Article

The Role of Galois Groups in TGD Framework

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Abstract

This article was inspired by the inverse problem of Galois theory. Galois groups are realized as number theoretic symmetry groups realized physically in TGD a symmetries of space-time surfaces. Galois confinement as an analog of color confinement is proposed in TGD inspired quantum biology. Two instances of the inverse Galois problem, which are especially interesting in TGD, are following: Q1: Can a given finite group appear as Galois group over Q? The answer is not known; and Q2: Can a given finite group G appear as a Galois group over some EQ? Answer to Q2 is positive as will be found and the extensions for a given G can be explicitly constructed. The TGD based formulation based on $M^8 - H$ duality in which space-time surface in complexified M^8 are coded by polynomials with rational coefficients involves the following open question. Q: Can one allow only polynomials with coefficients in Q or should one allow also coefficients in EQs? The idea allowing to answer this question is the requirement that TGD adelic physics is able to represent all finite groups as Galois groups of Q or some EQ acting physical symmetry group. If the answer to Q1 is positive, it is enough to have polynomials with coefficients in Q. It not, then also EQs are needed as coefficient fields for polynomials to get all Galois groups. The first option would be the more elegant one. In the sequel the inverse problem is considered from the perspective of TGD. Galois groups, in particular simple Galois groups, play a fundamental role in the TGD view of cognition. The TGD based model of the genetic code involves in an essential manner the groups A_5 (icosahedron), which is the smallest simple and non-commutative group, and A_4 (tetrahedron). The identification of these groups as Galois groups leads to a more precise view about genetic code.

1 Introduction

This article was inspired by the inverse problem of Galois theory [3] (`https://cutt.ly/jmjpyDS`). Galois groups are realized as number theoretic symmetry groups realized physically in TGD [13, 14]. Galois confinement as an analog of color confinement is proposed in TGD inspired quantum biology [20, 22, 32, 19, 30, 31, 27].

Two instances of the inverse Galois problem, which are especially interesting in TGD, are following:

Q1: Can a given finite group appear as Galois group over Q? The answer is not known.

Q2: Can a given finite group G appear as a Galois group over some EQ? The answer to this question is positive as will be found and the extensions for a given G can be explicitly constructed.

The formulation adelic physics [13, 14] is based on $M^8 - H$ duality in which space-time surface in complexified M^8 are coded by polynomials with rational coefficients. Adelic physics involves the following open question.

Q: Can one allow only polynomials with coefficients in Q or should one allow also coefficients in EQs?

The idea allowing to answer this question is the requirement that TGD adelic physics is able to represent all finite groups as Galois groups of Q or some EQ acting physical symmetry group.

If the answer to **Q1** is positive, it is enough to have polynomials with coefficients in Q. It not, then also EQs are needed as coefficient fields for polynomials to get all Galois groups. Needless to say, the first option would be the more elegant one.

In the sequel the inverse problem is considered from the perspective of TGD.

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1. M^8-H-duality, $H=M^4\times CP_2$ adelic physics [13, 14] based on the identification of space-time surfaces X^4 in the complexified M^8 identifiable as complexified octonions. Normal space of X^4 is required to be associative/quaternionic and to contain an integrable distribution of commutative 2-D sub-spaces. At the level of H, twistor lift of TGD implies partial differential equations defined by a variational principle based on action, which is sum of volume term and Kähler action.

   The spacetime surfaces are preferred extremals [25, 28] identifiable as minimal surfaces analogous to soap films except at the dynamically generated analogs of frames at which the minimal surface property fails and field equations hold true only for the full action. At the frame, the divergences of isometry currents for the volume term and Kähler action have delta function singularities which cancel each other.

   In M^8, space-time surfaces are determined as 4-D "roots" of polynomials of complex variable over rationals continued to an octonionic polynomial [16, 17, 23, 28]. The Galois group over Q acts as a physical symmetry group permuting the sheets of the X^4 defined by the 4-D roots. This action induces symmetry also at the H-side of the duality and the Galois group defines a new kind of symmetry, which distinguishes between TGD and competing theories.

2. The notion of infinite primes is inspired by TGD [7] and has a physical interpretation as a repeated second quantization of an arithmetic super-symmetric quantum field theory (QFT). This hierarchy corresponds to a hierarchy of multi-variable polynomials obtained by taking a polynomial of t_n and replacing its rational coefficients with rational functions of $t_1,..t_{n-1}$ with rational coefficients. One can assign a Galois group to these polynomials and therefore also to infinite primes and integers.

3. The fraction of 2-groups with order not larger than n approaches unity at the limit $n\rightarrow\infty$. All 2-groups act as Galois groups of space-time surfaces. p-Adic length scale hypothesis states that primes near power of 2, define physically preferred p-adic length scales. The special role of 2-groups might explain why the p-adic length scale hypothesis [5] is true.

4. Galois groups, in particular simple Galois groups acting on cognitive representations consisting of points, whose coordinates in a number theoretically preferred coordinate system of octonions belong to EQ, play a fundamental role in the TGD view of cognition [21]. The TGD based model of genetic code [8, 20] involves in an essential manner the groups A_5 (icosahedron (I)), which is the smallest simple and non-commutative group, and A_4 (tetrahedron (T)). Genetic code has as building bricks Hamiltonian cycles of I and T. Genetic code relates to information and therefore to cognition so that the interpretation of these symmetry groups as Galois groups is suggestive.

   The most recent step of progress was the realization that genetic code can be represented in terms of icosa-tetrahedral tesselation of a hyperbolic 3-space H^3 [22] and that the notion of genetic code generalizes dramatically. Also octahedron (O) is involved with the tesselation but plays a completely passive role. The question why the genetic code is a fusion of 3 icosahedral codes and of only a single tetrahedral code remained however poorly understood.

   The identification of the symmetry groups of the I, O, and T as Galois groups makes it possible to answer this question. Icosa-tetrahedral tesselation can be replaced with its 3-fold covering replacing $I/O/T$ with the corresponding symmetry group acting as a Galois group. T has only only a single Hamiltonian cycle and its 3-fold covering behaves as a single cycle.

2 Some background about Galois groups

2.1 Basic definitions

Galois extensions are by definition represented by the roots of polynomials with coefficients in K. By definition Galois group Gal leaves K invariant and permutes roots. For instance, complex conjugate roots are permuted.

There are two basic ways to construct Galois extensions L/K of a number field K.

1. The roots of irreducible polynomials over K (no rational roots in K) define a Galois extension. The order $ord(Gal)$ of Gal is equal to the dimension n of extension L: $ord(Gal) = n$.

2. If a number field L and its automorphism group Aut is known then any subgroup G of Aut defines a sub-field K^G invariant under G and L/K^G an extension having G as Galois group.

The functional composition $P_1 \cdot P_2$ creates an extensions for which the Galois group of P_2 is a normal subgroup if one has $P_i(0) = 0$ and $P_1 \circ P_2$ has also the roots of P_2.

Polynomial rings $K(t_1, ..., t_n)$ of several variables give rise to extensions via the roots $P(t) = 0$.

1. The permutation group S_n acts as automorphisms of $K(t_1, ...t_n)$. Especially interesting sub-field is the invariant field of S_n generated by polynomials generated by symmetric functions. Any finite group G is sub-group of some S_n and defines a G-invariant field of $K(t_1, ...t_n)$ as G-invariant rational functions containing the field generated by symmetric functions.

2. The completion of a number field is algebraically complete. For instance, for Q this completion $\overline{Q}$ consists of algebraic numbers. The Galois group of $\overline{Q}/Q$ is profinite, which means that it is infinite but effectively finite, and can be constructed by inverse limit construction for a sequence of extensions leading to algebraic numbers.

   The extensions of completions are necessarily transcendental. In the case of $\overline{Q}$ they involve addition of transcendentals the extensions.

2.2 Some results about Galois groups over rationals

It is good to start by listing some basic results related to the Galois groups [3].

1. Galois theorem states that polynomials are solvable for degree $d \leq 5$. In these cases the Galois group is solvable meaning EQ is extensions of extension .... of rationals and that Galois groups for EQ has a descending decomposition by normal groups H_i which are commutator groups of the normal group H_{i+1} at the previous level. Equivalently, the Galois group for an extension of an extension at level i is Abelian. For $d < 5$ Galois group is $A_5 = S_5/Z_2$ with 60 elements. This is the smallest non-Abelian simple group. By its definition, a simple group does not have a decomposition to normal groups.

2. Kronecker-Webber theorem states that any abelian group appears as a Galois group for Q. This was found by studying cyclic extensions.

3. Shafarevich proved that every solvable group appears as a Galois group for an EQ.

4. Scholz and Reichardt proved that for an odd prime p, every finite p-group occurs as a Galois group over Q. The order of each element of a simple group is a power of p. 2-groups which also appear as a Galois group over Q are of special interest since for given n, most groups with order smaller than n are 2-groups. This result is of special interest from the point of view of p-adic length scale hypothesis.

5. It has been conjectured that almost all finite groups can act as a Galois group over Q. (`https://cutt.ly/imjwDKC`).

6. Simple groups are primes for finite groups. Simple groups appear in the decomposition of EQ to a sequence of extensions with a simple Galois group represented by a hierarchy of polynomials. In TGD inspired theory of cognition simple groups are analogs of elementary particles [21, 26, 33, 23] so that they are of special interest.

Any finite group, in particular, any simple group, appears as a Galois group over Q. The open question is whether a given simple group can appear as a Galois group over Q.

Many simple groups appear as Galois groups over Q. The theorem of Malle and Matzat states that if p is an odd prime such that either 2, 3 or 7 is a quadratic non-residue modulo p (q is quadratic residue if $x^2 = q \ mod \ p$ has a solution) then $PSL(2,p)$ occurs as a Galois group of EQ.

Four of the Mathieu groups, namely M_{11}, M_{12}, M_{22} and M_{24}, occur as Galois groups of EQ. For M_{23} the situation is open. The theorem of Thompson states that the Monster group, the largest sporadic simple group, appears as a Galois group over Q.

2.3 Various problems related to inverse Galois problem

In [3] various problems related to inverse Galois problem are listed.

1. The general existence problem can be formulated as the following question. Given number field K and finite group G, can G act as a Galois group for some extension of K?

2. If the answer to the general existence problem is affirmative for given K and G, the explicit construction polynomials $P(t)$ is the next challenge.

   One can also consider polynomials $P(t_1, ..., t_n)$ of several arguments which could be regarded as parametric representation for a large number of polynomials. $P(t_1, ..., t_n)$ must be irreducible. If the restriction of arguments appearing as parameters to a specific value produces irreducible polynomial, one can hope that the Galois group over Q is same as that of the polynomial which is permutation group S_n of the arguments.

   This process is called specification, and Hilbert's irreducibility theorem states the conditions for when the irreducibility is preserved in the process so that Galois group is inherited. The conditions mean that the Galois group is almost independent on the parameter values and the loci were irreducibility fails are the places where this happens. Obviously they correspond to the occurrence of multiple roots. Hilbert proved that for S_n the conditions are satisfied so that they appear as Galois groups of some EQ.

   Also the subgroups $G \subset S_n$ act in $Q(t_1, ...t_n)$ and one can ask under what conditions one can find EQ for which G acts as a Galois group. It turns out that one can construct explictly an EQ, whose extension allows G as a Galois group and specify explicitly the conditions under which this EQ reduces to Q.

3. What is the smallest number of parameters for a generic polynomial $P(t_1, ...t_n)$?

Two special results are mentioned in [3]. G can be any finite group in the following cases involving only one parameter.

1. For $K = C(t)$ any finite group G appears as Galois group of some Galois extension (defined by polynomial) of $K = C(t)$. This is true also for Galois extensions of generalizes to $K = R(t)$ and $K = \overline{Q}(t)$.

   The result for $K = C(t)$ follows from Riemann existence theorem `https://cutt.ly/FmjsnPA`, which in its original form states that the space of functions on Riemann sphere having singularities at punctures can be regarded as space of analytic functions at Riemann surface obtained as a finite branched covering of S^2 with branchings at punctures.

   The absolute Galois group over $C(t)$ corresponding to an infinite covering of the puncture sphere is identifiable as homotopy group of infinite covering and is free profinite (infinite but effectively finite) group with infinite number of generators. This true when K is closed and therefore holds true for $\overline{Q}$. Extension of $K(t)$ is obtained by adding a parameter and finding the roots.

The homotopy group of a given finite covering corresponds to a braid group as a finite covering of S_n. This raises the idea that a given finite group having always a representation as subgroup of S_n could allow a construction giving G as Galois group of over Q.

2. If K is p-adic field Q_p, any finite group can act as a Galois group over $K = Q_p(t)$.

3 Methods

In this section various methods to answer to the question whether a given finite group can act as a Galois group for given number field K are briefly summarize. The discussion follows the discussion in [3].

3.1 Regular Inverse Galois Problem

Regular inverse Galois problem starts from an extension L/K, which is regular.

Regularity requires that K is algebraically closed in L - or L is purely transcendental extension of K. This means that the elements of K cannot be expressed as solutions of algebraic extensions in L. One example of a purely transcendental extension is extension of rationals by adding some transcendental numbers. If K is algebraically closed - this is the case for C and algebraic numbers - the condition is satisfied.

A further condition is that L is separable over K. For a physicist, this rather technical looking condition states that the number field $L \otimes_K \overline{K}$ is an integral domain meaning that it has no divisors of zero.

Transcendental extensions are regular. The so-called transcendence basis S consists of elements of L, which do not satisfy any algebraic equation in L. One can construct the field $K(S)$ by forming the product of basis for K and S and $L/K(S))$ is algebraic extension of $K(S)$.

Extensions of algebraic completions are regular/transcendental. The field defined by rational functions formed from S_n invariant polynomials in $K(t_1, ..., , t_n)$ define field K^G of symmetric rational functions which cannot be regarded as an algebraic extension of K. By Noether's theorem stating that K^G is isomorphic with $K(t_1, ..., t_n)$, $K(t_1, ..., t_n)/K^G$ defines an extension of K with the same Galois group.

Whenever one has a Galois extension $M/Q(t)$ (regular or not), it is an easy consequence of the Hilbert Irreducibility Theorem that there is a specialisation M/Q with the same Galois group. If $M/Q(t)$ is regular, one obtains such specialized extensions M/K over any Hilbertian field in $char = 0$, in particular over all algebraic number fields. Char denotes the integer n for which $nx = 0$ is true for all elements of the field. Finite fields F_p have $char = p$.

$C(t)$ and $\overline{Q}$(t) are algebraically closed and transcendental and any finite group defines a Galois group for the extensions of these fields are obtained from a polynomial of n variables by specification.

The problem is how to get down to $Q(t)$ from $\overline{Q}$. One must restrict the coefficients of polynomials to a sub-field and it is not clear what happens to the Galois group in this process. This is one case of specialization: one starts from a parametrized set of polynomials $C(t_1, ...t_n)$ or $\overline{Q}(t_1, ...t_n)$ and restricts the parameters $t_1, ..., t_{n-1}$ to say Q.

3.2 Hilbert's irreducibility theorem

Consider polynomials $f(t, x)$ with parameters $t = (t_1, ...t_r)$ and indeterminates $x = (x_1, ...x_n)$. Assume that f irreducible polynomial. Define Hilbert f-set H_f/K as the set of parameter values $t \in K^r$ for which the restriction is well-defined and irreducible. Define Hilbert g-set as a subset of the set of the parameter values $t \in K^r$ for which $g(t)$ is non-vanishing so that g has no zeros in the set defined by the points $(x_1, ..x_n)$: this is possible since K^r is a subset of all parameter values t. Define Hilbert set as an intersection of finitely many f-sets and finitely many g-sets.

The field K is said to be Hilbertian if Hilbert sets are nonempty for all r.

The above condition is rather abstract but the following characterization of Hilberianity is more concrete. For a field K with $char = 0$, K is Hilbertian if and only if the following condition holds true. If $f(t, X)$ has no roots in $K(t)$ then $f(a, X)$ has no roots in K.

For $K = Q$, the first condition means that there are no roots which are rational functions and the second condition means that $f(a, X)$ has no rational roots. Rational roots emerge when two or more algebraic roots coincide. In this situation, the irreducibility is preserved in the specification and the Galois group is inherited.

Hilbert also proved that for S_n acting as a Galois group for $Q(t_1, ..., t_n)$, it is possible to find an extension of rationals with the same Galois group by specification. The polynomials in question are invariant under S_n and generated by symmetric functions. If the specification has a root, the action of S_n to a root corresponds to the action of a permutation in the parameter space and creates a new root so that the Galois group is S_n.

Under the conditions stated by Noether, this generalizes to subgroups $G \subset S_n$.

3.3 Noether's problem

Algebraic numbers are algebraically complete and can have only transcendental extensions, say by addition of transcendental numbers.

On the case of polynomial algebra $Q(t_1, ..., t_n)$ the field of invariants Q^G, $G = S_n$ is generated by polynomials symmetric under permutations of n arguments acting as Galois group in the polynomial algebra $C(t_1, ...t_n) \equiv C(t)$. Q^{S_n} is transcendental in the sense that the generators do not satisfy polynomial conditions with coefficients in $C(t)$. This algebra has $C(t)$ as extension with Galois group S_n, which obvious commutes with the field operations.

One can consider also sub-groups of $G \subset S_n$ and analogous extensions. In this case it is not obvious that the algebra Q^G is rational which meas that $C(t)$ is purely transcendental extension of Q^G.

The theorem by Emmy Noether states the following: If G is finite and $Q(X)^G/Q$ is rational (purely transcendental), then there is a Galois field extension K/Q with group G.

The proof of the theorem involves Hilbert's irreducibility theorem, rationality property implying that Q^G is isomorphic with $Q(t_1, ...t_n)$, and primitive element theorem stating that the extensions of Q are generated by powers of primitive element. How G becomes the Galois group for Q has been already explained.

3.4 Rigidity method

Riemann existence theorem is an essential part of the rigidity method. One considers compactified plane with punctures allowing interpretation as a punctured sphere with origin as a marked point. Rieman proved that meromorphic functions singular at punctures functions can be regarded as regular functions in cover of S^2 branched at the punctures defining a Riemann surface.

The homotopy group of the sphere with n punctures has n generators $g_1, ..., g_n$ satisfying the relation $g_1...g_n = 1$, since the complement of the regions containing punctures contains no punctures.

There is an infinite number of coverings characterized by the number n of sheets. Intuitively they are analogous to functions $z^{1/n}$. The homotopy group gives rise to the homotopy group of n-fold covering acting also as a Galois group for extension of meromorphic functions induced by the cover. The Galois group serves also as a braid group defining n-fold covering group for the permutation group of the punctures.

Absolute Galois group is associated with the covering with $n = \infty$ and is pro-finite group (infinite but effectively finite). Also this group satisfies the analog of the relation $g_1...g_n = 1$. The absolute Galois group is obtained as an inverse limit of the groups associated with a sequence of extensions (`https://en.wikipedia.org/wiki/Inverse_limit`). The map h_{jk} from k:th level to $j \leq k$:th level is homomorphism but not isomorphism for $k > j$ because it is many-to-one. h_{ii} is identity homomorphism. Compatibility condition $h_{ik} = hijhjk$ is satisfied.

Since the braid group B_n is a covering of S_n, any finite group is sub-group of some B_n. Therefore subgroups of B_n could act as Galois group for Q under suitable conditions. Note that braid groups as coverings of S_n also relate to quantum groups and are therefore physically highly interesting.

According to [3], a considerable progress has been made in the realization of simple groups as Galois groups of regular extensions over $C(t)$ and, and by Hilbert's irreducibility theorem, over every number field.

The basic idea of the rigidity method is that every finite group is a Galois group of some covering of a polynomial field with coefficient in $C(t)$ and $\overline{Q}(t)$. What covering means that one has effectively many-valued polynomials (recall the analog with function $z^{1/n}$) and Galois group permutes the values at a given point. One must only identify the conditions, which ensure that the polynomial can be defined over $Q(t)$.

Rigidity method helps to get down to Q. It is shown that there exists an extension with a given finite Galois group G over some EQ. EQ is generated by the values of G characters for $r \geq 3$ classes of G. Already this is an interesting result. However, if the characters are rational valued, EQ reduces to that for Q and has G as Galois group permuting the copies of the many-valued function.

4 Connections with TGD

How Galois groups emerge in TGD framework, was discussed in the introduction. In this section the connections of the inverse Galois problem with TGD are discussed.

4.1 Why the inverse Galois problem is so relevant for TGD?

The formulation of TGD relies on $M^8 - H$ duality in which space-time surface in complexified M^8 with octonionic interpretation are coded by polynomials with rational coefficients involves the following open question.

Q: Can one allow only polynomials with coefficients in Q or should one allow also coefficients in EQs?

The condition allowing to answer this question is that adelic physics [13, 14] must be able to represent all finite groups as Galois groups over Q or some EQ as physical symmetry group. More generally, TGD Universe is able to physically represent all internally consistent mathematics.

1. If any finite groups can serve as a Galois group over Q, it is enough to have polynomials with coefficients in Q.

2. It this is not possible, then also EQs are needed as coefficient fields for polynomials to have all possible finite groups as Galois groups. The answer to this question is positive. One studies rational functions of a complex variable in S^2 having singularities at n punctures. The Galois group is identified as a braid group for n braids identifiable as a covering group of S_n. G is identified as a subgroup of the braid group.

   One constructs first an extension of certain EQ with Galois group G. As already explained, EQ can be explicitly constructed in terms of the characters of G assignable to $r \geq 3$ conjugacy classes of G and defined as traces of the matrices representing the group element. G acts as a conjugation. This extension of EQ has G as a Galois group. If the characters are rational, the extension is trivial and G acts as a Galois group over Q.

Needless to say, the first option would be the more elegant one.

The folklore is that the inverse Galois hypothesis is true for very many simple groups (this is true at least in the sense that almost all simple groups are 2-groups). Simple groups do not have a non-trivial normal subgroup decomposition so that the polynomial defining the extensions is not representable as a functional composite of polynomials. If all simple Galois groups can appear as Galois groups over rationals then extensions with non-simple Galois group could correspond to composite polynomials.

Note that the functional composition of polynomials yields a fractal structure at space-time level. The polynomial P_{n_1} resp. P_{n_2} corresponds to n_1- resp. n_2-sheeted and $P_{n_1} \cdot P_{n_2}$ corresponds to n_1-sheeted structure with each sheet consisting of n_2 sheets. The question whether functional iteration of a polynomial P_n could define an analog for the approach to chaos at space-time level in the sense of Mandelbrot and Julia fractals is discussed in [18]. Also the functional composites involving different polynomials P_{n_i} should lead to fractal-like structures at the space-time level.

For $P(0) = 0$, the roots of polynomial P are possessed also by its iterate and one could glue regions defined by m:th iterate and time reversal of n:th iterate at values $t = t_n$ corresponding to the roots of P to get a a sequence of iterates with various values of n [28]. In this case the roots are conserved and this brings in mind the notion of conserved genes. It is difficult to avoid the idea that genes could at the level of the magnetic body of the gene actually correspond to functional composites of polynomials P_i satisfying $P_i(0) = 0$.

4.2 Galois invariance as a physical symmetry in TGD

Adelic physics [12, 13] is a proposal for the physics of both sensory experience having real physics as correlate and cognition having various p-adic physics as correlates. Adele is a book-like structure formed by real numbers and the extensions of p-adic number fields induced by a given extension of rationals with the pages of the book glued together along its back consisting of numbers belonging to the extension of rationals. This picture generalizes to space-time level. Adelic physics relies on the notion of cognitive representation as a unique number theoretic discretization of the space-time surface. This discretization has also fermionic analog in terms of spinor structure associated with the group algebra of the Galois group over Q.

4.2.1 Adelic physics very briefly

Number theoretic vision leading to adelic physics [14] provides a general formulation of TGD complementary to the vision [6] (`http://tinyurl.com/sh42dc2`) about physics as geometry of world of classical words (WCW).

1. p-Adic number fields and p-adic space-time sheets serve as correlates of cognition. Adele is a Cartesian product of reals and extensions of all p-adic number fields induced by given extension of rationals. Adeles are thus labelled by extensions of rationals, and one has an evolutionary hierarchy labelled by these extensions. The large the extension, the more complex the extension which can be regarded as $n-D$ space in K sense, that is with K-valued coordinates.

2. Evolution is assigned with the increase of algebraic complexity occurring in statistical sense in BSFRs, and possibly also during the time evolution by unitary evolutions and SSFRs following them. Indeed, in [18] (`http://tinyurl.com/quofttl`) I considered the possibility that the time evolution of self in this manner could be induced by an iteration of polynomials - at least in approximate sense. Iteration is a universal manner to produce fractals as Julia sets and this would lead to the emergence of Mandelbrot and Julia fractals and their 4-D generalizations. In the sequel will represent and argument that the evolution as iterations could hold true in exact sense.

   Cognitive representations are identified as intersection of reality and various p-adicities (cognition). At space-time level they consist of points of imbedding space $H = M^4 \times CP_2$ or M^8 ($M^8 - H$ duality [9, 10, 11] allows to consider both as imbedding space) having preferred coordinates - M^8 indeed has almost unique linear M^8 coordinates for a given octonion structure.

3. Given extension of given number field K (rationals or extension of rationals) is characterized by its Galois group leaving K - say rationals - invariant and mapping products to products and sums to sums. Given extension E of rationals decomposes to extension E_N of extension E_{N-1} of ... of extension E_1 - denote it by $E \equiv H_N = E_N \circ E_{N-1} ... \circ E_1$. It is represented at the level of classical

space-time dynamics in M^8 (`http://tinyurl.com/quofttl`) by a polynomial P which is functional composite $P = P_N \circ P_{N-1} \circ ... \circ P_1$. with $P_i(0) = 0$. The Galois group of $G(E)$ has the Galois group $H_{N-1} = G(E_{N-1} \circ ... \circ E_1)$ as a normal subgroup so that $G(E)/H_{N-1}$ is group.

The elements of $G(E)$ allow a decomposition to a product $g = h_{N-1} \times h_{N-1} \times ...$ and the order of $G(E)$ is given as the product of orders of H_k: $n = n_0 \times .. \times n_{N-1}$. This factorization of prime importance also from quantum point of view. Galois groups with prime order do not allow this decomposition and the maximal decomposition and are actually cyclic groups Z_p of prime order so that primes appear also in this manner.

Second manner for primes to appear is as ramified primes p_{ram} of extension for which the p-adic dynamics is critical in a well-defined sense since the irreducible polynomial with rational coefficients defining the extension becomes reducible (decomposes into a product) in order $O(p) = 0$. The p-adic primes assigned to elementary particles in p-adic calculation have been identified as ramified primes but also the primes labelling prime extensions possess properties making them candidates for p-adic primes.

Iterations correspond to the sequence $H_k = G_0^{\circ k}$ of powers of generating Galois groups for the extension of K serving as a starting point. The order of H_k is the power n_0^k of integer $n_0 = \prod p_{0i}^{k_i}$. Now new primes emerges in the decomposition of n_0. Evolution by iteration is analogous to a unitary evolution as ex^{iHt} power of Hamiltonian, where t parameter takes the role of k .

4. The complexity of extension is characterized by the orders n and the orders n_k as also the number N of the factors. In the case of iterations of extension the limit of large N gives fractal.

5. At space-time level, Galois group acts in the space of cognitive representations and for Galois extensions for which Galois group has same order as extensions, it is natural do consider quantum states as wave functions in $G(E)$ forming n-D group algebra. Therefore Galois groups becomes physical symmetry groups.

   One can assign to the group algebra also spinor structure giving rise to $D = 2^{M/2}$ fermionic states where one has $N = 2M$ or $N = 2M + 1$). One can also consider chirality constraints reducing D by a power of 2. An attractive idea is that this spinor structure represents many-fermion states consisting of $M/2$ fermion modes and providing representation of the fermionic Fock space in finite measurement resolution.

Adelic physics [14], $M^8 - H$ duality [16, 17, 23, 28], and zero energy ontology lead (ZEO) [15, 25, 24] to a proposal [21] that the dynamics involved with small state function reductions (SSFRs) as counterparts of weak measurements could be basically number theoretical dynamics with SSFRs identified as reduction cascades leading to completely un-entangled state in the space of wave functions in Galois group of EQ identifiable as wave functions in the space of cognitive representations. As a side product a prime factorization of the Galois group to simple factors as normal subgroups is obtained.

The result looks even more fascinating if the cognitive dynamics is a representation for the dynamics in real degrees of freedom in finite resolution characterized by the extension of rationals. If cognitive representations represent reality approximately, this indeed looks very natural and would provide an analog for adele formula expressing the norm of a rational as the inverse of the product of is p-adic norms. The results can be appplied to the TGD inspired model of genetic code.

4.2.2 The notions of invariant field and Galois confinement

The physical meaning of the invariant field is interesting from the TGD point of view. I have proposed the notion of Galois confinement as a generalization of color confinement stating that physical states are invariant under Galois group. Color confinement would force hadrons to behave like single unit so that one cannot observe free quarks and the same would happen now.

For instance, in living systems the states of magnetic bodies could be Galois singlets with respect to appropriate Galois group. This would guarantee their stability. For istance, units consisting of 3 dark protons ($h_{eff} = nh_0 > h$) and of 3 dark phonons would represent genetic codons. Galois confinement would force them to behave like single unit. In double DNA strand the codon and is conjugate would form this kind of pair. Dark $3N$-protons and $3N$-photons would in turn represent genes [32, 30, 31, 20, 22].

Galois invariance means invariance under permutations of space-time sheets by the action of Gal and also the invariance of many-fermion states proposed to correspond to the $2^{[n/2]}$-D spinor space for the spinors assignable to the $n-D$ extension having interpretation as fermionic Fock states.

Galois confinement cannot be permanent. In transitions changing the value of h_{eff} it could be lost. For instance, gene can decay to codons and DNA strand could split during replications and transcription.

4.3 The physical interpretation of multi-variable polynomial rings in TGD

Consider the polynomial ring $Q(t_1, .., t_n)$ over Q defining a field of rational functions. This set could define a parametrized set of space-time surfaces. By solving the roots with respect to t_n by keeping $t_1, ...t_{n-1}$ as parameters, one obtains some number of roots. This gives rise to an extension of $Q(t_1, ..., t_{n-1})$ involving algebraic functions of $t_1, ..., t_{n-1}$. One can actually solve the roots of the polynomial equation with respect to any variable t_k. The degrees of the polynomial with respect to t_i are in general different and this would mean that the orders of the Galois group are different.

Here the Noether's theorem comes to the rescue. Instead of polynomials in $Q(t_1, ...t_n)$, one can consider symmetrized polynomials invariant under S_n for which the degree is same for all variables t_i and Galois groups have the same order. Therefore the specification with respect to any t_i can give rise to an irreducible polynomial with the same Galois group. In this case, the extensions defined by the roots of polynomials give also an extension of Q with the same Galois group except at points, where the restriction fails to be irreducible.

Noether's theorem considers a situation for a subgroup of $G \subset S_n$. If the G-invariant field Q^G is purely transcendental and therefore isometric with K, the Galois group of extension of Q^G defines a Galois group over Q. Hilbert's theorem states that this is the case for $G = S_n$.

The specification obtained by fixing the values of $t_1, ..., t_{n-1}$ must be irreducible. The Galois group for the restriction of the polynomials in Q^G is the same for all parameter values at which irreducibility is true.

4.3.1 Could multi-variable polynomials define sub-spaces of WCW with a given Galois group

Space-time surfaces are determined by polynomials of a complex variable with rational coefficients by algebraically continuing them to polynomials of a complexified octonion.

Polynomials with several variables play a central role in the theory of Galois groups. In the TGD framework the parameter type variables would give a parameterized set of space-time surfaces with the same Galois group except at the points at which the irreducibility fails. The order of parameters matters unless one considers only polynomials invariant under S_n or in some cases its sub-group G and inverse Galois theorem does not hold true as is clear from the fact that the dimension of the local Galois group depends on what variable one regards as the variable which is solved.

This kind of parameter sets would naturally define sub-WCW) (WCW is shorthand for "the world of classical worlds") and allow to define WCW spinor fields defining quantum superpositions of space-time surfaces with the same Galois group except at a the points at which the irreducibility of the restriction to a polynomial of a single variable fails.

At the level of sub-WCW, Galois invariance would mean a restriction to the S_n invariant field of a polynomial ring defined by symmetrized multi-variable polynomials gives a parametrized set of extensions of rationals for the "behavior" variable as a complex coordinate continued to complexified octonionic coordinate of M^8. One can also consider the restricted symmetry defined by $G \subset S_n$ encountered in Noether's theorem.

1. If the Galois group is S_n, a possible physical interpretation of Galois confinement would be as a realization of Bose-Einstein statistics in the bosonic degrees of freedom of WCW. Also fermionic statistics could allow a similar interpretation.

2. Could the Galois confinement with respect to a subgroup $G \subset S_n$ have an interpretation in terms of anyonic statistics and charge fractionalization? Could the condition that the invariant field defines a transcendental extension and hence G acts as a Galois group over Q serve as a physical constraint.

3. On the other hand, the fact that anyonic statistics is essentially a 2-D phenomenon associated with the braid group suggests that it could be assigned to the function field $Q(z)$ at a partonic 2-surface containing fermions as punctures. In $M^8 - H$ duality, the positions of fermions would have interpretation as singularities and their position would represent WCW coordinates of the space-time surface. If the strongest form of holography holds true, the position of these punctures could code for the space-time surface (real polynomials are determined by their values at a finite number of points).

4.3.2 The failure of specification, catastrophe theory, and quantum criticality

In Hilbert's irreducibility theorem the notion of specification is essential. Specification fails when it produces as a restriction a reducible polynomial decomposing into a product of polynomials.

1. From the factorization in terms of roots it follows that this occurs when 2 or more roots coincide. For instance, if the roots correspond to a conjugate pair of real or complex roots, they become degenerate and rational. In this case the order of the Galois group decreases.

2. The polynomial can decompose to a more general product meaning a decomposition of the Galois group to a product. One can imagine that there is a small term added to a product of polynomials which vanishes at the criticality. The corresponding space-time region decomposes to distinct regions which can intersect at discrete points. Note for rational polynomials the critical situation is not achieved by a smooth change of the parameters.

   Geometrically criticality means that the space-time surface decomposes to disjoint surfaces corresponding to the roots of the factors which define lower-D extensions of rationals with smaller Galois groups. The decay of the space-time surface occurs. Particle reactions in the geometric sense could correspond to this kind of critical situation.

For the first alternative, catastrophe theoretic analogy [2] (`https://cutt.ly/9mEG8gn`) helps to gain some physical intuition. In the simplest situations such as cusp catastrophe, one has one behavior variable x and some number of control parameters t_i. The roots are those of the gradient of the dV/dx. The equation $dV/dx = 0$ for equilibrium states gives rise to a catastrophe graph in the space defined by x and control parameters. One restricts to real roots so that the map decomposes to regions characterized by different numbers of real roots.

The cusp catastrophe is a simple example. In this case dV/dx has degree $d = 3$ allowing 3 real roots or 1 real root and complex conjugate pair or roots. By restricting x to be real, these correspond to regions which are 3 sheeted and 1-sheeted covers of the 2-D parameter space. At the different sides of the boundary two real *resp.* complex roots become degenerate and irreducibility fails.

This situation corresponds to a criticality at which sudden catastrophic changes can occur. Therefore also the lower-dimensional regions where irreducibility fails are physically highly interesting.

Self-organized criticality (SOC) (`https://cutt.ly/xmEHgcN`) is a real phenomenon but very difficult to understand in thermodynamics with a single arrow of time and should also have a number theoretical interpretation. Zero energy ontology (ZEO) [15, 25] is crucial for the formulation of quantum measurement theory in the TGD framework. This theory extends to a theory of consciousness and leads to a model of self-organized quantum criticality (SOQC) [32, 29, 24] .

One of the key predictions is that the arrow of time changes in the TGD counterparts of ordinary state function reductions (SFRs) - "big" SFRs (BSFRs). For the non-standard arrow of time, dissipative processes look like self-organization processes. This leads to an understanding of (SQOC). The state of the critical sub-system S_1 is unstable but in a time direction opposite to the arrow of time for the system (S). Hence the S_1 tends to criticality when viewed by S. The critical surfaces of the parameter space correspond to the analogs of catastrophes as a failure of reducibility: life seems to love catastrophes!

4.3.3 Hierarchies of parametrized polynomials and of infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy obtained by a repeated second quantization of arithmetic quantum field theory gave a strong boost for the speculations about TGD as a generalized number theory [7].

1. At the lowest level, ordinary primes p label bosonic and fermionic states of an arithmetic supersymmetric quantum field theory (QFT). The product $X_1 = \prod_p p$ of all finite primes is infinite as a real number but finite as a p-adic number for all primes p (the norm is $1/p$) and can be regarded as an analog of the Dirac sea.

   Infinite primes (having unit p-adic norm for every p) are created by kicking from the Dirac sea a set of negative energy fermions represented by the product $n = \prod_{k\in U} p_k$ primes p_k. This gives rise to an object $P = X_1/n + n$. It is easy to check that P is prime. One can also add bosons by the replacement $P \rightarrow kX_1/n + ln$ such that k does not divide n and l has a decomposition to primes appearing as prime factors of n. Altogher $m+l$ bosons have been added.

   The simplest infinite primes are linear in the formal variable X_1 as analogs of roots of monomials with rational coefficients and analogous to Fock states of free bosons and fermions. Besides analogs of Fock states, also analogs of bound states are obtained as infinite primes. They correspond to irreducible polynomials $P(X_1)$ of a single variable obtained. One can decompose P just like an ordinary polynomial to factors corresponding to the roots of $P(X_1) = 0$.

   Infinite primes have therefore an interpretation as many-particle states of a supersymmetric QFT with bound states included and represented in terms of extensions of rationals. There is no need to emphasize that bound states represent a basic problem of QFTs.

2. At the next level of the hierarchy infinite primes X_1 is replaced with X_2 as a product of infinite primes at the first level of the hierarchy and the construction can be continued. The formal variables X_i characterizing various levels of the hierarchy correspond to the infinite (in real sense) numbers defined by the products of all primes at the previous level.

   It is possible to decompose the polynomials at level n to products of monomials defined by the roots of the polynomial equation with X_n as an independent variable to be solved. The infinite primes at the first level become single particle states and second quantization is repeated. At n:th level, one can also construct irreducible polynomials of $X_1, X_2, ..X_n$ and obtains analogs of bound states. One can decompose these polynomials just like one decomposes polynomials of ordinary variables x_i. This gives rise to algebraic extension associated with the rational field defined by polynomials of $X_1, ..., X_n$.

3. The polynomials associated with infinite primes are ordered. The polynomial at n_{th} level has polynomials at $n-1$:th level as coefficients. The nature of the construction as a hierarchy of second quantizations gives rise to states formed from states formed from ...

   This suggests that the symmetrization with respect to variables X_i leading to S_n invariant field realized in terms of symmetric functions does not make sense physically. Notice that the polynomial obtained by the symmetrization does not represent an infinite prime. Physically the different levels in the quantization could correspond to space-time surfaces, whose size scales increase with n. Space-time surfaces at level $n-1$ would be glued by wormhole contacts to the space-time surfaces at the level n.

4. In M^8 picture, infinite primes mapped to polynomials of several variables could be interpreted as representations for a parametrized set of space-time possibly representable as space-time surfaces in complexified M^8 (complexified octonions) and mappable to H by M^8-H duality. They would define a sub-WCW in the "world of classical worlds" (WCW) consisting of space-time surfaces with the same Galois group defined by the roots of X_n. WCW spinor fields would be restricted to this sub-WCW as fermionic Fock states associated with the corresponding space-time surfaces.

   Symmetric polynomials do not correspond to infinite primes but one can wonder whether one could construct WCW spinor fields invariant with respect to S_n or its subgroup. A possible interpretation of S_n in terms of Bose-Einstein statistics and of $G \subset S_n$ in terms of anyonic statistics was already mentioned.

This picture strengthens the hope that TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of rational complexified quaternions and complexified octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

4.4 About possible physical implications

4.4.1 The order of Galois group equals to the dimension of extension

The order of the Galois group is equal to the dimension of the extension. For S_n the order is $n!$ and for the simple group $A_n = S_n/Z_2$, is $n!/2$. The dimension $n! \simeq \sqrt{2\pi n}(n/e)^n$ as the number of space-time sheets increases roughly exponentially with n (`https://cutt.ly/Omk8K6E`). For instance, the order of A_{34} is $34!/2 \simeq 1.4710^{38}$ whereas Mersenne prime M_{127} is $M_{127}2^{127}-1 \simeq 1.7\times10^{38}$. There might be a correlation between p-adic length scales and dark scales proportional to n reflecting resonant coupling between phases with different values of n when dark scale and p-adic length scale are nearly identical.

The Galois group need not act in CP_2 direction and the orbits of the Galois group can quite well be in M^4 direction. Coherent structures of parallel flux tubes with a very large number of flux tubes are suggestive.

The gravitational Planck constant $h_{eff} = h_{gr} = GMm/v_0$, where v_0 is a parameter with dimensions of velocity, has very large values, and extensions with very large dimensions of extension could be assigned with gravitational flux tube bundles.

Galois group acts on the Fock states of fermions. A natural expectation is that this space is effectively finite-dimensional and can be regarded as the $2^{n/2}$-dimensional space of spinors for an n-dimensional extension.

The hierarchy of infinite primes, which correspond to a hierarchy of polynomials in n variables with a natural action of S_n. n has a logarithmic dependence on the order of the Galois group. The Galois groups representable as sub-groups of S_n are representable also as sub-groups of S_{n+1} so that there is an inclusion hierarchy.

4.4.2 Most groups of order at most n are 2-groups

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All p-groups can act as Galois groups of EQ. Most groups of order at most n are 2-groups (`https://en.wikipedia.org/wiki/P-group`). This result is highly interesting from the TGD point of view.

p-Adic length scale hypothesis $p \simeq 2^k$ if the order $n = 2^k$ for Galois group correlates with p-adic prime $p \simeq 2^k$. There is also support for a more general form of hypothesis for primes near a power of 3 are involved. Could p-adic length scale hypothesis relate directly to the fact that most Galois groups are 2-groups?

The order n of the Galois group corresponds to effective Planck constant $h_{eff} = nh_0$ proportional to the number of sheets of the covering defined by Galois extension. Dark Compton length scales are proportional to n.

I have proposed that dark scales for h_{eff} and p-adic length scales for $h_{eff} = h$ interact in the sense that the transitions between states with $h_{eff} = nh_0$ and $h_{eff} = h = n_0h_0$ occur resonantly when the p-adic length scale defined for $h_{eff} = h$ is equal to the dark scale [4]. A kind of frequency or wavelength resonance would take place.

If this proposal is correct, the p-adic length scale hypothesis could be understood as a poorly group theoretical fact. A given element of a 2-group has order 2^m, where m depends on the element, and the order 2-group is 2^k. Could these facts have some interpretation in terms of Boolean algebra of 2^k elements? Could group multiplication and collections of elements coming as powers of a generating element have some interpretation in terms of Boolean algebra and provide it with an additional group structure.

Interestingly, the spinor space of the extension would be 2^{2^k}-dimensional if the Galois group is 2-group. This might relate to the proposal that the Combinatorial Hierarchy, which consists of Mersenne primes M(k+1)=$M_{M_k} - 1$, $M^k = 2^k - 1$. One has $M(1) = 3$, $M(2) = 2^3 - 1 = 7$, $M(3) = 2^7 - 1 = 127$, $M(4) = 2^{127} - 1$. It is not known whether the subsequent Mersenne numbers are primes. M^3 is assigned to genetic code and $M(4)$ to possible memetic code. The number of genetic codons is $2^{M_3-1} = 2^6$ and the number of memetic codons would be $2^{M_7-1} = 2^{126}$.

4.4.3 Braids, knots, and Galois groups

The polynomials defining the space-time surface are polynomials of a complex variable. Could one also consider rational functions with n punctures as poles having Taylor or Laurent expansion. This would bring in the braid group B_n as a covering group of S_n and make it possible to find extensions of Q or some EQ in terms of the values of the characters of $G \subset N_n$.

Braid groups also emerge in another way. Partonic 2-surfaces contain fermions as punctures, which suggests that this approach applies to partonic 2-surfaces and that there is a connection with fermionic braids at the light-like orbits of partonic 2-surfaces and that the Galois group represents the braid group.

4.5 Galois groups and genetic code

Abelian groups Z_p, p prime, are simple and the alternating group A_5 with order 60 is the smallest non-Abelian simple group. All groups A_n, $n \geq 5$ are simple and have $n!/2$ elements. A_5 corresponds to the icosahedral group isomorphic with the symmetry group of the dodecahedron.

The TGD based model of genetic code [8, 20, 22] involves in an essential manner the groups A_5 (icosahedron) and A_4 (tetrahedron). Simple groups play a fundamental role in the TGD view of cognition. Could this mean that genetic code represents the lowest level of an infinite cognitive hierarchy?

4.5.1 The TGD inspired model model of genetic code, cognition, and Galois groups

TGD based model of bioharmony [8, 20, 22] provides a model of genetic code as a fusion of 3 icosahedral Hamiltonian cycles and the unique tetrahedral Hamiltonian cycle (what "fusion" precisely means is far from clear and I have considered several options).

Icosahedral Hamiltonian cycles is a non-self-intersecting path at icosahedron connecting nearest points if icosahedron going through all 12 points of the icosahedron. It is interpreted as a representation of a 12-note scale with a scaling by quint assigned to a given step along the cycle. For a given Hamiltonian cycle, the allowed 3-chords of icosahedral harmony are identified as chords defined by the triangular faces of the icosahedron.

Remark: In the sequel I will use the shorthands IH, OH, and TH for icosahedral, octahedral, and tetrahedral harmonies. Also the notation $I/O/T$ will be used for icosahedron/octahedron/tetrahedron unless there is a danger of confusing them with their symmetry groups with identical shorthand notations.

Galois groups are essential for cognition in the TGD framework. In particular, simple groups as primes for groups are also primes for cognition [21]. Genes represent information and Galois groups are crucial for cognition in the TGD framework. Genes would correspond to sequences of 3-chords of bioharmony. This raises several questions.

Could genetic code relate to Galois group A_5 as the smallest simple non-abelian Galois group (and also to the fact that the only polynomials of order smaller than 5 are generically solvable)? Could genetic code correspond to the lowest level in a hierarchy of cognition and of analogs of genetic code?

The order $n = 60$ for A_5 suggests a fusion of 3 icosahedral codes to give 20+20+20 = 60 codons.

1. 3 Platonic solids, - icosahedron (I), tetrahedron (T) , and octahedron (O) - which have triangles as faces so that one can consider the possibility of constructing a lattice like structure by gluing these Platonic solids together along their faces. Hyperbolic space H^3 indeed allows isosa-tetrahedral tessellation, which also involves O:s. I have proposed that this allows a realization of genetic code and also of genes [22]. The notion of gene generalizes so that genes can also be 2- or 3-D lattice-like structures.

2. A_5 has $A_3 = Z_3$ as a subgroup and I(cosahedron) corresponds to A_5/Z_3. I has several Hamiltonian cycles having as a symmetry group Z_6, Z_4 or Z_2. Z_2 can act either as rotations or reflections.

   Q: Could A_5 as a Galois group as 3-fold covering of I make it possible to understand why the fusion of just 3 icosahedral codes is possible?

3. Tetrahedral group T corresponds to the alternating group $A_4 = S_4/Z_2 = Z_4 \times Z_3$ with 12 elements and tetrahedron identification as A_4/Z_3. The tetrahedral Hamiltonian cycle (4-scale) is unique and has 4 3-chords. The 3-fold copy would correspond to A_4. Information about the unique Hamiltonian cycles of O and T can be found in [1] (`https://cutt.ly/9mlMiV8`).

   Q: Could the factor that there is only one tetrahedral cycle explain why only a single tetrahedron contributes?

4. Octahedral group O has 24 elements and is the wreath product of Z_3 and Z_2^3 and has also the decomposition $O = S_2 \times S_4$. Octahedron can be identified as O/Z_3. Also octahedral Hamiltonian cycle representing 8-scale with 8 chords is unique.

   Q: Why don't octahedral codons contribute?

4.5.2 A model of the genetic code based on icosa-tetrahedral tessellation of hyperbolic 3-space

TGD leads to a proposal for a geometric representation of the genetic code in terms of icosa-tetrahedral tessellation of the hyperbolic 3-space H^3 (mass shell or light-cone proper time $a = constant$ hyperboloids of M^4) [22]. Both I, O, and T having triangular faces appear in the tessellation. Recall that the corresponding harmonies are denoted by IH, OH and TH.

I do not completely understand the details of the icosa-tetrahedral tessellation. The following picture satisfies the constraints coming from the notion of harmony but I have not proven that it is correct. Here the help of a professional geometrician knowing about tessellations of H^3 would be needed.

1. The analog of the discrete translational symmetry for lattices can be assumed: all I:s , O:s and T:s are equivalent as far as common faces with neighboring Platonic solids are considered.

2. The term icosa-tetrahedral tessellation suggests that all octahedral faces are glued to tetrahedral and icosahedral faces so that octahedral chords reduce to either icosahedral or tetrahedral chords. OH would not be an independent harmony. This requires that the number of common faces between two O:s vanishes: $n_O^O = 0$.

3. T shares at least 1 face with a given I so that the number of tetrahedral chords is reduced to at most 3 for given T. 4 purely tetrahedral faces (not shared with I) are needed. I would have $n_{IT} \leq 4$ purely tetrahedral faces in such a way that the total number of purely tetrahedral 3-chords is 4.

   The simplest possibility is that I shares a common face with 2 T:s. Each T shares 2 faces with O providing 2 purely tetrahedral 3-chords and shares the remaining 2 faces with distinct I:s. One would have $n_T^I = 2$, $n_T^O = 2$, $n_T^T = 0$.

   Since each I defines independently 20 chords, 2 I:s cannot have common faces. One would have $n_I^T = 2$, $n_I^I = 0$ and $n_I^O = 18$ to give $n_I^T + n_I^O + n_I^I = 2 + 18 + 0 = 20$.

4. What remains to be fixed are the numbers n_O^I and n_O^T satisfying $n_O^I + n_O^T = 8$. The conditions $n_O^T \geq 1$ and $n_O^I \geq 1$ must be satisfied since both T and I share faces with Os.

   Music comes to rescue here. The 8 3-chords of OH could define OH sub-harmony of IH. Analogously, the 4 3-chords of TH could define TH as a sub-harmony of OH.

   Could IH sharing 18 3-chords with OH contain 2 transposed copies of OH plus 2 chords of TH? IH cannot of course contain the entire TH as a sub-harmony.

   Could OH contain one copy of TH? This would give $n_O^I = n_O^T = 4$. Could the IH part of OH actually be TH as a sub-harmony of IH so that OH would reduce to 2 copies of TH?

To sum up, if the answers to the questions are positive, the incidence matrix n_i^j, $i, j \in \{I, T, O\}$, telling how many faces i shares with j would be given by

$$\begin{bmatrix} n_I^I & n_I^O & n_I^T \\ n_O^I & N_O^O & n_O^T \\ n_T^I & N_T^O & n_T^T \end{bmatrix} = \begin{bmatrix} 0 & 18 & 2 \\ 4 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix} . \tag{4.1}$$

4.5.3 3-fold cover of the icosa-tetrahedral tessellation

The proposed model does not yet explain the fusion of 3 icosahedral Hamiltomnian cycles. A 3-fold cover of the icosa-tetrahedral tessellation which replaces Platonic solids with their symmetry groups is highly suggestive. This raises a series of questions.

1. How could this representation relate to a possible interpretation in terms of the Galois groups $I = A_5$ and $O = S_2 \times S_4$ and $T = A_4$? Z_3 appears as a sub-group of all these groups and these Platonic solids are coset spaces I/Z_3, O/Z_3, and T/Z_3.

2. Could one lift the icosa-tetrahedral tessellation to a 3-sheeted structure formed by the geometric representations of the Galois groups of this structure acting as symmetry groups? Platonic solids would be replaced with their symmetry groups acting as Galois groups.

3. Could the 3 different icosahedral Hamiltonian cycles correspond to different space-time sheets - roughly CP_2 coordinates as 3-valued functions of M^4 coordinates whereas 20 regions representing icosahedral vertices would correspond to different loci of $E^3 \subset M^4$ just as one intuitively expects?

4. Same should apply to the tetrahedral and octahedral parts of the tessellation. But don't the 3 identical copies of the tetrahedral Hamiltonian cycle give 64+8=72 codons? How can one overcome this problem?

The following is a possible answer to these questions.

1. $h_{eff} = 60h_0$ corresponds to 60-sheeted space-time (here also $60k$-sheeted space-time is possible if 60-D extension of k-dimensional extension is in question). For T and O an analogous picture would apply. One could say that the projections of I and O and T are in M^4. At each sheet one would have icosa-tetrahedral tessellation.

2. I has 3 types of Hamiltonian cycles with symmetry groups Z_6, Z_4, and Z_2 and can give 3 different copies. However, only a single copy of tetrahedral harmony appears in the model: otherwise the number of codons would be larger than 64. Could the 3 identical Hamiltonian cycles for T and O effectively correspond to a single Hamiltonian cycle?

3. The fusion of Hamiltonian cycles is analogous to a formation of many-boson states. For T and O all Hamiltonian cycles would be identical: one would have only one Hamiltonian cycle effectively. The 3-chords associated with the 3 octahedral and tetrahedral cycles are identical so that only single tetrahedral harmony would be present.

To sum up, the lift of the icosa-tetrahedral complex to that defined by the respective Galois groups could explain why just 3 icosahedral Hamiltomian cycles and effectively only 1 tetrahedral cycle.

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References

[1] Sequin CH. Hamiltonian Cycles on Symmetrical Graphs, 2015. Available at: https://cutt.ly/EmQFtF8.

[2] Zeeman EC. *Catastrophe Theory*. Addison-Wessley Publishing Company, 1977.

[3] Rajnbar S Ranjbar F. Inverse Galois Problem and Significant Methods, 2015. Available at: https://arxiv.org/pdf/1512.08708.pdf.

[4] Pitkänen M. Magnetic Sensory Canvas Hypothesis. In *TGD and EEG*. Available at: http://tgdtheory.fi/pdfpool/mec.pdf, 2006.

[5] Pitkänen M. Massless states and particle massivation. In *p-Adic Physics*. Available at: http://tgdtheory.fi/pdfpool/mless.pdf, 2006.

[6] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry*. Available at: http://tgdtheory.fi/pdfpool/wcwnew.pdf, 2014.

[7] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I*. Available at: http://tgdtheory.fi/pdfpool/visionc.pdf, 2019.

[8] Pitkänen M. Geometric theory of harmony. Available at: http://tgdtheory.fi/public_html/articles/harmonytheory.pdf., 2014.

[9] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part I. Available at: http://tgdtheory.fi/public_html/articles/ratpoints1.pdf., 2017.

[10] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part II. Available at: http://tgdtheory.fi/public_html/articles/ratpoints2.pdf., 2017.

[11] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part III. Available at: http://tgdtheory.fi/public_html/articles/ratpoints3.pdf., 2017.

[12] Pitkänen M. p-Adicization and adelic physics. Available at: http://tgdtheory.fi/public_html/articles/adelicphysics.pdf., 2017.

[13] Pitkänen M. Philosophy of Adelic Physics. In *Trends and Mathematical Methods in Interdisciplinary Mathematical Sciences*, pages 241–319. Springer.Available at: https://link.springer.com/chapter/10.1007/978-3-319-55612-3_11, 2017.

[14] Pitkänen M. Philosophy of Adelic Physics. Available at: http://tgdtheory.fi/public_html/articles/adelephysics.pdf., 2017.

[15] Pitkänen M. Some comments related to Zero Energy Ontology (ZEO). Available at: http://tgdtheory.fi/public_html/articles/zeoquestions.pdf., 2019.

[16] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part I. Available at: http://tgdtheory.fi/public_html/articles/M8H1.pdf., 2020.

[17] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part II. Available at: http://tgdtheory.fi/public_html/articles/M8H2.pdf., 2020.

[18] Pitkänen M. Could quantum randomness have something to do with classical chaos? Available at: http://tgdtheory.fi/public_html/articles/chaostgd.pdf., 2020.

[19] Pitkänen M. Generalization of Fermat's last theorem and TGD. Available at: http://tgdtheory.fi/public_html/articles/FermatTGD.pdf., 2020.

[20] Pitkänen M. How to compose beautiful music of light in bio-harmony? https://tgdtheory.fi/public_html/articles/bioharmony2020.pdf., 2020.

[21] Pitkänen M. The dynamics of SSFRs as quantum measurement cascades in the group algebra of Galois group. Available at: http://tgdtheory.fi/public_html/articles/SSFRGalois.pdf., 2020.

[22] Pitkänen M. Is genetic code part of fundamental physics in TGD framework? Available at: https://tgdtheory.fi/public_html/articles/TIH.pdf., 2021.

[23] Pitkänen M. Is $M^8 - H$ duality consistent with Fourier analysis at the level of $M^4 \times CP_2$? https://tgdtheory.fi/public_html/articles/M8Hperiodic.pdf., 2021.

[24] Pitkänen M. Negentropy Maximization Principle and Second Law. Available at: https://tgdtheory.fi/public_html/articles/nmpsecondlaw.pdf., 2021.

[25] Pitkänen M. Some questions concerning zero energy ontology. https://tgdtheory.fi/public_html/articles/zeonew.pdf., 2021.

[26] Pitkänen M. The idea of Connes about inherent time evolution of certain algebraic structures from TGD point of view. https://tgdtheory.fi/public_html/articles/ConnesTGD.pdf., 2021.

[27] Pitkänen M. Time reversal and the anomalies of rotating magnetic systems. Available at: https://tgdtheory.fi/public_html/articles/freereverse.pdf., 2021.

[28] Pitkänen M. What could 2-D minimal surfaces teach about TGD? https://tgdtheory.fi/public_html/articles/minimal.pdf., 2021.

[29] Pitkänen M and Rastmanesh R. Homeostasis as self-organized quantum criticality. Available at: http://tgdtheory.fi/public_html/articles/SP.pdf., 2020.

[30] Pitkänen M and Rastmanesh R. New Physics View about Language: part I. Available at: http://tgdtheory.fi/public_html/articles/languageTGD1.pdf., 2020.

[31] Pitkänen M and Rastmanesh R. New Physics View about Language: part II. Available at: http://tgdtheory.fi/public_html/articles/languageTGD2.pdf., 2020.

[32] Pitkänen M and Rastmanesh R. The based view about dark matter at the level of molecular biology. Available at: http://tgdtheory.fi/public_html/articles/darkchemi.pdf., 2020.

[33] Pitkänen M and Rastmanesh R. Why the outcome of an event would be more predictable if it is known to occur? https://tgdtheory.fi/public_html/articles/scavhunt.pdf., 2021.