Article

TGD & Condensed Matter: Basic Notions

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Abstract

This article is the first one in a series of 2 articles devoted to the possible applications of TGD to condensed matter physics. The series of 3 articles about TGD as it is towards the end of 2021 is recommended as a background. In the first article the basic notions of condensed matter physics are considered from the TGD point of view:

1. TGD is analogous to hydrodynamics in the sense that field equations at the level of H reduce to conservation laws for isometry charges. The preferred extremal property meaning that space-time surfaces are simultaneous extremals of volume action and Kähler action allows interpretation in terms of induced gauge fields. The generalized Beltrami property implies the existence of an integrable flow serving as a correlate for quantum coherence. Conserved Beltrami flows currents correspond to gradient flows. At the QFT limit this simplicity would be lost.
2. The fields H, M, B and D, P, E needed in the applications of Maxwell's theory could emerge at the fundamental level in the TGD framework and reflect the deviation between Maxwellian and the TGD based view about gauge fields due to CP_2 topology.
3. The understanding of macroscopic quantum phases improves. The role of the magnetic body carrying dark matter is central. The understanding of the role of WCW degrees of freedom improves considerably in the case of Bose-Einstein condensates of bosonic particles such as polaritons. M^8 picture allows us to understand the notion of skyrmion. The formation of Cooper pairs and analogous states with higher energy would correspond to a formation of Galois singlets liberating energy used to increase h_{eff}. What is new is that energy feed makes possible supra-phases and their analogs above the critical temperature. TGD description of Cooper pairs is consistent with fermion number conservation and the notion of Bogoliubov quasiparticles does not seem necessary or relevant in TGD.
4. Fermi surface emerges as a fundamental notion at the level of M^8 but has a counterpart also at the level of H. Galois groups would be crucial for understanding braids, anyons and fractional Quantum Hall effect. Galois confinement suggests a universal mechanism for the formation of bound states. Space-time surface could be seen as a curved quasicrystal associated with the lattice of M^8 defined by algebraic integers in an extension of rationals. Also the TGD analogs of condensed matter Majorana fermions emerge.
5. TGD also provides new insights about topological physics and space-time topology provides a new element of topological physics.

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1 Introduction

The purpose of this article is to consider the possible applications of TGD in condensed matter physics at the general level. It must be emphasized that TGD is only a vision, not a theory able to provide precise rules for calculating scattering amplitudes. A collective theoretical and experimental effort would be needed to achieve this. The proposal for a model of superconductivity [36] provides a representative example about what TGD could possibly give for condensed matter physics.

It is perhaps good to explain what TGD is not and what it is or hoped to be. The article [35] gives an overview of various aspects of TGD and is warmly recommended. The appendix of this article provide figures illustrating the basic notions.

1. "Geometro-" refers to the idea about the geometrization of physics. The geometrization program of Einstein is extended to gauge fields allowing realization in terms of the geometry of surfaces so that Einsteinian space-time as abstract Riemann geometry is replaced with sub-manifold geometry. The basic motivation is the loss of classical conservation laws in General Relativity Theory (GRT). Also the interpretation as a generalization of string models by replacing string with 3-D surface is natural.

   Standard model symmetries uniquely fix the choice of 8-D space in which space-time surfaces live to $H = M^4 \times CP_2$ [22]. Also the notion of twistor is geometrized in terms of surface geometry and the existence of twistor lift fixes the choice of H completely so that TGD is unique [26, 28].

The geometrization applies even to the quantum theory itself and the space of space-time surfaces - "world of classical worlds" (WCW) - becomes the basic object endowed with Kähler geometry. General Coordinate Invariance (GCI) for space-time surfaces has dramatic implications. Given 3-surface fixes the space-time surface almost completely as analog of Bohr orbit (preferred extremal).This implies holography and leads to zero energy ontology (ZEO) in which quantum states are superpositions of space-time surfaces.

2. Consider next the attribute "Topological". In condensed matter physical topological physics has become a standard topic. Typically one has fields having values in compact spaces, which are topologically non-trivial. In the TGD framework space-time topology itself is non-trivial as also the topology of $H = M^4 \times CP_2$.

   The space-time as 4-surface $X^4 \subset H$ has a non-trivial topology in all scales and this together with the notion of many-sheeted space-time brings in something completely new. Topologically trivial Einsteinian space-time emerges only at the QFT limit in which all information about topology is lost.

   Practically any GCI action has the same universal basic extremals: CP_2 type extremals serving basic building bricks of elementary particles, cosmic strings and their thickenings to flux tubes defining a fractal hierarchy of structure extending from CP_2 scale to cosmic scales, and massless extremals (MEs) define space-time correletes for massless particles. World as a set or particles is replaced with a network having particles as nodes and flux tubes as bonds between them serving as correlates of quantum entanglement.

   "Topological" could refer also to p-adic number fields obeying p-adic local topology differing radically from the real topology.

3. Adelic physics fusing real and various p-adic physics are part of the number theoretic vision, which provides a kind of dual description for the description based on space-time geometry and the geometry of "world of classical" orders. Adelic physics predicts two fractal length scale hierarchies: p-adic length scale hierarchy and the hierarchy of dark length scales labelled by $h_{eff} = nh_0$, where n is the dimension of extension of rational. The interpretation of the latter hierarchy is as phases of ordinary matter behaving like dark matter. Quantum coherence is possible in all scales.

   The concrete realization of the number theoretic vision is based on $M^8 - H$ duality. The physics in the complexification of M^8 is algebraic - field equations as partial differential equations are replaced with algebraic equations associating to a polynomial with rational coefficients a X^4 mapped to H by $M^8 - H$ duality. The dark matter hierarchy corresponds to a hierarchy of algebraic extensions of rationals inducing that for adeles and has interpretation as an evolutionary hierarchy.

   $M^8 - H$ duality provides two complementary visions about physics, and can be seen as a generalization of the q-p duality of wave mechanics, which fails to generalize to quantum field theories (QFTs).

4. In Zero energy ontology (ZEO), the superpositions of space-time surfaces inside causal diamond (CD) having their ends at the opposite light-like boundaries of CD, define quantum states. CDs form a scale hierarchy.

   Quantum jumps occur between these and the basic problem of standard quantum measurement theory disappears. Ordinary state function reductions (SFRs) correspond to "big" SFRs (BSFRs) in which the arrow of time changes. This has profound thermodynamic implications and the question about the scale in which the transition from classical to quantum takes place becomes obsolete. BSFRs can occur in all scales but from the point of view of an observer with an opposite arrow of time they look like smooth time evolutions.

   In "small" SFRs (SSFRs) as counterparts of "weak measurements" the arrow of time does not change and the passive boundary of CD and states at it remain unchanged (Zeno effect).

This work led to considerable progress in several aspects of TGD.

1. The mutual entanglement of fermions (bosons) as elementary particles is always maximal so that only fermionic and bosonic degrees can entangle in QFTs. The replacement of point-like particles with 3-surfaces forces us to reconsider the notion of identical particles from the category theoretical point of view. The number theoretic definition of particle identity seems to be the most natural and implies that the new degrees of freedom make possible geometric entanglement.

   Also the notion particle generalizes: also many-particle states can be regarded as particles with the constraint that the operators creating and annihilating them satisfy commutation/anticommutation relations. This leads to a close analogy with the notion of infinite prime.

2. The understanding of the details of the M^8-H duality forces us to modify the earlier view. The notion of causal diamond (CD) central to zero energy ontology (ZEO) emerges as a prediction at the level of H. The pre-image of CD at the level of M^8 is a region bounded by two mass shells rather than CD. M^8-H duality maps the points of cognitive representations as momenta of quarks with fixed mass in M^8 to either boundary of CD in H.

3. Galois confinement at the level of M^8 is understood at the level of momentum space and is found to be necessary. Galois confinement implies that quark momenta in suitable units are algebraic integers but integers for Galois singlet just as in ordinary quantization for a particle in a box replaced by CD. Galois confinement could provide a universal mechanism for the formation of all bound states.

4. There is considerable progress in the understanding of the quantum measurement theory based on ZEO. From the point of view of cognition BSFRs would be like heureka moments and the sequence of SSFRs would correspond to an analysis having as a correlate the decay of 3-surface to smaller 3-surfaces.

The improved vision allows us to develop the TGD interpretation for various condensed matter notions.

1. TGD is analogous to hydrodynamics in the sense that field equations at the level of H reduce to conservation laws for isometry charges. The preferred extremal property meaning that space-time surfaces are simultaneous extremals of volume action and Kähler action allows interpretation in terms of induced gauge fields. The generalized Beltrami property implies the existence of an integrable flow serving as a correlate for quantum coherence. Conserved Beltrami flows currents correspond to gradient flows. At the QFT limit this simplicity would be lost.

2. The fields H, M, B and D, P, E needed in the applications of Maxwell's theory could emerge at the fundamental level in the TGD framework and reflect the deviation between Maxwellian and the TGD based view about gauge fields due to CP_2 topology.

3. The understanding of macroscopic quantum phases improves. The role of the magnetic body carrying dark matter is central. The understanding of the role of WCW degrees of freedom improves considerably in the case of Bose-Einstein condensates of bosonic particles such as polaritons. M^8 picture allows us to understand the notion of skyrmion. The formation of Cooper pairs and analogous states with higher energy would correspond to a formation of Galois singlets liberating energy used to increase h_{eff}. What is new is that energy feed makes possible supra-phases and their analogs above the critical temperature.

4. Fermi surface emerges as a fundamental notion at the level of M^8 but has a counterpart also at the level of H. Galois groups would be crucial for understanding braids, anyons and fractional Quantum Hall effect. Space-time surface could be seen as a curved quasicrystal associated with the lattice of M^8 defined by algebraic integers in an extension of rationals. Also the TGD analogs of condensed matter Majorana fermions emerge.

2 Some notions of condensed matter physics from the TGD point of view

Before continuing I must emphasize that I am not a condensed matter physicist and have no practical experience about experimental physics. Therefore I cannot propose any experimental protocols. I dare to hope that the new vision about space-time and quantum theory could inspire people who are doing real condensed matter physics.

2.1 The notion of Brillouin zone from the TGD viewpoint

In condensed matter physics the notions of lattice, reciprocal lattice, unit cell and Brillouin zone at its counterpart in reciprocal lattices are central notions.

The reciprocal lattice in momentum space is the dual of the lattice in 3-space. This follows automatically from the periodicity of properties of wave functions in the lattice : they force wave vectors to be in the reciprocal lattice. The diffraction amplitude has peaks at the photon momenta in the reciprocal lattice.

M^8-H duality can be seen as the counterpart of position-momentum duality. Therefore it is interesting to look at these notions from the point of view of M^8_H duality. Recall that 4-surfaces in $H=M^4\times CP_2$ is identified as space-time whereas the 4-surface in $M^8=M^4\times E^4$ is analogous to momentum space with slicing induced by the mass shells (hyperboloids) of M^4. In H the corresponding slicing is by CDs inside CDs with size given by the Compton length associated with mass m.

1. At the level of H, periodic minimal surfaces would nicely produce lattice-like structures and the momenta associated with the peaks of Fourier transforms would belong to the reciprocal lattice. I have considered the construction of also more general structures in [42].

2. At the level of M^8, the allowed momenta as points of $X^4\subset M^8$ belong to cognitive representations: the momentum components are algebraic integers in the extension defined by the polynomial defined the 4-surface in M^8. This guarantees the theoretical universality of the adelic physics [24, 25]) so that the points make sense also as points of the p-adic variants of space-time surface defining geometric correlates of cognition.

   Lattice-like structures are naturally associated with the lattice of algebraic integers and one obtains a hierarchy of lattices. The lattices can be seen as products of ordinary lattices in E^3 and lattices in the extension of rationals having dimension n: this feature is completely new.

2.1.1 Construction of bound states

Number theoretic vision suggests a universal way to construct bound states as Galois confined states. This would mean that many quark states in M^8 consisting of points of cognitive representation carrying quark are Galois singlets. In the case of momentum degrees of freedom this would mean that the total momentum is (rational) integer.

The physical motivation for Galois confinement is that periodic boundary conditions require integer value 4-momenta which are rational integers using a suitable momentum unit determined by the size scale of CD (Compton length $\hbar_{eff}/m$ for some particle would be in question for $\hbar_{eff}=\hbar_{gr}=GMm/v_0$ the gravitational Compton length $\Lambda_{gr}=GM/v_0=r_a(M)/2v_0$ would not depend at all on mass of the particle.

1. The condition that the total 4-momentum is integer-valued poses a strong condition on the bound states.

2. Second condition is that the inner products of the momenta (algebraic integers which can have an imaginary part) defining number theoretical metric are real valued. This poses strong quantization

conditions, and one obtains also lattice structures in the lattice defined by the unit vectors of extension and by 3-space. These lattice structures are sublattices of lattice E^3, whose points are n-D number theoretical lattices defined by the unit vectors of the extension of rationals.

3. The fundamental entities are quarks and the construction gives a hierarchy of increasingly complex bound states of them. One obtains also atoms and their lattices. Quasi-crystals are obtained as cut and project construction and it is feasible that number theoretical lattices makes them possible also now.

4. The lattices in M^8 involving particles with the same mass are actually lattices in 3-D hyperbolic space and called tessellations. In good approximation they are lattices in E^3 since H^3 can be approximated by E^3 below length scale given by h_{eff}/m which is Λ_{gr} for $\hbar_{gr}$ (.9 cm for Earth and of the order of radius of Earth for Sun).

   The structure of tessellations is extremely rich and perhaps the simplest tessellations known as icosa-tetrahedral tessellations involve all basic Platonic solids and are proposed to give rise to universal realization of genetic code having chemical realization only as a special case and having besides DNA also higher dimensional realizations [38].

   M^8 picture allows also universal 6-D brane-like solutions with a topology of 6-sphere, whose projection to CD is its intersection with 3-D hyperplane E^3 of constant energy. This plane would allow many quarks states with an ordinary lattice structure. There both hyperbolic tesselations and Euclidian lattices would be allowed.

5. Even the lattice formed by atoms would be a bound state of this kind. The reciprocal lattice in M^8 has an interpretation in terms of cognitive representation in M^8 mapped to H by $M^8 - H$ duality defined by particle momenta, which are basically bound states of quarks (also leptons).

2.1.2 $M^8 - H$ duality and the relation between lattices and reciprocal lattices

M^8 and H descriptions are related by $M^8 - H$ duality as an analog for momentum-position duality. Uncertainty Principle (UP) must be respected but what does this really require is not quite clear. The map of $X^4 \subset M^8$ to $X^4 \subset H$ is certainly involved. This would be the $M^8 - H$ duality for space-time surfaces. This description is not enough: $M^8 - H$ duality is required also at the level of "world of classical worlds" (WCW).

1. $M^8 - H$ duality at the level of 4-surfaces

Consider first the $M^8 - H$ duality at space-time level.

1. Uncertainty Principle (UP) is the basic constraint on $M^8 - H$ duality and fixes the form of $M^8 - H$ duality at the space-time level.

   One takes the momentum projection p in M^4 - an algebraic integer for cognitive representations and quarks are at these points, not all - and maps it to a point of $M^4 \subset M^4 \times CP_2$ that is to a point of $X^4 \subset H$. One assigns to p a geodesic line in the direction of momentum beginning at the common center of all CDs. In this way the slicing by mass shells of $M^4 \subset M^8$ is mapped to a slicing by CDs inside CDs (Russian doll-like structure).

2. p is mapped to the intersection of this geodesic line with the boundary of CD. One obtains the analog of the pattern produced by diffraction from the lattice. In particular, the intersections of the geodesics with the $t = T$ plane above the center point of CD form a reciprocal lattice, whose projection to the 2-D surface of a large 2-sphere corresponds to the standard diffraction pattern. One would be happy if one would obtain a lattice, rather than its reciprocal.

As if there were a lattice around the center of the ball producing the diffraction pattern as a projection of the reciprocal lattice to the heavenly sphere. Intuition would suggest that this must be the case but one must be very cautious.

3. The momenta of quarks (or atoms) are therefore mapped to the light-cone boundaries of CD and basically define boundary values for the induced quark fields for quarks composing both proton, nuclei, and even electrons. These fields would be localized at these points at the boundary of CD and disperse in the interior. Induced spinor fields are second quantized H-spinor fields restricted to space-time surface and obeying modified Dirac equation for induced geometry and determined by variational principle.

   One can assign to the points at the boundary of CD corresponding to the image of the reciprocal lattice localized states of atoms of the lattice (many-quark states). At quark level this corresponds to a superposition of spinor harmonics of H localized to the point of the boundary (this corresponds to so-called light-cone quantization). This would dictate the time evolution of the induced spinor field inside the space-time surface and it would reflect the data coding for the reciprocal lattice.

4. Does this mean the emergence of lattice (as desired) or of reciprocal lattice in the interior? Since the lattice points by definition would correspond to peaks of plane waves generated by the reciprocal lattice at the boundary of CD would expect that the peak positions define the lattice.

One can also wonder whether one could one define M^8-H duality so that it would take momentum lattice in M^8 to its dual in H? The notion of dual lattice makes sense for the lattice defined by the extension. If one defines the cognitive representation in M^8 by selecting a tessellation at the mass shell of M^8 (this might follow the conditions for bound states), one could map the momenta of tessellations to their duals and would obtain the desired result in H. It is however not clear whether the map of tesselation to its dual (if it exists) can be completed to a continuous map of H^3 to itself.

2. M^8-H duality at the level of WCW

It seems that the proposed description need not be enough to realize UP at the level of H and the "world of classical worlds" (WCW) . The objection is that localized states in M^8 correspond to delocalized states at the level of H.

The above description maps quarks at points of $X^4 \subset M^8$ to states of induced spinor field localized at the 3-D boundaries of CD but necessarily delocalized into the interior of the space-time surface $X^4 \subset H$. This is analogous to a dispersion of a wave packet. One would obtain a wave picture in the interior and the lattice should emerge.

1. The basic observation leading to TGD is that in the TGD framework a particle as a point is replaced with a particle as a 3-surface, which by holography corresponds to 4-surface.

   Momentum eigenstate corresponds to a plane wave. Now planewave could correspond to a delocalized state of 3-surface associated with a particle in M^4 and by holography that of 4-surface.

2. A generalized plane wave would be a quantum superposition of shifted space-time surfaces with a phase factor determined by 4-momentum. This suggests that M^8-H duality should map the point of M^8 containing an object with momentum p to a generalized plane wave and this is assumed.

   This would also define WCW description. Recent physics relies on the assumption about single background space-time: WCW is effectively replaced with M^4 since 3-surface is replaced with point and CP_2 is forgotten so that one must introduce gauge fields and metric as primary field variables.

3. For cognitive representations, momenta are given by algebraic integers. Lattice plane waves can be idealized as waves in a discrete lattice. This would suggest that the plane wave is replaced by a discretized plane wave corresponding to the points of H at which the plane wave has the same value. One can say that one counts only the wave crests and thus only the information about wavelength and frequency.

4. For reciprocal momenta, one obtains a wave function in H for the shifted images of the 3-surface/4-surface labelled by a vector of the reciprocal lattice in H and this wave function can be regarded as a wave function with the periodicities of lattice.

The WCW picture is necessary if one wants to take into account WCW degrees of freedom. In the approximate description of phenomena involving only elementary particles constructible from quarks, WCW is not absolutely necessary.

2.1.3 Galois confinement and lattice like structures

It is interesting to look more explicitly at the conditions for the Galois confinement.

Single quark states have momenta, which are algebraic integers generated by so called integral basis (`https://cutt.ly/SRuZySX`) analogous to unit vectors of momentum lattice but for single component of momentum as vector in extension. There is a theorem stating that one can form the basis as powers of a single root. It is also known that irreducible monic polynomials have algebraic integers as roots.

1. In its minimal form Galois confinement states that only momenta, which are rational integers, are allowed by Galois confinement. Note that for irreducible polynomials with rational coefficients one does not obtain any rational roots. If one assumes that single particle states can have an arbitrary algebraic integer as a momentum, one also obtains rational integers for momentum values. These states are not at mass - or energy shell associated with the single particle momenta.

2. A stronger condition would be that also the inner products of the momenta involved are real so that one has $Re(p_i) \cdot Im(p_j) = 0$. For $i = j$ this gives a condition possible only for the real roots for the real polynomials defining the space-time surface.

To see that real roots are some facts about the realization of the co-associativity condition [32] are necessary.

1. The expectation is that the vanishing condition for the real part (in a quaternionic sense) of the octonionic polynomial gives a co-associative surface. By the Lorentz symmetry one actually obtains as a solution a 6-D complex mass shell $m_c^2 \equiv m_{Re}^2 - m_{Im}^2 + 2iRe(p) \cdot Im(p) = r_1$, where the real and imaginary masses are defined are $m_{Re}^2 = Re(p)^2$ and $m_{Im}^2 = Im(p)^2$ and r_1 is some root for the odd part of the polynomial P assumed to determining the 4-surface.

2. This surface can be co-associative but would also be co-commutative. Maximally co-associative surface requires quaternionic normal space and the proposal is that the 6-surface having a structure of S^2 bundle defines as its base space quaternionic 4-surface. This space would correspond to a gauge choice selecting a point of S^2 at every point of M^4. To a given polynomial one could assign an entire family of 4-surfaces mapped to different space-time surfaces in H. A possible interpretation of gauge group would be as quaternionic automorphisms acting on the 2-sphere.

Concerning Galois confinement, the basic result is that for complex roots r_1 the conditions $Re(p_i) \cdot Im(p_i) = 0$ cannot be satisfied unless one requires that r_1 is real. Therefore the stronger option makes sense for real roots only.

Despite this one can also consider the strong option for real roots. There are two cases to consider. The first case corresponds to complex 4-surfaces for which complex mass squared is equal to a root of the odd part of the polynomial determining the space-time surface. The real part of these surfaces in the sense that the imaginary part of mass squared vanishes is 4-D.

These conditions lead to a spectrum of 4-momenta and masses with each mass involving a subset of momenta. One can form Galois singlets also from states with different masses.

1. One can assign to each algebraic integer n_A a Galois invariant defined as the determinant $det(N(n_A))$ of the matrix $N(n_A)$ of the linear transformation defined by a multiplication of the units of algebraic integers by n_A. The algebraic integers n_A with the same value of $det(N(n_A))$ can belong to the orbit of Galois group. Physical intuition suggests that the values of mass squared (energy) are the same for these integers in the case of H^3.

2. One expects that the group $SL(2, Z_A)$, where Z_A denotes algebraic integers associated with the polynomial defining the space-time surface produces new solutions from a given solution. This would be a discrete version of Lorentz invariance. Tessellations of H^3 are highly suggestive as bound states.

3. Since Galois group is finite, the only possibility is that Galois groups corresponds to a subgroup of rotations permuting algebraic integers with the same time-component of 4-momentum. Therefore the discrete subgroups of $SO(3)$ associated with the inclusions of hyper-finite factors of type II_1 would emerge.

The situation for the surfaces $E = E_n$, E_n the root of the polynomial P defined the 4-surface situation is different.

1. Single particle states correspond to a discrete set of in general complex mass values extending from E_n to 0. The number of momenta with given m is finite and one obtains a slicing of the space of 3-momenta by spheres $S^2(m)$ with constant mass having the allowed points of $S^2(m)$ at the orbits of Galois group. Also now single particle states are impossible but one obtains many-particle states and also lattice like structures are expected. A given mass m can correspond to several energies $E_n(m)$ giving this value of mass.

2. Also now it is possible to construct Galois singlets as many-particle states and these have rational integer valued momenta. In condensed matter, one has energy bands such that the energy inside the band depends on the momentum k. Could one think that the values of energy form bands decomposing to discrete energy levels?

Two further remarks are in order.

1. Besides the simplest realization also a higher level realization is possible: Galois singlets are not realized in the space of momenta but in the space of wavefunctions of momenta. States of an electron in an atom serve as an analogy. Origin is invariant under the rotation group and electron at origin would be classical analog of rotationally invariant state. In quantum theory, this is replaced with an s-wave invariant under rotations although its argument is not.

   In the recent situation, one would have a wave function in the space of algebraic integers representing momenta which are not Galois invariants but if one has Galois singlet, the average momentum as Galois invariant is ordinary integer. Also single-quark states could be Galois invariant in this sense.

2. The proposal inspired by TGD inspired quantum biology is that the polynomials defining 4-surface in M^8 vanish at origin: $P(0) = 0$. One can form increasingly complex 4-surfaces in M^8 by forming composite polynomials $P_n \circ P_{n-1} \circ ... \circ P_1$ and these polynomials have roots of P_1....and P_{n-1} as their roots. These roots are like conserved genes: also the momentum spectra of Galois singlets are analogous to conserved genes. This construction applies to Galois singlets in both classical and quantal sense.

   At the highest level one can construct states as singlets under the entire Galois group. One can use non-singlets of previous level as building bricks of these singlets.

2.2 Topological condensed matter physics and TGD

Topological considerations have become an essential part of condensed matter physics. In condensed matter physics the topology of patterns of order parameters and of Fermi surface play a key role. In the TGD framework the topology of space-time surface in X^4 and the dual 4-surface in M^8 having an interpretation as an analog of momentum space are non-trivial and the question how this could reflect itself in condensed matter physics.

2.2.1 Topology of the energy bands in solids

The notions of 2-D face states, edge states, and corner states seem to be behind many topological states. It is interesting to see what they could correspond to in the TGD framework.

One can imagine two alternative guesses.

1. At H level 4-surfaces as analogous to 4-D complexified momentum space are algebraic surfaces, that is 4-D "roots" of polynomials.nThese algebraic surfaces have singularities at the level of H mapped to singularities at the level of H. They can have corners, edges, and intersection points, 2-D singular surfaces. At the level of H they correspond to strings, string world sheets, and light-like orbits of partonic 2-surfaces: in this case the line singularity is blown up to a 3-D singularity.

2. These singularities need not however correspond as such to the above listed singularities since the active points of cognitive representation defined by mometa which are algebraic integers do not correspond as such to the physical states. Rather, physical states are Galois confined bound states of quarks for a given extension of rationals and it is the energy and momentum spectrum of these states which is relevant.

The second guess is based on the idea that the energy bands correspond to substructures formed by discrete 4-momenta of Galois confined states.

1. Cognitive representation consists of momenta for which momentum components are algebraic integers. Some of these points are occupied by quarks, they are "active" (this brings in mind Bohm's notion of active information).

   Physical states must have total momentum which is rational integer using the unit defined by the largest CD involved defining IR cutoff. Smallest CD defines the UV cutoff. This means Galois confinement in momentum degrees of freedom. Same happens also in spinorial degrees of freedom.

2. Bloch waves are of the form $exp(ikx)u(x)$ where u is a periodic function with the periods of lattices and k is continuous pseudo-momentum. k can be restricted to the first Brillouin zone defined as the counterpart of a lattice cell in momentum space. For Bloch states the translational symmetry is broken down to a discrete subgroup of the translation group acting as symmetries of lattices and therefore of u.

   For Bloch waves, the wave vectors and also energies would be quantized by periodic boundary conditions which would mean in the TGD framework that the momenta are integer valued using a suitable unit. The phase factors $epx(iknL)$ would be roots of unity and therefore number theoretically universal. This requires that $kL = m$ is a rational integer.

3. Mass shells as hyperboloids $H^3(m)$ are of special interest as are also the 3-D M^4 projections of 6-D universal brane-like entities. The latter are 3-surfaces $E = E_n$ where E_n is the root of the polynomial defining the 4-surface in M^8. Hyperboloid allows tessellations and the Euclidean 3-space E_3 defined $E = E_n$ surfaces inside light-cone allows lattices expected to emerge naturally from Galois confinement.

4. This picture suggests that each $E = E_n$ shell gives rise to real energy shells with rational integer valued energy and momentum components as sums of the multiples of algebraic integers for quarks. The allowed momenta for given total energy would correspond to states assignable to a given total energy analogous to a given $E = constant$ 2-surface of an energy band. The singular topologies could correspond to intersections or touchings of these bands.

One cannot exclude the possibility that the states with quarks with momenta at the singular pieces of 4-surfaces (touching along 0,1, or 2-D surface) could correspond to these singularities. For instance, the touching of two energy bands could correspond to this kind of singularity.

The article of Carpentier [1] gives a nice introduction to the topology of bands in solids and it is interesting to see the situation from the TGD point of view. Topological insulators, semimetals, so called Majorana fermions, etc. involve singular situations in which energy bands touch each other and the question is what this means at the level of M^8.

Can one have a situation in which different energy bands touch each other at a single point or possibly along 1-D or 2-D (discrete) surfaces? The discussion is very similar for mass shells $H^3(m)$ and energy bands $E^3(E_n)$ so that only the case of E^3 is discussed.

1. Consider first energy bands E_n. For a given mass m, one obtains a set of energies E_n corresponding to the roots of P. When two roots co-incide, entire energy bands coincide. This would be however the situation for single quark states which are not possible by Galois confinement for irreducible polynomials with rational coefficients.

2. Two Galois confined states belonging to different energy bands E_n have energies, which are sums of the integer combinations of rational parts of energies E_n of single particle states. These sums are identical for some states associated with E_n and E_n.

   One can imagine that these bound states energies are the same for two different values of E_n so that bands formed by bound states can touch. Even higher-dimensional intersections can be considered. Similar situation might occur for the Galois confined states associated with different mass hyperboloids.

3. In condensed matter situation momenta are defined only modulo the addition of lattice momentum, which is multiple of $\hbar_{eff}/a = N\hbar_{eff}/L$ where a and L are UV and IR length scale cutoffs defined by the smallest and largest CDs. This condition would loosen the conditions for touching.

2.2.2 Topological insulators in the TGD framework

There is a nice summary by Suichi Murakami about topological insulators [2] helpful for a newcomer to the field.

Let us summarize the basic physical properties of spin waves.

1. Topological insulator is an insulator in the bulk and therefore has a gap between valence and conduction bands. TIs have conducting surface states, which can be edge states for 2-D TIs and surface states for 3-D TIs (Dirac cone in momentum space). The edge/surface states correspond to edges/surfaces in x-space. Fig. 1 of [2] `https://cutt.ly/yRGDV1U`) provides an illustration of edge and surface states. As Fig.3 associated with a simple model for surface states illustrates, edge and surface states have a finite penetration depth to the bulk.

   For 2-D TIs, valence and conduction bands touch in 1-D k-space (see Fig. 2 of `https://cutt.ly/yRGDV1U`), which also illustrates the Dirac cone). The states with degenerate energies correspond to pairs of electrons with opposite spins and momenta related by the condition $k_1 = -k_2$ modulo lattice momentum. The electrons at opposite edges/surfaces move in opposite directions and have opposite spins. The net charge current vanishes but there is net spin current.

2. Spin orbit coupling is present. Orbital momentum is mathematically like magnetic field B effectively replaced with angular momentum L. The analog of torque for B is replaced with torque $s \times L$. This gives rise to counter propagating opposite spins and spin currents.

3. For TIs, T is not violated but PT and P are violated. The presence of magnetic fields breaking T thus destroys the edge/surface conductivity. The states are helical and have no definite parity since P changes the helicity. Superposition of states with opposite momenta and spins occurs so that spin current is formed. By the absence of magnetic field back scattering destroying the conductivity is not possible since this would require change of both spin direction and momentum direction.

   Spin orbit ($L \cdot S$) interaction is required for the formation of spin currents. L comes from the rotational motion of electrons along the surface or edge; it tends to turn L and S in the same direction so that spin waves emerge.

4. Z_2 topological quantum number Z is conserved and reflects time reflection invariance. Z can be understood from the graph of energy at the conduction band, which has suffered splitting due to the spin orbit interaction so that energy is reduced in the conduction band. The graph of energy has two topologically non-equivalent forms. The graph either connects valence and conduction bands or not. In the latter case one has an ordinary insulator (I). In the first case one has TI.

   For I, the graph has 2 or 0 intersections with the graph for the lower energy of the spin-slit state. TI has only one intersection. Perturbations invariant under time reversal do not affect the situation. More general formulation for the Z_2 invariance is in terms of the odd/even character of intersections.

Could TGD add something interesting to the notion of TI?

1. Mathematically the spin-orbit interaction is analogous to that between magnetic moment and magnetic field except that it couples orbital motion and spin and forces the correlation between spin direction and momentum and therefore the formation of a spin wave. Magnetic field does not cause this although it would parallelize spins with itself.

2. In Quantum hydrodynamics (QHD) according to TGD [41], the circular orbital motion could be accompanied by a Kähler magnetic field in the direction of angular motion possibly assignable to a monopole flux tube.

   Could this make sense now? There would be 2 magnetic fields of opposite direction associated with the two directions of rotation of electrons. They should reside at different space-time sheets. At QFT limit the net B would vanish but make itself visible as a spin current. The effect could be therefore seen as evidence in favour of many-sheeted space-time.

3. One can also consider variants of this picture. Kähler magnetic field at flux tubes would be an essential element. This can come from both M^4 and CP_2 and one can ask whether only M^4 contribution is present. Velocity of current flow would be proportional to Kaehler gauge potential which would be of opposite sign a 2 space-time sheets. This would not break T at the QFT limit.

   Note that neutrinos would experience this contribution and this provides an experimental test: could the strange behavior of solar solar neutrinos and also in laboratory be understood in terms of M^4 Kähler field in Sun or in laboratory?

2.2.3 Discrete symmetries at the level of M^8

Discrete symmetries T, PT, and CP and their violations are closely involved with the phenomena of topological condensed matter physics. The challenge is to understand T, PT, and CP violations at the level of M^8.

The definition of discrete symmetries in $H = M^4 \times CP_2$ was discussed already in my thesis [4, 22] about TGD. In particular, geometrically C corresponds to a complex conjugation in CP_2. At the level of

M^8, these discrete symmetries should allow a realization as symmetries of the polynomials defining the space-time surface.

P changes the direction of 3-momenta. The counterpart of the Fermi surface should therefore become reflection asymmetric in the violation of P. The reflections are with respect to the middle point of the CD. T changes the sign of energy and half cones of CD in H and mass shells with opposite sign in M^8 are permuted. Also the time reversed classical time evolutions are different if T is violated. One can ask whether the violation of P implies a compensating violation of T (by CPT)?

Both M^4 and CP_2 contributions to Kähler magnetic field could induce T violation and M^4 contribution could do this in long scales. If T violation takes place at the fundamental level, topological instanton term which is divergence of axial current appearing in Kähler action could induce it. The analogs of instantons induce a violation of the conservation of monopole charge. This is possible only if the M^4 projection of the space-time surface is 4-dimensional. Analogous statement applies in the case of CP_2 and CP_2 type extremals have indeed 4-D CP_2 projection.

C involves a complex conjugation and changes the signs of charges. What does this mean in M^8? The normal spaces of 4-surface in M^8 containing a preferred complex plane or having integrable distribution of them are labelled by CP_2 coordinates. They are mapped to their complex conjugates.

What happens to the polynomial defining the space-time surface? Polynomial itself is real and cannot change but its algebraic continuation to an octonionic polynomial can be different. Indeed, real function can be algebraically continued to a complex function or its conjugate.

1. The complexified octonions involve a commutative imaginary unit i. Complex conjugation with respect to i leaving the real polynomial invariant but leading to a complex conjugate of the 4-surface looks like a reasonable first guess. One can however argue that the conjugation with respect to i is associated with T.

   Recall, that the proposal [32] that co-associative 4-surfaces in M^8_c, having an interpretation as an analog of momentum space, correspond to 4-surfaces identifiable as roots of complexified octonionic polynomials yielded a cold shower. Due to Lorentz symmetry, naive counting of dimensions fails and one obtains 2 polynomial equations with complexified mass as argument stating that the mass squared is a complex root of the polynomial. The solutions correspond to common roots and are 6-D.

   The solution of the problem would be that 4-surface is the intersection of 6-surface and its complex conjugate with respect to the commuting imaginary unit i. The common root must be real but the points in the intersection can be complex. Hence the action of T on X^4 is in general non-trivial and a spontaneous violation of T is possible at momentum space level.

2. Also octonions allow conjugation. In M^4 sector conjugations for octonionic units this would give rise to P and T. In the complement E^4 the conjugations for 2-D subspaces are also possible.

   Could C relate to the commutative normal spaces of 6-D surfaces labelled by points of the CP_2 twistor space $SU(3)/U(1) \times U(1)$. Could the complex conjugation in the 2-D $U(1) \times U(1)$ fiber of this space, correspond to C. The complex conjugation would therefore act on the (integrable distribution of) 2-D normal spaces of these 6-D surfaces and would not act in $M^4 \subset M^8$.

3. At the level of H, C and P are violated for the Dirac equation for a fixed H-chirality of quarks spinors and also for the modified Dirac equation, which corresponds to the octonionic Dirac equation in M^8. Also CP is violated for the modified Dirac equation in H if the action contains topological Kähler instanton terms. This violation should have a counterpart for the octonionic Dirac equation. Since this equation selects a single point at 4-surface, the CP violation for the 4-surface could induce CP violation.

2.2.4 Instantons in the TGD framework

Instantons induce violations of CP and therefore of T in gauge theories such as QCD.

It is interesting to consider the interpretation of Q as an instanton number.

1. Montonen-Olive duality (https://cutt.ly/HE6gMX6) is associated with a gauge theory in which magnetic and electric charges are rotated so that the coefficient of YM action in the action exponential is replaced with the quantity $\tau = \theta/2\pi + 4\pi i/g^2$.

2. τ is invariant under modular transformations $SL(2,Z)$ generated by a shift $\tau \rightarrow \tau+1$ and $\tau \rightarrow 1/\tau$. The inversion symmetry has strong implications for the understanding of the strong coupling phases of quantum field theories, in which magnetic monopoles replace particles as elementary objects.

3. In the gauge theory θ is analogous to momentum. The vacuum state is plane-wave like superposition $\sum_N exp(iN\theta/2pi))|N\rangle$ of vacuum states differing by a topologically non-trivial gauge transformation as a map $S^3 \rightarrow G$. Note that ball B^3 is effectively S^3 if the gauge transformations are trivial at its boundary. The homotopy equivalence classes of gauge transformations are labelled by the winding number N. N characterizes instantons changing the magnetic charge by N units so that the ground state is a superposition of states with varying values of N transforming by a phase factor under a topologically non-trivial gauge phase transformation.

Consider now the situation in the TGD framework.

1. There are differences between TGD and gauge theory context. Gauge group is replaced with $U(1)$ having a trivial third homotopy group.

   Could a localized version of the quaternionic automorphism group $SO(3)$ serve as a counterpart of a gauge group. The surfaces in M^8 can be indeed thought of as maps from M^4 to the quaternionic automorphism group G_2.

2. The non-trivial gauge transformations - $U(1)$ instantons - are clearly possible. The non-trivial gauge transformation could correspond to a tological non-trivial gauge transformation $A + nd\phi$, where ϕ is angle coordinate around axis going through a line singularity as a puncture in 3-space associated with the time-like line connecting the tips of CD. Note however that color gauge action reduces to the Kähler action so that both interpretations might make sense.

3. Kähler action generalizes to

$$S_K = \frac{1}{\alpha_K}\int J\wedge *J\sqrt{g} - (\frac{}{i}\theta/2\pi)\int J\wedge J\sqrt{g} \quad . \tag{2.1}$$

   Since only the exponent of S_K matters in the vacuum functional, I contributes a non-trivial phase factor to the Kähler function only for $exp(i\theta/2\pi) \neq 1$ ($\theta \neq n2\pi$). One can assign θ to both M^4 and CP_2 parts of Kähler action. The value of instanton term characterizes the non-conservation of the axial (monopole) current having instanton term as divergence.

   If one assumes self-duality of the gauge field true for instantons interpreted as gauge fields in S^4, the action reduces to ordinary Kähler action with coefficient proportional to τ. Interestingly, the quaternionic projective space M^4/Q can be regarded as S^4 so that Hamilton-Jacobi structures of M^4 proposed to serve as moduli space for the self-dual Kähler fields in M^4 could appear naturally.

4. $I(CP_2)$ is non-trivial due to the non-trivial homology of CP_2. $I(CP_2)$ gives a 3-D contribution, which appears at the boundaries between Minkowskian and Euclidean regions of the space-time surface as a topological Chern-Simons term and affecting the boundary conditions at the light-like orbits of partonic 2-surfaces in this way. These boundaries have interpretations as light-like parton orbits carrying quarks lines.

5. If CD contains a time-like "hole" along the axis connecting the tips of CD, also $I(M^4)$ is non-trivial. One can imagine extremals for which a genuine hole in the metric sense is generated along the M^4 time axis. What is required is that the induced metric using M^4 coordinates is of the form $dt^2 - dr^2 - (r^2 + r_0^2)d\Omega^2$. These holes should correspond to "blow-ups" of singularities of the algebraic surface in M^8. Now the 3-D tangent spaces would have no special direction at the singular points. For CP_2 type extremals the same would hold true at the level of M^8. Could this "hole" be the TGD counterpart of the blackhole of GRT and could it serve as a signature of CD?

6. $J \wedge J$ is non-vanishing only if the M^4 *resp.* CP_2 projection is 4-D. This does not guarantee self-duality unless also the induced metric reduces to the metric of M^4 *resp.* CP_2. This is true for the canonical embedding of M^4 and for CP_2 type extremals having light-like M^4 projection. Self-duality is true for the Kähler forms of M^4 and CP_2 but not for the induced Kähler forms $J(M^4)$ and $J(CP_2)$. Therefore classical gravitation breaks the self-duality and Montonen-Olive duality in the TGD framework. The possibility of extremals with M^4 and CP_2 projections smaller than $D = 4$ implies that θ is effectively vanishing for them.

$\theta(M^4)$ and $\theta(CP_2)$ as fundamental parameters obeying number theoretical coupling constant evolution would imply a violation of CP symmetry in both M^4 and CP_2 sector. Are the instanton terms present at the fundamental level or are they present only at the QFT limit and induced as a description of spontaneous violation of CP and T? Indeed, as in the condensed matter systems, CP violation could be caused by the magnetic part of the generalized Kähler action even without instanton term.

1. The strong CP problem of QCD is due to instanton inducing an instanton term in effective color YM action. The parameter characterizing the violations should be very small.

   In the TGD framework, a proposal for a solution of this problem could be that the counterpart of the color gauge field does not allow instantons. Here one must be cautious however. The components of the proposed classical color gauge field are proportional to the products of Hamiltonians of color isometries and Kähler form and instanton terms for the induced Kähler form would induce a CP violation. Indeed, Kähler action can be also regarded as a color gauge action and therefore instanton term makes sense for it.

2. Could $\theta(M^4)$ and $\theta(CP_2)$ induce a CP violation consistent with the observed CP violation in hadron physics or does one encounter the strong CP problem also in the TGD framework?

   If hadrons are string-like objects, they correspond to flux tubes as deformations of strings. For deformations with dimension $D < 4$, instanton term vanishes. Could this be the reason for the small violation of CP at the level of M^4? For CP_2 type extremals, $I(CP_2)$ is non-vanishing but equal to the Kähler action and non-dynamical for the basic CP_2 extremals since dynamics in in M^4 degrees of freedom with CP_2 taking the role of arena of physics. Could these effects make the hadronic CP violation small?

3. Matter-antimatter asymmetry is a CP violation, which does not look small at all. If the mechanism is actually a small CP violation implying that rate for the condensation of antiquarks to leptons is slightly larger than that for the condensation of quarks to antileptons, the matter antimatter symmetry could emerge during a very early period of the cosmic evolution when leptons were formed.

4. There are also further questions. Could the QCD instantons have TGD counterparts as Hamilton-Jacobi structures and also as analogs of S^4 instantons in the quaternionic projective space of octonions which would be 4-D mass hyperboloid H^4 as Minkowski analog of S^4 but with space-like signature. Could the parameter θ in the instanton term of Kähler action induce the formation of the ground state (θ vacuum) as a superposition of space-time surfaces with various instanton numbers in the sector of WCW consting space-time surface with 4-D M^4 and/or CP_2 projection?

2.3 The new view about classical fields

The TGD view about classical gauge fields differs in many aspects from the Maxwellian and gauge theory view since the classical fields associated with the system define a geometric what I call its field body (magnetic body (MB)) is the term that I have used. MB can carry also electric fields very closely related to magnetic fields unless the corresponding space-time surface is static. MB consists of flux tubes and flux sheets.

There are 2 kinds of cosmic strings: with monopole flux or without it: see the TGD glossary and the ilustration of monopole flux tube in appendix of the article "TGD as it is towards end of 2021" [43] (`http://tgdtheory.fi/public_html/articles/TGD2021.pdf`). The simplest cases correspond to Y^2, which is either a homologically non-trivial or trivial geodesic sphere of CP_2.

This predicts two kinds of magnetic flux tubes and two kinds of magnetic and electric fields. This suggests a possible interpretation for the fields H, M, B appearing in Maxwell's theory as field H carrying monopole flux requiring no current as source, magnetization M as non-monopole part induced by H , and $B = H + M$ as their sum experienced by test particle in many-sheeted space-time. The same would apply to D, P and E. If this interpretation is correct, TGD would have been secretly present in Maxwell's theory from the beginning.

The proposal that MB serves as a seat for dark matter as $h_{eff} = nh_0$ phases is central in the TGD inspired theory of consciousness and living matter. MB would be the boss and receive sensory input from ordinary biomatter and control it. This would happen in terms of dark photons with frequencies in EEG range and also in other ranges. The energies would be in the visible and UV range assigned to biophotons to which the dark photons would transform.

Magnetic flux tubes could accompty quantum vortices appearing in various macroscopic quantum phases. Even the hydrodynamical vortices in macroscopic scales could correspond to quantum coherent magnetic flux tubes with a large value of h_{eff} acting as a master forcing the coherent dynamics or ordinary matter. In hydrodynamics the classical Z^0 magnetic field, which in situations allowing skyrmions, is proportional to the induced Kähler form, could be important. Large parity breaking effects would be the prediction.

Also the view about radiation fields changes. Massless extremals (MEs)/topological light rays are counterparts for massless modes. They allow a superposition of modes with a single direction of massless momentum. The ordinary superposition of gauge potentials in gauge theory is replaced with union of space-time surfaces with common M^4 projection. The test particle experiences the sum of gauge potentials associated with various space-time sheets so that the gauge potentials effectively superpose. Ideal laser beam is a convenient analogy.

MEs are ideal for precisely targeted communications without dispersion and dissipation. MEs are soliton-like entities and one can ask whether MEs could provide a model for solitons or accompany solitons. TGD based model for nerve pulse involves Sine-Gordon solitons with large h_{eff} assigned to the cell membrane and dark Josephson radiation would have MEs as space-time correlate [11, 7, 12].

MEs do not allow standing waves possible in Maxwell theory but a set theoretic union of parallel MEs can effectively give rise to standing waves. Lorentz transformations give rise to waves moving with arbitrary sub-luminal velocity. Even a superposition in which fields effectively sum up to zero but there is a non-vanishing energy density as sum of energy densities for the two MEs, is possible.

2.4 About quantum criticality in TGD

In TGD number theoretical vision about physics brings a new view about quantum criticality.

1. Quantum criticality is actually the basic assumption of TGD: the Kähler coupling strength α_K appearing in the classical action principle of TGD would be analogous to a critical temperature and have a discrete spectrum. This would make the theory unique. All space-time sheets are quantum critical but at QFT limit this is of course masked by the replacement of sheets with a single region of M^4 made curved.

2. At the number theoretical M^8 side there is no action principle. The universality of the dynamics could be seen as a manifestation of quantum criticality. Can α_K emerge at M^8 level somehow from scattering amplitudes in M^8 and have a number theoretical origin [40].

   At the level of H coupling constants are visible only at the level of frames defining the space-time as an analog of soap film. The parts of the frame are images of singularities for the X^4 in M^8. The challenge is to understand how the singularities of the space-time surfaces determine α_K already at the level of M^8?

   p-adic thermodynamics for mass squared predicts a spectrum of temperatures with values coming as inverse integers [9, 5]. Also this temperature quantization could be seen as a counterpart for the quantum criticality.

3. Quantum criticality involves long range correlations and the hierarchy of Planck constants characterizing them [16, 17, 18]. h_{eff} corresponds to a dimension of extension of rationals characterizing the space-time surfaces. At criticality there is quantum superposition of space-time surfaces with various values of h_{eff} corresponding to polynomials defining the X^4 and one value of h_{eff} is selected in state function reduction.

2.5 What infinite-volume limit could mean in TGD?

Infinite volume limit corresponds to both thermodynamic and QFT limit and should be understood in the TGD framework. The questions are what it means if the infinite volume limit is actually realized and whether this has practical consequences.

1. At the level of ZEO infinite volume limit means that the size of causal diamond (CD) as an analog of Nature given quantization volume becomes infinite. The scattering amplitudes coded by zero energy states conserve Poincare quantum numbers at this limit.

2. At the level of H the volume action vanishes since the p-adic length scale dependent cosmological constant $\Lambda \propto 1/L_p^2$ approaches zero at the limit when the p-adic length scale L_p characterizing the X^4 becomes infinitely large.

   If $\Lambda = 0$ phase is real, the action would reduce to mere Kähler action containing both M^4 contribution and CP_2. In this case, one would also have extremals of form $X^2 \times Y^2$ for which CP_2 projection if the Lagrangian manifold with vanishing induced Kähler form.

3. In the number theoretic picture infinite volume limit in H could mean that polynomials defining $X^4 \subset M^8$ mapped to H are replaced with analytic functions with rational coefficients.

   Polynomials are assumed to vanish at origin (this guarantees that roots are "inherited" in their functional composition) and so should also the analytic functions. The inverse 1/f is infinite at origin and does not belong to the set so that one does not have a function field. Since one has only multiplication, one can speak about functional primes as in the case of polynomials.

   One can ask whether they should satisfy conditions guaranteeing that they can be regarded as polynomials of infinite order. Could one speak about polynomials of infinite degree as the limit of functional composites of polynomials with finite degree. As a matter of fact, infinite Galois groups are profinite groups and this requires this kind of inverse limit definition [37].

   A concrete example is provided by the iteration of a polynomial of finite degree [37]. In this case the spectrum of roots contains a continuous part at the limit so that complex numbers as completion of rationals would emerge at the infinite volume limit much like the continuum spectrum of momenta emerges from a discrete spectrum.

2.6 The notions of geometric phase, Berry curvature, and fidelity in TGD?

Non-contractible ground state Berry phase in the loop over the parameter space is associated with QPTs and is associated Berry curvature defining non-trivial U(1) holonomy (`https://cutt.ly/RWy7Deq`) Geometric phase (`https://cutt.ly/6Wy7GIT`) is a more general notion. It can be associated with homotopically non-trivial loops. For homotopically trivial loop geometric phase is due to non-trivial holonomy manifesting itself as Berry curvature. The Aharonov-Bohm effect represents an example about non-trivial holonomy. Electrons pass along paths closing together a region containing a magnetic field, which vanishes at the paths. Berry phase can be associated with loops in the parameter space for the Hamiltonian modelling the system.

Fidelity [3] (`https://cutt.ly/VWy5sVj`) defines a metric in the space of parameter dependent quantum states. It could be induced from metric of the parameter space. The abrupt changes of fidelity serve as a signature of quantum criticality.

Is this possible at the level of WCW?

1. WCW is a Kähler manifold [14, 8]. Finite-dimensional Kähler manifolds have a trivial homotopy group. Complex coordinates of WCW contributing to Kähler form and metric correspond to complex coordinates. In these degrees there should be no homotopically trivial loops so that topological phase is not possible. The curvature of the Kähler form can however have effects.

2. The remaining degrees of freedom are zero modes and define the analog of the base space in bundle theory. They appear as parameters - essentially classical background fields - in the Kähler metric and Kähler form. The topology in the zero modes can have non-trivial homotopy. Geometric phase could be assigned with homotopically trivial loops in the zero modes.

   At the infinite-volume limit the sub-WCW defined by the degenerate ground states with a Lagrangian manifold Y^2 as CP_2 projection (vanishing Kähler form and color gauge fields but non-vanishing weak gauge fields) is highly interesting. The preferred extremal property could exclude these space-time surfaces.

It seems that TGD could provide a unified description of all these exotic quantum coherent phases.

2.6.1 How the description in terms of Berry phase and fidelity could relate to TGD?

Consider first the identification of the TGD counterparts of Berry phase and fidelity.

1. In TGD the ground states are defined as space-time surfaces/3-surfaces and quantum states are their superpositions. The Kähler metric defines the analog of the quantum metric and the Kähler form corresponds to Berry curvature.

   The fidelity of two quantum states $\Psi(\lambda)$ and $\Psi(\lambda+\delta\lambda)$ is defined as the overlap $\langle\Psi(\lambda)\rangle\Psi(\lambda+\delta\lambda)\rangle$ in parameter space. The fidelity for nearby states is expected to change dramatically at singularity.

   Fidelity at the level of WCW - rather than WCW spinor fields representing quantum states - would mean disappearance of appearance of quantal WCW degrees of freedom as zero models transform to dynamical quantal degrees of freedom or vice versa. This change would make itself visible at the level of quantum states whose inner product depends on the WCW Kähler metric.

2. WCW also allows spinor connection with some gauge group acting as non-abelian holonomies. This corresponds to non-Abelian Berry phase Kac-Moody algebras of H isometries are an excellent candidate in this respect. WCW allows super-symplectic group as isometries.

3. WCW metric has also zero modes, which do not contribute to the WCW metric. Any symplectic invariant associated with X^4 defines such an invariant and the induced CP_2 Kähler form is invariant under the symplectic transformations of CP_2 and can be said to define a continuum of this kind of

invariants. This could induce a geometric phase, which is not due to a holonomy but non-trivial homotopy.

Kähler magnetic fluxes over 2-surfaces define such invariants. For closed surfaces these invariants reduce to quantized magnetic fluxes. Also M^4 Kähler form defines such invariants. At the boundary of CD the sphere S^2 (light-like radial coordinate =constant) has symplectic structure and also this defines solid angles assignable to 3-surfaces as seen from the tip of the CD as invariants.

2.6.2 Could the singularity of the quantum metric relate to number theoretical physics?

The singularity of the quantum metric would mean a reduction of the number of the dynamical quantum degrees of freedom contributing to the WCW metric meaning that the rank of the WCW metric tensor decreases. At criticality complex coordinates would transform to zero modes. Some complex coordinates of WCW would reduce to real coordinates. This would correspond to quantum criticality. In a concrete mechanical system some eigen modes would vanish and corresponding frequencies would become zero.

Since the TGD Universe is quantum critical and this is expected to be a generic phenomenon. Quantum criticality involves long range fluctuations which would correspond to large values of h_{eff} and therefore space-time surfaces which are algebraically complex. Could these long range fluctuations relate to almost zero modes with small frequencies and large wave lengths?

These phase transitions could be number theoretic. They would change the polynomial defining the X^4 (recall that quantum state is the superposition of space-time surfaces in ZEO). The dimension n for the extension of rationals is equal to the order of the Galois group and would change. Galois symmetries would act as zero mode symmetries. The dimensions of the representations of the Galois group in terms of quarks would also change. The change in the number of degrees of freedom would change the fidelity.

n defines also the algebraic dimension of the integers extended to algebraic integers for extension as a space regarded as a ring of integers. If algebraic integers can define components of the momenta, the dimension of the momentum space with integer components of momentum increases from 3 to $3n$ as the dimension of the Galois group increases by factor n. This increase occurs in the transitions in which the polynomial Q defining the space-time region is replaced with $P \circ Q$ such that P defines n-dimensional extension.

This would have rather dramatic effects since the radius of the Fermi ball with radius would be reduced by factor $1/n$ and could contain the same number of states as ordinary Fermi ball: this would mean an increase of density by factor n^3 corresponding to n sheets. Quasicrystal structure in both $X^4 \subset M^8$ and its images in $X^4 \subset H$ is also suggestive.

2.6.3 Does infinite volume limit have spin-glass type degeneracy?

One can look at the situation also at the infinite volume limit. At the infinite volume limit the action is expected to reduce to Kähler action. Whether this implies ground state degeneracy depends on whether preferred extremal property allows it.

1. In the original picture there was only CP_2 contribution to Kähler action. This implies huge vacuum degeneracy of CP_2 Kähler action. Any X^4 with CP_2 projection which is 2-D Lagrangian manifold is a vacuum extremal. WCW metric becomes singular if its inverse does not exist: this means singularity and the existence of zero modes. 4-D spin variant of glass degeneracy (`https://cutt.ly/ORuZfgu`) and classical non-determinism emerge. Classical non-determinism does however not look physically acceptable.

2. The twistor lift forces the Kähler action to have also an M^4 part obtained by analytical continuation from E^4. Does the resulting Kähler action have ground state degeneracy at infinite volume limit?

   The simplest extremals are of the form $X^4 = X^2 \times Y^2$, X^2 a minimal surface in M^4 and Y^2 a Lagrangian manifold in CP_2. Symplectic transformations in CP_2 degrees act like $U(1)$ gauge transformations on CP_2 Kähler gauge potential and do not affect either Kähler form nor the Lagrangian manifold property.

Only the induced metric is affected so that the effects are purely gravitational. This gives rise to the ground state degeneracy. The area of CP_2 projection is not changed and the action is affected only by the change of the induced metric. Conserved quantities are modified only by gravitational effects and are non-vanishing. The extremals are deterministic and apart from gravitational effects one has a huge ground state degeneracy analogous to spin glass degeneracy.

Apart from gravitation, the WCW Kähler metric receives contributions only from M^4 degrees of freedom, which are not affected under these deformations. Could one say that CP_2 degrees have transformed to zero modes?

3. One can also have surfaces $X^2 \times Y^2 \subset M^4 \times CP_2$ such that both X^2 and Y^2 are Lagrangian manifolds at infinite volume limit. These would be vacuum extremals. Preferred extremal property should exclude them. Could the interpretation be that all quantum degrees of freedom have transformed to zero modes?

4. One can invent objections against this proposal.

   (a) Negative energies might emerge from the electric energy in M^4 degrees of freedom. Electric field gives a negative contribution to energy density. Signature is Minkowskian for M^2 subset $M^2 \times E^2$. The M^2 part of Kähler form is obtained from its E^2 variant by multiplication with factor i. This might cause problems.

   (b) These surfaces are extremals but the preferred extremal property could fail since the needed 4-D analog of complex structure is missing since Y^2 as a Lagrangian manifold is not a complex surface of CP_2.

   (c) There is however also an argument in favor of this picture. Ordinary Maxwellian magnetic fields correspond to a homologically trivial geodesic sphere of CP_2 and they are Lagrangian submanifolds. Therefore one cannot exclude the proposal.

2.6.4 The parameters of the effective Hamiltonian from the TGD point of view

Could the parameters of effective Hamiltonians have counterparts at the level of WCW?

1. 4-surfaces as WCW points define parameters in the analogs of eigenvalues of observables. Both supersymplectic and Kac-Moody algebras have as parameters the parameters coding the point of WCW and Kac-Moody algebra. Number theoretic coding of ground states based on the Galois group as a symmetry group and p-adic primes defining p-adic length scale is what comes to mind.

   The preferred 4-surfaces would naturally correspond to the maxima of Kähler function. It is quite possible that Kähler coupling constant is complex so that the complex number defining the exponent of Kähler function has phase $\pm pi/2$. The phase of the exponent is different and maxima are also stationary points. This would make possible interference effects central in QFTs. This is implied by the condition that classical conserved charges are apart from a phase factor real and can therefore be made real.

   If M^8 space-time sheets are defined as "roots" of polynomials with rational coefficients [32, 33], WCW becomes discrete and has the coefficients of polynomials as coordinates of a given point (X^4). An open question is why the maxima of Kähler function should correspond to rational polynomials with rational coefficients.

2. Super-symplectic transformations [6, 14] as isometries of WCW are symmetries and can be regarded as a generalization of Kac-Moody type symmetries. The complex coordinate z and light-like radial coordinate r of the light-cone boundary are in the role of parameters. Analog of 3-D gauge group

but gauge group replaced with the symplectic group of $S^2 \times CP_2$ is in question. The light-like orbits of partonic surfaces could naturally carry Kac-Moody algebra representations of isometries - at least at infinite volume limit.

Non-negative conformal weights parameterize the representations of this algebra. The construction of states would be as follows. A sub-algebra $SCA_{n_{max}}$ with conformal weight larger than n_{max} and its commutator with the entire algebra annihilate states. Only the states with conformal weight smaller than n_{max} remain. Other degrees of freedom are effectively gauge degrees of freedom. n_{max} is expected to depend on the polynomial, its Galois group and degree. A huge reduction of degrees of freedom takes place. The remnant of the super-symplectic group would act as dynamical symmetries.

Same could occur in the symplectic degrees of freedom labelled by Hamiltonians which are products of S^2 and CP_2 Hamiltonians. The only non-trivial normal subalgebra corresponds to isometries and states would be annihilated by the generators in the complement of this algebra.

Rational coefficients of a polynomial defining the X^4 serve as the parameters characterizing the ground state. Higher level description is in terms of the Galois group which depends only weakly on the polynomial.

3. What about the description at the level of X^4? The solutions of modified Dirac action for induced spinor fields depend on the parameters characterizing the space-time surface.

2.7 Quantum hydrodynamics in TGD context

In the standard picture quantum hydrodynamics is obtained from the hydrodynamic interpretation of the Schrödinger equation. Bohm theory involves this interpretation. (`https://cutt.ly/cWy3O9Ts`).

1. Quantum hydrodynamics appears in TGD as an *exact* classical correlate of quantum theory [15]. Modified Dirac equation forces as a consistency condition classical field equations for X^4. Actually, a TGD variant of the supersymmetry, which is very different from the standard SUSY, is in question.

2. TGD itself has the structure of hydrodynamics. Field equations for a single space-time sheet are conservation laws. Minimal surfaces as counterparts of massless fields emerge as solutions satisfying simultaneously analogs of Maxwell equations [42]. Beltrami flow for classical Kähler field defines an integrable flow [36]. There is no dissipation classically and this can be interpreted as a correlate for a quantum coherent phase.

3. Induced Kähler form J is the fundamental field variable. Classical em and Z^0 fields have it as a part. For $S^3 \subset CP_2$ em and Z^0 fields are proportional to J: which suggests large parity breaking effects. Hydrodynamic flow would naturally correspond to a generalized Beltrami flow and flow lines would integrate to a hydrodynamic flow.

4. The condition that Kähler magnetic field defines an integrable flow demands that one can define a coordinate along the flow line. This would suggest non-dissipating generalized Beltrami flows as a solution to the field equations and justifies the expectation that Einstein's equations are obtained at QFT limit.

5. If one assumes that a given conserved current defines an integrable flow, the current is a gradient. The strongest condition is that this is true for all conserved currents. The non-triviality of the first homotopy group could allow gradient flows at the fundamental level. The situation changes at the QFT limit.

6. Beltrami conditions make sense also for fermionic conserved currents as purely algebraic linear conditions stating that fermionic current is a gradient of some function bilear in oscillator operators. Whether they are actually implied by the classical Beltrami conditions, is an interesting question.

7. Minimal surfaces as analogs of solutions of massless field equations and their additional property of being extremals of Kähler action gives a very concrete connection with Maxwell's theory [42].

2.8 Length scale hierarchies

The length scale hierarchy associated with the hierarchy of Planck constants and p-adic length scale hierarchy lead to the proposal that one has quantum coherence and supra phase always realized in some scale and the loss of say superconductivity means only the reduction of this scale.

Also dark variants of valence electrons make sense and there is evidence for them. When looking at the definition of say exciton, one cannot avoid the impression that something is missing. Electrons and holes are assumed to have incredibly small effective masses. The very notion of effective mass is in conflict with the idea that one has a fundamental quantum theory description.

One also introduces in the Schrödinger equation dielectric constant which comes from macroscopic description. Why doesn't one do the same in the case of ordinary atoms. This kind of mixing of phenomenological descriptions with a fundamental description is to me a deadly sin.

One cannot avoid the crazy looking question whether exciton could be a valence electron which is dark with $h_{eff} = k \times h$ and binds with an atom. It would be automatically accompanied by a hole. The binding energies would be scaled like $1/k^2$ and one would obtain the energies which can be 3 orders of magnitude smaller than those for hydrogen.

2.9 A general model of macroscopic quantum phases

2.9.1 Hierarchy of quantizations at the level of WCW

Before saying anything about macroscopic quantum phases, one must define what many-particle states correspond at the level of WCW.

1. The combination of UP with $M^8 - H$ duality leads to the view that many particle states at the level correspond to many-fermion (quarks actually) such that the momenta of quarks correspond to momenta as points of $X^4 \subset M^8$ with components, which are algebraic integers. In TGD framework, where all particles, also bosons, are composites of fermions. At M^8 level Cooper pairs would correspond to pairs of occupied points of a mass shell $H^3 \subset M^8$. The image of the region of momentum space in H corresponds for quarks of given mass m corresponds to a region at the boundary of sub-CD with size given by Compton length $L = \hbar_{eff}/m$.

2. At the level of WCW, the analog of the many-quark state associated with a given quark mass corresponds to the analog of plane wave inside a large $CD \subset H$ defined by the smallest mass involve but with point-like particle replaces with space-time surface inside sub-CD (CD(m)) carrying zero energy state characterized by quark momenta at opposite boundaries of CD(m) having opposite sign of energy.

3. The entanglement between these states due to Fermi statistics is however maximal and SFRs are not possible. How can one construct entangled states. The answer is simple perform the analog of second quantization at the level of WCW. One can form the analogs of 2-particle states by taking two CDs with specified quark content and assign to both the analogs of plane waves. If the CDs correspond to different extensions of rationals so that the effective Planck constants are different, one can entangle these states in WCW degrees of freedom. One can construction N-particle states by using the same recipe.

4. To each many quark state one can assign odd or even boson number and regard this state as analog of elementary fermion or boson. This is what is indeed done quite generally. Could this operation have deeper meaning. Could one require that the many-quark operators indeed commute or anticommute mutually. This condition cannot hold true generally but could be posed as an additional condition

to the physical states: the commutator/anticommutation would be proportional $h_{eff}I$, I identity matrix.

This construction would be third quantization. And nothing prevents from performing also fourth quantization within even larger CD. This hierarchy of quantizations brings in mind the basic hierarchical structures of the TGD Universe: many-sheeted space-time characterized by p-adic and dark length scale hierarchies, and also the hierarchy of infinite primes which corresponds to a repeated second quantization of supersymmetric arithmetic QFT [20] conjecture to correspond to the hierarchy of space-time sheets.

2.9.2 WCW description of BECs and their excitations as analogs of particles

Fermi statistics requires that the BEC correspond to a distribution of correlated momentum pairs with the sum of the momenta equal to the momentum of the boson. Cooper pairs also have binding energy so that the mass of the pairs is slightly smaller than the particle mass so that the Cooper pairs belong to different $H^3 \subset M^8$ than the free fermions.

For the excitations of BEC condensate giving rise to supracurrents and superflows, some momenta of fermions are different from the common momentum of BEC, usually larger than the common momentum of BEC. The image of excitation of BEC in H would be a pair at proper time=constant hyperboloid in H and the map of momentum to position would be linear inside $CD(m)$. BEC would look very much the same at both M^8 and H side of duality.

The space-time surface $X^4 \subset CD(m)$ should correspond to a minimal surface and to a generalized Beltrami flow defining an integrable coordinate along the flux lines. In the case of conserved current gradient flow (vortex flow is an example of this). All many-particle states would be of this kind in the scale of $CD(m)$. These multi-BEC states would be analogs of many-particle states and one would have many-particle states of BECs and their condensates, which could entangle in WCW degrees of freedom. For instance, the entanglement between geometric representations of Galois groups is possible. In the TGD inspired quantum biology the multi-BEC like states are proposed to play a key role [34, 38].

2.9.3 Superconductivity and superfluidity in TGD framework

The TGD based view about superconductivity and fluidity [36] differs in many respects from BCS theory.

1. In the BCS theory superconducting state does not have a well defined fermion number and this leads to a somewhat questionable notion of coherent state of Cooper pairs. The Bogobliubov transformation creates the diagonalizable oscillator operator basis by mixing creation and annihilation operators. The resulting operators create superpositions of electrons and holes.

   In the TGD framework, the interpretation would be that the hole actually corresponds to dark fermion with $h_{eff} > h$ at dark space-time sheet so that fermion number conservation is not lost. Bogoliubov operators would be replaced with superpositions of creation/annihilation operators associated with different space-time sheets and create states which are superpositions of state at the two space-time sheets. Effective Hamiltonian would include diagonalizable kinetic parts assignable to both space-time sheets, and the terms quadratic in creation/annihilation operators breaking fermion number conservation would be replaced with pairs of creation and annihilation operators associated with different space-time sheets describing the transfer of electron between the space-time sheets.

2. In the BSC theory Cooper pairs are carriers of supra current. In the TGD framework, dark electrons at dark spacetime-sheets could be the carriers. The binding energy of Cooper pairs liberated in their formation would provide the energy needed to transform ordinary electrons to dark electrons (the energies of particle states typically increase with h_{eff}). This makes possible superconductivity driven by energy feed possible also above critical temperature.

3. Can one describe supra currents and supra flows in terms of a single space-time surface as the classical space-time view based on Beltrami currents would suggest? This would mean that supracurrent would correspond to a collection of momenta of dark electrons at $H^3 \subset M^8$ in the proposed TGD based model or collection of Cooper pairs with $h_{eff} = h$ as in the standard description. The current carriers would have fixed momenta at the two boundaries of $CD(m)$ corresponding to the analogs of initial and final state momenta. Is this all that one can say at the quantum level and is the description as a flow only a classical description. At quantum level one could only deduce the change of the positions for the group of particles defining the flow. This indeed conforms with the UP.

2.9.4 WCW level is necessary for the description for purely geometric bosonic excitations

The quantum description of sound requires WCW description since the phonons as oscillations of relative position of particles cannot be described in terms of quark-antiquark pairs. The description of exotic supra flows like that associated with magnon BEC in say ^3He supra fluid allowing orbital magnetization requires WCW. A good manner to clarify thoughts is to look at what this means in the case of magnons.

1. Standard classical description (`https://cutt.ly/HRuZh53`) suggests a direction of magnetization M which has changed due to the presence of external field H. This leads to the Landau-Lifschitz equation for the magnetization.

   The Fock space picture about magnons is as a plane wave for which the argument is the position of spin whose direction has changed. The quantization is described by introducing a Hamiltonian for spins. The relationship between these descriptions is somewhat obscure.

2. In TGD the fermionic Fock space description is not possible. Bosonic creation and annihilation operators would be needed but one cannot construct bosonic operators with a vanishing fermion number from quarks. Therefore magnons should correspond to WCW degrees of freedom.

3. In the TGD description, M would correspond at space-time level to the magnetic field at a non-monopole flux tube and H possibly at a monopole flux tube inducing the magnetization. Magnons would correspond to magnetization waves, as kinks propagating along magnetic flux tubes for M. Magnon should correspond to space-time surface H and this would determine its M^8 pre-image. If these excitations behave like identical particles, one can assign to them wave vectors and classical momenta.

4. Also the notion of BEC makes sense at WCW level since one can construct the counterparts of genuine bosonic oscillator operators. Super-symplectic and Kac-Moody algebras of WCW acting at the boundaries of CD indeed include purely bosonic operators. Similar description at WCW level applies also to phonons as quanta.

   Cooper pair BECs allow approximate description in terms of fermion pairs with given total momentum but with members having different momenta. One cannot however exclude the possibility that there purely bosonic BEC at WCW level such that each Cooper pair is associated with a bosonic excitation of space-time surface.

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