beta 1}}(s)\right\| = c$ say. Now the normal compotent ${\beta_f}^{N_{\beta 1}}(s)$ is given by

$$\beta_f(s) = F(s)T_\beta(s) + {\beta_f}^{N_{\beta 1}}(s)$$

∀s∈ I. Thus we find

$$\langle \beta_{f(s)}, \beta_{f(s)}\rangle = \langle \beta_{f(s)}, T_\beta(s)\rangle^2 + c^2$$

∀s∈ I. Differentiating last equation and using (1), we get

$$\langle \beta_{f(s)}, N_{\beta 1}(s)\rangle = 0$$

∀s∈ I. This means that β is an f-rectifying curve in $\mathbb{R}^4$ according to type-2 quaternionic frame.

Finally, we suppose that expression (iv) is valid. At the time we find

$$\left\langle \beta_{f(s)}, N_{\beta 2}(s) \right\rangle = \frac{-\tau_\beta(s)}{K_\beta F(s)} \tag{11}$$

$$\left\langle \beta_{f(s)}, N_{\beta 3}(s) \right\rangle = \frac{1}{(K-k)_\beta(s)} \frac{d}{ds}\left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right) \tag{12}$$

$\forall$s$\in I$. Differentiating (11) and using (1), we get

$$\tau\left\langle \beta_{f(s)}, N_{\beta 1}(s) \right\rangle + (K-k)_\beta \left\langle \beta_{f(s)}, N_{\beta 3}(s) \right\rangle = \frac{d}{ds}\left(\frac{-\tau_\beta(s)}{K_\beta F(s)}\right)$$

$\forall$s$\in I$. From the equations (12) and previous, we have

$$\left\langle \beta_{f(s)}, N_{\beta 1}(s) \right\rangle = 0$$

s$\forall \in I$. Therefore β is an f-rectifying curve in $\mathbb{R}^4$ according to type-2 quaternionic frame.

4. Some Open Problems

In this chapter, we suggest to research the following open problem with the attempting towards classication of the rectifying curves which are mostly based on their parametrizations.

Problem 4.1: Let that $\beta: I \to \mathbb{R}^4$ be a unit-speed curve (parametrized by arc length function s), having nonzero curvatures $K_\beta, \tau_\beta, (K-k)_\beta$. Furthermore, let $f: I \to \mathbb{R}$ be a nonzero integrable function in parameter s with at least twice differentiable primitive function F. Then β is an f-rectifying curve in $\mathbb{R}^4$ according to type-2 quaternionic frame if and only if, up to parametrization, its f-position vector β_f is given by

$$\beta_f(\mathrm{r}) = \mathrm{x}(\mathrm{r}) \cdot \frac{c}{\cos(r + arctan\frac{F(s_0)}{c})}$$

for all r$\in J$, where c is a positive constant, $s_0 \in I$ and x:J$\to S^3(1)$ is a unit speed curve having r:I$\to J$ as arc length function based at s_0.

Received September 22, 2022; Accepted December 02, 2022

References

[1] Hamilton W.R. Elements of Quaternions I, II and III, Chelsea, New York.1899.

[2] Bharathi K, Nagaraj M. Quaternion Valued Function Of a Real Variable Serret-Frenet Formulae. Ind. J. P. App. Math. 1987; 18(6): 507-511

[3] Aksoyak F.K. A New Type of Quaternionic Frame in R⁴.International Journal of Geometric Methods in ModernPhysics. 2019;16(06):1950084.

[4] Chen B.Y. When does the position vector of a space curves always lie in its rectifying plane. Amer. Math. Monthly. (2003); 110: 147-152.

[5] Chen B.Y,Dillen F. Rectifying curves as centrodes and extremal curves. Bull. Inst.Math.Academia Sinica. 2005; 33(2):77-90.

[6] Güngör MA, Tosun M. Some characterizations of quaternionic rectifying curves. Differential Geometry, Dynamical Systems. 2011; 13:102-113.

[7] Iqbal Z, Sengupta J. On f-rectifying curves in the Euclidean 4-space. Acta Universitatis Sapientiae, Mathematica. 2021; 13(1): 192-208.

[8] Iqbal Z, Sengupta J. Non-null (spacelike or timelike) f-rectifying curves in the Minkowski 3-space E_1^3., Eurasian Bul. Math. 2020; 3(1): 38-55.

[9] Iqbal Z, Sengupta J. Null (lightlike) f-rectifying curves in the Minkowski 3-space E_1^3. Fundam. J. Math. Appl. 2020; 3 (1): 8-16.

[10] Iqbal Z, Sengupta J. Differential geometric aspects of lightlike f-rectifying curves in Minkowski space-time, Diff. Geom. - Dyn. Syst. 2020; 22: 113-129.

[11] Ward J.P. Quaternions and Cayley Numbers, Kluwer Academic Publishers. Boston, London. (1997).

[12] Hacısalihoğlu H.H. Motion Geometry and Quaternions Theory. Faculty of Sciences and Arts. University of Gazi Press. (1983).