Article

Some Characterizations of Curves with Fractional Derivatives in Higher Dimensional Euclidean Spaces

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Abstract

In this paper, curves are examined by using fractional derivatives in general. Especially, by considering the Caputo fractional derivative, the relations between the standard Frenet curvatures and fractional curvatures of curves are obtained. Then, the characterizations of some special curves are given.

Keywords: Caputo fractional derivative, Frenet-Serret formulas, curvature, torsion.

1. Introduction

The concept of fractional derivative was first introduced in the 17th century and with an increasing number of studies, it has become the focus of attention for many researchers in many fields. Fractional analysis has recently become one of the important fields of study in differential geometry. While, in the classical sense, the differential and integral are calculated by integer order, in fractional calculus the orders of the differential and integral are not necessary integers but any real number. That is, fractional calculus is the generalization of ordinary differential and integral to arbitrary order. The difference of the fractional derivative from the integer derivative is that it is given by the integration of a function. Numerous studies have been conducted on this subject, and it can be found in detail [1-4]. We can also say that a non-local fractional derivative of function is related to a past history or a space-range interaction. Furthermore, fractional calculus has many applications to viscoelastic [5-11], analytical mechanics [12-14] and dynamical systems [15-19].

Fractional analysis has also started to be studied from a differential geometry perspective in recent studies. There are many types of fractional operators, but it is recommended to study the geometry of curves and surfaces mostly based on the Caputo fractional derivative [20]. However, the Caputo fractional derivative is not yet directly used to formulate the differential geometry of curves. Using the Caputo fractional derivative is more appropriate than other fractional derivative operators for formulating a geometric theory since the fractional derivative of the constant function is zero [21-25]. Based on the advantages of the Caputo fractional derivative, it is discussed in [22,24] as a quantification of Lagrangian mechanics and in the theory of gravity [21,23,26]. In general, the concepts of Leibnitz rule and derivative of the composite function are

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needed when studying fractional differential geometry. However, within the scope of fractional analysis, these concepts are obtained with infinite series and are used in impact situations at the initial moment and after a long period [3,4].

Leibnitz's rule and derivative of the composite function can be given as follows for two functions *f(x)* and *g(x)* [27]:

$$(D_x^\alpha fg)(x) = \sum_{i=0}^{\infty}\binom{\alpha}{i}\frac{d^i f}{dx^i}\left(D_x^{\alpha-i}g\right)(x) - \frac{f(0)g(0)}{\Gamma(1-\alpha)}x^{-\alpha}$$

and

$$(D_x^\alpha f)\big(g(x)\big) = \sum_{i=1}^{\infty}\binom{\alpha}{i}\frac{x^{i-\alpha}}{\Gamma(i-\alpha+1)}\frac{d^i f(g(x))}{dx^i} + \frac{f\big(g(x)\big)-f(g(0))}{\Gamma(1-\alpha)}x^{-\alpha}. \tag{1.1}$$

This different form of the integer derivative presents a challenge for deriving geometric concepts such as the curvature of a curve and the unit tangent vector. So a certain simplification of the infinite series is used to construct the geometric theory of the derivative. With this simplification, most fundamental terms are removed from the infinite series, which retain the properties of the fractional derivative. Hence, with $t = g(x)$, the following equality is achieved [28]:

$$(D_x^\alpha f)\big(g(x)\big) = \frac{\alpha x^{1-\alpha}}{\Gamma(2-\alpha)}\frac{df}{dt}\frac{dg}{dx}. \tag{1.2}$$

This simplification formula is obtained by taking only the i=1 term of the infinite series in equation (1.1). This formula actually gives a partial effect of the fractional derivative and is expressed by the ordinary derivative. After this simplification, the construction of the fractional geometric theory based on the direct Caputo derivative can be expected using the simplified Leibnitz rule and the derivative of the composite function. In other words, using the Caputo derivative researchers have an advantage when studying the differential geometry of curves and surfaces, especially since it is ineffective on a constant function. Throughout the study, the derivative formula given by (1.2) will be discussed.

In this study, the fractional invariants of curves in higher dimensional Euclidean spaces are obtained and these invariants are interpreted geometrically. In addition, the theorems obtained about the curvature of some special curves are given.

2. Preliminaries

In this section, we will talk about some basic concepts that we will use in the following sections. More detailed information on the following topics can be found in [28-32].
Let the scalar product and reduced norm in R^n be denoted as $<.,.>$ and $\|\,.\,\|$, respectively. Take a regular curve $\mathbf{y}: \tilde{I} \to \mathrm{R}^n$, that is, let $\|\dot{\mathbf{y}}\| \neq 0$ for each $t \in \tilde{I}$ with $\dot{\boldsymbol{y}} = \frac{dy}{dt}$. Throughout this section, the derivative with respect to the *t* parameter will be represented by a dot.

Unit speed parameterization $\mathbf{x} = \mathbf{y} \circ u^{-1}: I \to \mathrm{R}^n$ of the curve $\mathbf{y}$ curve exists as

$$u(t) = \int_{t_0}^{t}\left\|\frac{dy}{d\sigma}\right\| d\sigma, t_0 \in \tilde{I} \tag{2.1}$$

and $\left\|\frac{d\mathbf{x}}{du}\right\| = 1$. Here the parameter u is called the arc-length and remains invariant under the Euclidean motion of R^n. We can separate the formulas to study the local properties of curves into two parts.

Firstly, let $\mathbf{x}: I \to \mathrm{R}^2$ be a plane curve with unit speed. Then the curvature κ of curve $\mathbf{x}$ at $u \in I$ is defined as $\kappa(u) = \left\|\frac{d^2\mathbf{x}}{du^2}\right\|$. The curvature κ and the Frenet formulas of the curve $\mathbf{x}$ for an arbitrary parameter t can be given as follows

$$\kappa(t) = \frac{<\ddot{\mathbf{x}}(t), J(\dot{\mathbf{x}}(t))>}{\|\dot{\mathbf{x}}(t)\|^3}$$

and

$$\frac{d\mathbf{t}}{du} = \kappa\mathbf{n}$$
$$\frac{d\mathbf{n}}{du} = -\kappa\mathbf{t}\,.$$

Now, let $\mathbf{x}: I \to \mathrm{R}^3$ be a space curve with unit speed. Then the curvature κ of curve $\mathbf{x}$ at $u \in I$ is defined as $\kappa(u) = \left\|\frac{d^2\mathbf{x}}{du^2}\right\|$. Frenet-Serret frame of $\mathbf{x}$ at $u \in I$ is given by

$$\mathbf{t}(\boldsymbol{u}) = \left.\frac{d\mathbf{x}}{d\boldsymbol{u}}\right|_{\boldsymbol{u}}, \mathbf{n}(\boldsymbol{u}) = \frac{\mathbf{1}}{\boldsymbol{\kappa}(\boldsymbol{u})}\left.\frac{\boldsymbol{d}^2\mathbf{x}}{\boldsymbol{du}^2}\right|_{\boldsymbol{u}}, \boldsymbol{\kappa}(\boldsymbol{u}) \neq \mathbf{0} \text{ and } \mathbf{b}(\boldsymbol{u}) = \mathbf{t}(\boldsymbol{u}) \times \mathbf{n}(\boldsymbol{u})$$

where $\times$ is the cross product in $\mathbf{R}^3$. Thus Frenet-Serret formulas of $\mathbf{x}$ is given by

$$\frac{d\mathbf{t}}{du} = \kappa\mathbf{n}$$
$$\frac{d\mathbf{n}}{du} = -\kappa\mathbf{t} + \tau\mathbf{b}$$
$$\frac{d\mathbf{b}}{du} = -\tau\mathbf{n}$$

where τ is called torsion of $\mathbf{x}$ at $u \in I$. Respectively, the curvature and torsion of $\mathbf{x}$ is writed by

$$\kappa(t) = \frac{\dot{\mathbf{x}}(t) \times \ddot{\mathbf{x}}(t))}{\|\dot{\mathbf{x}}(t)\|^3}, \tau(t) = \frac{<\dot{\mathbf{x}}(t) \times \ddot{\mathbf{x}}(t), \dddot{\mathbf{x}}(t))}{\|\dot{\mathbf{x}}(t) \times \ddot{\mathbf{x}}(t)\|^2} \tag{2.2}$$

where t is an arbitrary parameter.

3. Some Characterizations of Curves with Fractional Derivatives

Let $\mathbf{x}: I \subset R \to \mathrm{R}^n$ be a curve and u be arc-length of curve $\mathbf{x}$. Consider another parameter s of $\mathbf{x}$ is given by

$$u \to s = \left[\frac{\alpha^2}{\Gamma(2-\alpha)}u\right]^{\frac{1}{\alpha}}, \tag{3.1}$$

where Γ is Euler gamma function and $0 < \alpha \leq 1$. Because of (2.1) , s can be considered as a function of t. We can write it as $s = h(t)$. From (3.1), $h(t)$ can be written as

$$h(t) = \left(\frac{\alpha^2}{\Gamma(2-\alpha)} \int_{t_0}^{t} \left\| \frac{d\mathbf{x}}{d\sigma} \right\| d\sigma \right)^{\frac{1}{\alpha}}.$$
(3.2)

Considering the last equality, we get

$$\dot{h} = \frac{dh}{dt} = \frac{\alpha h^{1-\alpha}}{\Gamma(2-\alpha)} \|\dot{\mathbf{x}}\|, \tag{3.3}$$

where $\dot{h}$ is positive for each t and $t = h^{-1}(s)$ is inverse function of $\dot{h}$.

In this article, for the α-order Caputo fractional derivative we will use the notation

$$({}^{C}D_{0+}^{\alpha} f)(t) = \frac{d^{\alpha} f(t)}{dt^{\alpha}}$$

We will also denote the derivative with respect to the parameter s as " ´ ".

For a simpler version of the Caputo fractional derivative the following equality can be used [28]:

$$\frac{d^{\alpha}\mathbf{x}h(t)}{ds^{\alpha}} = \frac{\alpha h^{1-\alpha}}{\Gamma(2-\alpha)} (h^{-1})'\dot{\mathbf{x}}\ . \tag{3.4}$$

Considering (3.2) and (3.3) into (3.4), we obtain $\left\| \frac{d^{\alpha}\mathbf{x}}{ds^{\alpha}}\Big|_{s} \right\| = 1$ for each s i.e. s is the arc-length parameter of **x**. Hence **x** is a unit speed curve.

In order to get the Frenet-Serret formulas with fractional order α we firstly point out that the Frenet-Serret frame $\{\mathbf{t}, \mathbf{n}, \boldsymbol{b}_1, \dots, \boldsymbol{b}_{n-2}\}$ is independent of choice of parametrization, i.e.

$$Span\left\{\frac{d^{\alpha}\mathbf{x}}{ds^{\alpha}}, \frac{d}{ds}\left(\frac{d^{\alpha}\mathbf{x}}{ds^{\alpha}}\right), \dots, \frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\alpha}\mathbf{x}}{ds^{\alpha}}\right)\right\} = Span\left\{\frac{d\mathbf{x}}{du}, \frac{d^2\mathbf{x}}{du^2}, \dots, \frac{d^n\mathbf{x}}{du^n}\right\}$$

which means that $\boldsymbol{t}(s) = \frac{d^{\alpha}\mathbf{x}}{ds^{\alpha}}\Big|_{s}$ is the unit tangent vector of **x** at *s*.

Now, using the arc-length parameter s, the Frenet-Serret formulas of fractional order α can be obtained.

Let $\mathbf{x}: I \subset R \to \mathrm{R}^n$ be a parametrized curve by (3.1) with (3.4). Then we have

$$\begin{aligned} \boldsymbol{t}' &= \kappa_1 \mathbf{n} \\ \boldsymbol{n}' &= -\kappa_1 \mathbf{t} + \kappa_2 \boldsymbol{b}_1 \\ &\dots \\ {\boldsymbol{b}_{n-3}}' &= -\kappa_{n-2} \boldsymbol{b}_{n-4} + \kappa_{n-1} \boldsymbol{b}_{n-2} \\ {\boldsymbol{b}_{n-2}}' &= -\kappa_{n-1} \boldsymbol{b}_{n-3} \end{aligned} \tag{3.5}$$

where $\{\kappa_1, \dots, \kappa_{n-1}\}$ and $\{\mathbf{t}, \mathbf{n}, \boldsymbol{b}_1, \dots, \boldsymbol{b}_{n-2}\}$ are curvatures and Frenet- Serret frame of **x**, respectively. Also " ´ " is the derivative with respect to the parameter *u*. Here for $i = 3, \dots, n-1$, we can write

$$\kappa_1 = < \boldsymbol{t}', \mathbf{n} >, \kappa_2 = < \boldsymbol{n}', \boldsymbol{b}_1 >, \dots, \kappa_i = < {\boldsymbol{b}_{i-2}}', \boldsymbol{b}_{i-1} >.$$

Definition 3.1. Let $\mathbf{x}: I \subset R \rightarrow \mathrm{R}^n$ be parametrized by the arc-length u. Also, let the frenet frame of $\mathbf{x}$ be $\{\mathbf{t}, \mathbf{n}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-2}\}$ and its curvatures be $\{\kappa_1, \ldots, \kappa_{n-1}\}$. In this case $\kappa_1{}^{(\alpha)}$ is given as

$$\kappa_1{}^{(\alpha)} =< \frac{d\boldsymbol{t}}{ds}, \mathbf{n} >$$

and called the first α-fractional curvature of $\mathbf{x}$. Since

$$\kappa_1 =< \frac{d\boldsymbol{t}}{ds}\frac{ds}{du}, \mathbf{n} >= \frac{ds}{du} < \frac{d\boldsymbol{t}}{ds}, \mathbf{n} >,$$

we get

$$\kappa_1{}^{(\alpha)} = \frac{\Gamma(2-\alpha)}{\alpha} s^{\alpha-1}\kappa_1. \tag{3.6}$$

Definition 3.2. Let $\mathbf{x}: I \subset R \rightarrow \mathrm{R}^n$ be parametrized by the arc-length u. Also, let the frenet frame of $\mathbf{x}$ be $\{\mathbf{t}, \mathbf{n}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-2}\}$ and its curvatures be $\{\kappa_1, \ldots, \kappa_{n-1}\}$. In this case $\kappa_2{}^{(\alpha)}$ is given as

$$\kappa_2{}^{(\alpha)} =< \frac{d\boldsymbol{n}}{ds}, \boldsymbol{b}_1 >$$

and called the second α-fractional curvature of $\mathbf{x}$.

Then we get

$$\kappa_2{}^{(\alpha)} = \frac{\Gamma(2-\alpha)}{\alpha} s^{\alpha-1}\kappa_2. \tag{3.7}$$

Definition 3.3. Let $\mathbf{x}: I \subset R \rightarrow \mathrm{R}^n$ be parametrized by the arc-length u. Also, let the frenet frame of $\mathbf{x}$ be $\{\mathbf{t}, \mathbf{n}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-2}\}$ and its curvatures be $\{\kappa_1, \ldots, \kappa_{n-1}\}$. In this case $\kappa_i{}^{(\alpha)}$ is given as

$$\kappa_i{}^{(\alpha)} =< \frac{d\boldsymbol{b}_{i-2}}{ds}, \boldsymbol{b}_{i-1} >, i = 3, \ldots, n-1$$

and called the i-th α-fractional curvature of $\mathbf{x}$. Here we get

$$\kappa_i{}^{(\alpha)} = \frac{\Gamma(2-\alpha)}{\alpha} s^{\alpha-1}\kappa_i, i = 3, \ldots, n-1\,. \tag{3.8}$$

Then we can give the following lemma.

Lemma 3.1. Let $\mathbf{x}: I \subset R \rightarrow \mathrm{R}^n$ be a regular curve parameterized by the arc-length. The following relation exists between the curvatures fractional order α of $\mathbf{x}$.

$$\kappa_j{}^{(\alpha)} = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} \kappa_j, j = 1, \ldots, n-1 \tag{3.9}$$

We can give a result of this Lemma as follows.

Corollary 3.1. Let $\mathbf{x}: I \subset R \rightarrow \mathrm{R}^n$ be a regular parametric curve with constant curvature $\kappa_i{}^{(\alpha)} = \lambda$. Then we have

$$\kappa_i = \frac{\lambda\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\,.$$

Proof. Putting λ into equation (3.9) we prove the result.

Theorem 3.1. Let **x** be a regular parametric curve in R^n. Then the arc-length parameter and the fractional α curvature $\kappa_i{}^{(\alpha)}$ remain invariant under the Euclidean motions of R^n.

Proof. Considering (3.1) and (3.9), the proof of the theorem is obvious.

After the definitions and explanations above, theorems containing geometric interpretations for curves with fractional curvature can be given below.

Theorem 3.2. For a regular parametric curve in R^n to be a straight line a necessary and sufficient condition is that the first curvature of the α fractional order of this curve is identically zero.

Proof. Considering an curve $\mathbf{x}: I \subset R \to \mathrm{R}^n$ parameterized with the arc-length parameter, the first α-fractional curvature of this curve **x** will be as follows:

$$\kappa_1{}^{(\alpha)} = \frac{\Gamma(2-\alpha)}{\alpha} s^{\alpha-1}\kappa_1$$

This last equality proves the theorem.

For a curve $\mathbf{x}: I \subset R \to \mathrm{R}^n$, considering the α-fractional order curvatures of this curve, we can give the following lemma.

Lemma 3.2. For a regular curve $\mathbf{x}: I \subset R \to \mathrm{R}^n$ parameterized with the α-fractional order arc-length parameter, we have the following equation:

$$\frac{\kappa_i{}^{(\alpha)}}{\kappa_{i+1}{}^{(\alpha)}} = \frac{\kappa_i}{\kappa_{i+1}}, i = 1,2,\dots,n-2, n \geq 3\,.$$

Proof. From Definition 3.1 and 3.2, it can be written by

$$\kappa_1{}^{(\alpha)} = \frac{\Gamma(2-\alpha)}{\alpha} s^{\alpha-1}\kappa_1$$

and

$$\kappa_2{}^{(\alpha)} = \frac{\Gamma(2-\alpha)}{\alpha} s^{\alpha-1}\kappa_2.$$

If these last two equations are proportional to each other, then

$$\frac{\kappa_1{}^{(\alpha)}}{\kappa_2{}^{(\alpha)}} = \frac{\kappa_1}{\kappa_2}.$$

From Definition 3.3. we have

$$\kappa_i{}^{(\alpha)} = \frac{\Gamma(2-\alpha)}{\alpha} s^{\alpha-1}\kappa_i, i = 3,\dots,n-1\,.$$

Hence

$$\frac{\kappa_i{}^{(\alpha)}}{\kappa_{i+1}{}^{(\alpha)}} = \frac{\kappa_i}{\kappa_{i+1}},$$

$i = 3,\dots,n-1$. This completes the proof.

Theorem 3.2. For a regular parametric curve to have a constant curvature ratio in R^n, a necessary and sufficient condition is that

$\frac{\kappa_1{}^{(\alpha)}}{\kappa_2{}^{(\alpha)}}$=const., $\frac{\kappa_2{}^{(\alpha)}}{\kappa_3{}^{(\alpha)}}$= const., …, $\frac{\kappa_{n-2}{}^{(\alpha)}}{\kappa_{n-1}{}^{(\alpha)}}$= const.

Proof. The proof is obvious if the given definitions of curvatures are taken into account.

Remark 3.1. If $\mathbf{x}(u)$ and $\mathbf{x}^*(u^*)$ are the pair of involute curves in R^3, then

$$d\big(\mathbf{x}(u),\mathbf{x}(u^*)\big) = |c-u|, \forall u \in I, c = const.$$

holds [33], where d is the distance between the points $\mathbf{x}(u)$ and $\mathbf{x}(u^*)$. Also κ_1, κ_2 and $\kappa_1{}^*, \kappa_2{}^*$, being the standard curvatures of the curves $\mathbf{x}(u)$ and $\mathbf{x}^*(u^*)$, respectively, the following equality can be used [33]:

$$\kappa_1{}^{*2}(u^*) = \frac{\kappa_1{}^2(u)+\kappa_2{}^2(u)}{\kappa_1{}^2(u)(c-u)^2} \tag{3.10}$$

and

$$\kappa_2{}^*(u^*) = \frac{\frac{d}{du}\left(\frac{(\kappa_2(u))}{(\kappa_1(u))}\right)\kappa_1(u)}{(\kappa_1{}^2(u)+\kappa_2{}^2(u))(c-u)}. \tag{3.11}$$

We can now give the following theorems about pairs of involute curves.

Theorem 3.3. Let $\mathbf{x}^*$ be the involute of a parametric curve $\mathbf{x}$ in R^3. Then between the first curvatures of these curves hold

$$\left(\kappa_1{}^{(\alpha)^*}\right) = \frac{\big(\Gamma(2-\alpha)\big)}{\alpha(s^*)^{1-\alpha}} \frac{\sqrt{(\kappa_1{}^{(\alpha)})^2 + (\kappa_2{}^{(\alpha)})^2}}{(\kappa_1{}^{(\alpha)})\left|c - \frac{\Gamma(2-\alpha)s^\alpha}{\alpha^2}\right|}, c \in R,$$

where s, $\kappa_1{}^{(\alpha)} \neq 0$, $\left(\kappa_2{}^{(\alpha)}\right)$ and s^*, $\kappa_1{}^{(\alpha)^*}$ are parameters and curvatures with fractional order α of $\mathbf{x}$ ve $\mathbf{x}^*$, respectively.

Proof. If (3.6) and (3.7) are substituted in (3.10) , the proof can be done by direct calculation.

Theorem 3.4. Let $\mathbf{x}^*$ be the involute of a parametric curve $\mathbf{x}$ in R^3. Then between the second curvatures of these curves hold

$$\left(\kappa_2{}^{(\alpha)^*}\right) = \frac{\big(\Gamma(2-\alpha)\big)}{\alpha(s^*)^{1-\alpha}} \frac{\frac{d}{ds}\left(\frac{\kappa_2{}^{(\alpha)}}{\kappa_1{}^{(\alpha)}}\right)\left(\kappa_1{}^{(\alpha)}\right)}{\left(c - \frac{\Gamma(2-\alpha)s^\alpha}{\alpha^2}\right)(\kappa_1{}^{(\alpha)})^2 + (\kappa_2{}^{(\alpha)})^2}, c \in R,$$

where s, $\kappa_1{}^{(\alpha)} \neq 0$, $\left(\kappa_2{}^{(\alpha)}\right)$ and s^*, $\kappa_2{}^{(\alpha)^*}$ are parameters and curvatures with fractional order α of $\mathbf{x}$ ve $\mathbf{x}^*$, respectively.

Proof. If (3.6) and (3.7) are substituted in (3.11) , the proof can be done by simple calculations.

Remark 3.2. If $\mathbf{x}(u)$ and $\mathbf{x}^*(u^*)$ are a Bertrand pair in R^3, then the following relations hold:

$$\kappa_1{}^* = \frac{c\kappa_1 - sin^2\theta}{c(1-c\kappa_1)},\ c \in R \tag{3.12}$$

and

$$\kappa_2{}^* = \frac{1}{c^2\kappa_2} sin^2\theta,\ c \in R \tag{3.13}$$

where κ_1, κ_2 and ${\kappa_1}^*, {\kappa_2}^*$ being the standard curvatures of the $\mathbf{x}(u)$ and $\mathbf{x}^*(u^*)$ curves, respectively, and θ are the angle between the tangent vectors of the curves, [34].

Thus, we can give the following theorem.

Theorem 3.5. If $\mathbf{x}(u)$ and $\mathbf{x}^*(u^*)$ are a Bertrand pair in R^3, then the following relations hold:

$$ {\kappa_1}^{(\alpha)^*} = \frac{\Gamma(2-\alpha)}{\alpha(s^*)^{1-\alpha}} \frac{c \dfrac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} {\kappa_1}^{(\alpha)} - sin^2\theta}{c(1 - c\dfrac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} {\kappa_1}^{(\alpha)})}, c \in R $$

and

$$ {\kappa_2}^{(\alpha)^*} = \frac{\left(\Gamma(2-\alpha)\right)^2}{c^2 \alpha^2 (ss^*)^{1-\alpha} {\kappa_2}^{(\alpha)}} sin^2\theta $$

where s, ${\kappa_1}^{(\alpha)}$, ${\kappa_2}^{(\alpha)}$ and $s^*, {\kappa_1}^{(\alpha)^*}$, ${\kappa_2}^{(\alpha)^*}$ are parameter and curvatures with fractional α-order of $\mathbf{x}$ and $\mathbf{x}^*$ curves, respectively, θ is the angle between the tangent vectors of the curves.

Proof. If (3.6) and (3.7) are substituted in (3.12) and (3.13) , the proof can be done by direct calculations.

4. **Conclusion**

In the article, the Caputo fractional derivative is considered and the relations between the standard curvature and fractional curvature of the curves are obtained. It has been observed that these relations geometrically overlap with the results obtained using the derivative in the classical sense, and it has been obtained that there are some differences with the effect of the fractional derivative.

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