Article

Multifractal Modeling of Black Hole Thermodynamics

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Abstract

The gravitational dynamics of many-body systems is a replica of Hamiltonian chaos, whose phase-space description requires the tools of multifractal analysis. Multifractals are collections of self-similar sets and their distribution of continuous dimensions follows a *singularity spectrum.* Building on the multifractal characterization of Black Holes (BH), this brief report argues that a) the gravitational mass of BH reflects an attribute of the singularity spectrum, b) shifting the maximum of the singularity spectrum produces effects analogous to the Hawking radiation.

Keywords: Hamiltonian chaos, multifractals, thermodynamics, black Holes, Hawking radiation.

This report is a sequel to [1], where *multifractal geometry* is used to characterize the thermodynamics of BH horizons.

Consider the expression of the line interval in spherically symmetric spacetime and natural units ($k_B = c = 1$)

$$ds^2 = -g_{00}(r)\,dt^2 + \frac{dr^2}{g_{00}(r)} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{1}$$

$$g_{00}(r) = 1 - \frac{2G_N M}{r} \tag{2}$$

To facilitate the later comparison with multifractals, we normalize the metric parameters using the Planck scale ($M_{Pl}^{-1} \propto G_N^{1/2}$), namely,

$$g_{00}(r^0) = 1 - 2\frac{M\, G_N^{1/2}}{r^0} \tag{3}$$

$$r^0 = r M_{Pl} = r\, G_N^{-1/2} \tag{4}$$

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Einstein equation assumes the form [2-3]

$$(1-g_{00})-r\frac{dg_{00}}{dr}\propto -G_N P r^2 \tag{5}$$

where the pressure P coincide with radial component of the stress-energy tensor

$$P=T_r^r \tag{6}$$

Aside from some constant factors, one obtains by (3)-(5)

$$\left(\frac{dg_{00}}{dr^0}\bigg|_{a^0}\right) d(a^0)^2 - d a^0 \propto G_N P d(a^0)^3 \tag{7}$$

in which $a^0 = aM_{Pl}$ denotes the normalized zero of the function $g_{00}(r^0)$ [2]. Here, the normalized temperature and the differentials in entropy, energy and volume are given by, respectively,

$$T^0 \propto \frac{dg_{00}}{dr^0}\bigg|_{a^0} \tag{8}$$

$$dS \propto d(a^0)^2 \tag{9}$$

$$dU \propto d(a^0)\,;\;\; dV \propto d(a^0)^3 \tag{10}$$

(8) to (10) satisfy the thermodynamic equation of state in standard form, i.e.,

$$-PdV = dU - TdS \tag{11}$$

Consider next the side-by-side multifractal analogy of Thermodynamics displayed in Tab. 1 of the Appendix. Since negative pressure corresponds to volume reduction and positive pressure to dilation, one has [4]

$$PdV = -dF = -d\left[\frac{\tau(q)}{q}\right] \tag{12}$$

where F stands for "free energy" and, per (A10) of the Appendix,

$$d\tau(q) \propto q d\alpha - df(\alpha) \tag{13}$$

Tab. 1 also indicates that,

$$dU = d\alpha \tag{14}$$

$$T = \frac{1}{q} \tag{15}$$

$$dS = df(\alpha) \tag{16}$$

Comparing (8)-(11) to (12)-(16), reveals the following relationships connecting multifractals with the Thermodynamics of BH's

$$U \Leftrightarrow \alpha \Leftrightarrow a^0 \tag{17}$$

$$S \Leftrightarrow f(\alpha) \Leftrightarrow (a^0)^2 \tag{18}$$

$$V \Leftrightarrow (a^0)^3 \tag{19}$$

By (3)-(19) and following [4-6], the steepest descent method applied to the multifractal partition function (A4) renders the BH mass in the form

$$\boxed{M_{BH} \propto \alpha_0^2 G_N^{-1/2} [\frac{\partial f(\alpha)}{\partial \alpha}\Big|_{\alpha_o}]^{-1}} \tag{20}$$

(20) shows that the BH mass may be associated with the singularity spectrum, and it diverges in the regime of *unbounded temperatures*,

$$q \to 0 \Leftrightarrow T \to \infty \Leftrightarrow [\partial f(\alpha)/\partial \alpha]_{\alpha_0} = 0 \tag{21}$$

Shifting the maximum of the singularity spectrum $\alpha = \alpha_0 + \delta\alpha$, $\left|\partial f(\alpha)/\partial \alpha\right|_{\alpha} > 0$, produces effects analogous to BH evaporation through the *Hawking radiation*. By (15) and (A8), cooling off begins with letting q turn positive ($q > 0$), which entails a drop in the maximal BH mass.

(17) highlights the formal analogy between energy and the dimensional parameter α. One is then tempted to speculate that the continuous dimensional ensemble of $\alpha' s$ can *gravitationally collapse* and enable the formation of *Cantor Dust*, a large-scale condensed phase describing Dark Matter halos [7-8].

Appendix

Fractal measures generalize the concept of integer measures from ordinary geometry, which quantifies lengths, area, and volumes. Following [6], let a set Σ supporting a fractal measure be covered with a collection of boxes of size $\varepsilon << 1$. The number of boxes needed to cover the set is defined through the scaling

$$N(\varepsilon) \sim \varepsilon^{-D_H} \tag{A1}$$

where D_H is the Hausdorff dimension of the set, which is adequate for characterization of *mono-fractals*. In general, the quantitative description of *multifractal* measures requires replacing D_H with a continuous dimensional parameter α according to

$$\mu \sim \varepsilon^{\alpha} \ , \quad 0 < \alpha \in \left[\alpha_{\min}, \alpha_{\max}\right] < \infty \tag{A2}$$

The number of boxes of size ε having the dimension α is given by

$$N_{\varepsilon}(\alpha) \sim \varepsilon^{-f(\alpha)} \tag{A3}$$

where the distribution of dimensions follows the *singularity spectrum* $f(\alpha)$. The meaning of (A3) is that there are infinitely many subsets of boxes characterized by α in the limit $\varepsilon \to 0$.
By analogy with equilibrium statistical mechanics and Quantum Field Theory, multifractal analysis is based on a *partition function* defined as

$$Z_q(\varepsilon) = \sum_{i=1}^{N(\varepsilon)} \mu_i^q, \ \ q \in \mathbf{R} \tag{A4}$$

By (A2), the measures assigned to boxes $i = 1, 2, ..., N(\varepsilon)$ is $\mu_i = \varepsilon^{\alpha_i}$. Assuming that the number of boxes for which $\alpha < \alpha_i < \alpha + d\alpha$ is $N_{\varepsilon}(\alpha)d\alpha$, the contribution of the subset of boxes with $\alpha_i \in [\alpha, \alpha + d\alpha]$ to the partition function is $N_{\varepsilon}(\alpha)(\varepsilon^{\alpha})^q d\alpha$ and thus

$$Z_q(\varepsilon) = \int N_{\varepsilon}(\alpha)(\varepsilon^{\alpha})^q d\alpha \tag{A5}$$

By (A3) and (A5), we obtain

$$Z_q(\varepsilon) = \int \varepsilon^{q\alpha - f(\alpha)} d\alpha \tag{A6}$$

In the limit $\varepsilon \to 0$, the prevailing contribution to the integral (A6) arises from those values of α that minimize the sum $q\alpha - f(\alpha)$. Such minimum exists if

$$\frac{\partial}{\partial\alpha}\{q\alpha - f(\alpha)\} = 0 \tag{A7}$$

which implies two conditions, namely

$$q = \frac{\partial}{\partial\alpha} f(\alpha), \quad \alpha = \alpha(q) \tag{A8}$$

and

$$\frac{\partial^2}{\partial\alpha^2} f(\alpha) < 0, \ \ \alpha = \alpha(q) \tag{A9}$$

The sum $q\alpha - f(\alpha)$ defines a new exponent $\tau(q)$ written as [5-6]

$$\tau(q) = q\,\alpha(q) - f(\alpha(q)) \tag{A10}$$

It can be shown that classical Thermodynamics offers a straightforward analog of multifractal analysis. With reference to Tab. 1, temperature (T), internal energy (U), entropy (S) and free energy (F) are respectively echoed in multifractal theory by $q^{-1}, \alpha,\ f(\alpha)$ and $\tau(q)/q$ [5-6].

Multifractals	Thermodynamics
q	$1/T$
$\alpha(q)$	U
$\tau(q)/q$	F
$f(\alpha)$	S

Table 1: Mapping Multifractals to Thermodynamics

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