Article

Microscopes and Telescopes for Theoretical Physics: How Rich Locally and Large Globally is the Geometric Straight Line?

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Dedicated to Marie-Louise Nykamp

Abstract

One is reminded in this paper of the often overlooked fact that the *geometric straight line*, or GSL, of Euclidean geometry is not necessarily identical with its usual Cartesian coordinatisation given by the real numbers in $\mathbb{R}$. Indeed, the GSL is an *abstract idea*, while the Cartesian, or for that matter, any other specific coordinatisation of it is but one of the possible *mathematical models* chosen upon certain reasons. And as is known, there are a a variety of mathematical models of GSL, among them given by nonstandard analysis, reduced power algebras, the topological long line, or the surreal numbers, among others. As shown in this paper, the GSL can allow coordinatisations which are *arbitrarily more rich locally* and also *more large globally*, being given by corresponding linearly ordered sets of no matter how large cardinal. Thus one can obtain in relatively simple ways structures which are more rich locally and large globally than in nonstandard analysis, or in various reduced power algebras. Furthermore, vector space structures can be defined in such coordinatisations. Consequently, one can define an extension of the usual Differential Calculus. This fact can have a major importance in physics, since such locally more rich and globally more large coordinatisations of the GSL do allow new physical insights, just as the introduction of various microscopes and telescopes have done. Among others, it and general can reassess special relativity with respect to its independence of the mathematical models used for the GSL. Also, it can allow the more appropriate modelling of certain physical phenomena. One of the long vexing issue of so called "infinities in physics" can obtain a clarifying reconsideration. It indeed all comes down to looking at the GSL with suitably constructed microscopes and telescopes, and apply the resulted new modelling possibilities in theoretical physics. One may as well consider that in string theory, for instance, where several dimensions are supposed to be compact to the extent of not being observable on classical scales, their mathematical modelling may benefit from the presence of infinitesimals in the mathematical models of the GSL presented here.
However, beyond all such particular considerations, and not unlikely also above them,

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is the following one : theories of physics should be not only background independent, but quite likely, should also be independent of the specific mathematical models used when representing geometry, numbers, and in particular, the GSL.
One of the consequences of considering the essential difference between the GSL and its various mathematical models is that what appears to be the definitive answer is given to the intriguing question raised by Penrose : “Why is it that physics never uses spaces with a cardinal larger than that of the continuum ?".

"History is written with the feet ..."

Ex-Chairman Mao, of the Long March fame ...

"Science is nowadays not done scientifically, since it is mostly done by non-scientists ..."

Anonymous

A "mathematical problem" ?
For sometime by now, American mathematicians
have decided to hide their date of birth
and not to mention it in their academic CV-s.
Why ?
Amusingly, Hollywood actors and actresses have their
birth date easily available on Wikipedia.
Can one, therefore, trust American
mathematicians ?
Why are they so blatantly against transparency ?
By the way, Hollywood movies have also for long
been hiding the date of their production ...

A bemused non-American mathematician

0. Preliminaries

Let us start with three statements which may elicit strong - and quite likely wrong - reactions :

(0.1) We do not - and can never ever - know what *geometry* is !

(0.2) We do not - and can never ever - know what *numbers* are !

(0.3) We do not - and can never ever - know what the *geometric straight line*, or in short, the GSL is !

It often happens among those who use mathematics that they fail to recognize the *fundamental difference* between *abstract ideas*, and on the other hand, one or another of their *mathematical model* which happens to be chosen upon specific reasons, or rather, upon mere historical circumstance.

Recently, [17-25], a good illustration of that fundamental difference between an abstract idea and its various mathematical models was given with respect to the abstract idea of *numbers*, and on the other hand, what is suggested to be called as one or another specific *numeral system* which is but one of the many particular ways to model mathematically numbers.

As far as the GSL is concerned a variety of rather different mathematical models are known for it, among them those given by nonstandard analysis, [3,5], reduced power algebras, [6-14], the topological long line, [26], or the surreal numbers, [2], the latter giving in fact a GSL which is no longer a set but much larger still, being a proper class. And obviously, the abstract idea of the GSL is a particular instance of a mixture between the abstract ideas of geometry and numbers. Since, however, it plays such a fundamental role in the construction of a large number of theories of physics, it may indeed be useful to focus on the issue of the GSL all on its own, as done in the sequel.

Amusingly, that fundamental difference between an abstract idea and its possible various mathematical models is on occasion known and accepted by various users of mathematics, among them physicists, for instance. Such is the case, among others, with the abstract idea of *geometry* which is by now well understood - and not only in general relativity - *not* to refer to, and *not* to be identical with any one single mathematical model, but to have a rather large number of quite different such mathematical models.

However, it should be recalled that less than two centuries ago, following a philosophy of the calibre of that of Kant, the abstract idea of geometry was seen as being reduced to, and perfectly identical with, one single mathematical model, namely, that of Euclidean geometry. And as if to aggravate the error in such a view, Kant considered Euclidean geometry as an a priori concept.
As it happened nevertheless in the 1820s, non-Euclidean geometry was introduced by Lobachevski and Bolyai, followed not much later by Riemannian geometry, and more

near to our times, by the more or less explicit recognition of the immense difference between the abstract idea of geometry, and on the other hand, any of its mathematical models.
And still, the errors in approaching the concept of geometry were to continue for a while longer. The "Erlangen Program", for instance, published by Felix Klein in 1872, saw geometry as reduced to the rather narrow framework of the study of properties invariant under certain group transformations ...

In this regard, one may simply say that one does not - and in fact, can never ever - know what geometry, or for that matter, numbers are. And all one can know instead are merely various mathematical models of geometry, or of numbers.

Here, in the above spirit, we highlight the fundamental difference between the abstract idea of the GSL, and on the other hand, its various mathematical models.
In other words, we can never ever really know what the GSL may be. Instead, we may only try to represent it with one or another mathematical model ...

The importance of stressing this fundamental difference is to make theoretical physicists aware of the most simple and elementary - yet so widely missed - fact that confining all mathematical models used in physics to those based on the usual Cartesian coordinatisation of the GSL, a coordonatisation given by the customary field $\mathbb{R}$ of real numbers, is a highly dubious and detrimental approach.
Indeed, as seen in the sequel, the GSL can admit other mathematical models which are *far more rich locally*, as well as *far more large globally*. Thus they allow the modelling of realms of physics which have so far escaped attention, or have been modelled inappropriately due to the relative poverty of the usual Cartesian coordinatisation given by $\mathbb{R}$.

The coordinatisations constructed in the sequel are not only linear orders, but can also have useful commutative group, and in fact, vector space structures. Consequently, they allow the extension of the usual Differential Calculus. Thus one can obtain in relatively simple ways commutative group and vector space structures, together with a Differential Calculus, which may be arbitrarily more rich locally, and also arbitrarily more large globally than those constructed earlier in the literature, [2,3,5,26].

1. The View of the GSL through Microscopes and Telescopes ...

The GSL, that is, the straight line in Euclidean geometry, has since Descartes been coordinatised by the usual field $\mathbb{R}$ of real numbers. This association - so critically useful in modern physics, starting with Newtonian mechanics - has ended up with a tacit and rather universal *identification* between the GSL and the field $\mathbb{R}$ of usual real numbers. And as it happens, that situation is further entrenched by the rather simple mathematical fact that $\mathbb{R}$ is the only field which is linearly ordered, complete and Archimedean. As for the Archimedean Axiom, it again happens to be another quite universally and tacitly entrenched assumption, ever since the time of Euclid in ancient Egypt, more than two millennia ago, an assumption which, however, does not seem to have any known motivation in modern physics, [6-14].

Amusingly however, since the 1960s, the Nonstandard Analysis of Abraham Robinson, [3,5], has convincingly shown that the GSL can be endowed with a coordinatisation by a far larger field than $\mathbb{R}$, namely, that given by the field $^*\mathbb{R}$ of nonstandard real numbers. And such a coordinatisation is considerably more sophisticated than the usual one, both locally and globally, [2,3,5,26]. Also, it does no longer satisfy the Archimedean Axiom. And in fact, it simply cannot satisfy that axiom in view of the above remark on the unique position of $\mathbb{R}$.
Yet with very few exceptions, the fact of the existence of such a considerably more sophisticated coordinatisation of the GSL seems not to have registered in the least either with mathematicians, or with physicists, in spite of its manifest advantages, among others in the study of continuous time stochastic processes which have importance also in physics.

The crucial issue - so far quite widely missed - is that in physics one operates, as with a basic theoretical instrument, *not* so much with the GSL itself, but rather with its specific *coordinatisation* which, as mentioned, is nearly universally and also quite stubbornly stuck to the field $\mathbb{R}$ of usual real numbers.
And to the extent that such a coordinatisation is not sophisticated enough, one can expect two major negative outcomes which, so far, have passed rather unnoticed :

- fundamental physical phenomena may escape the general awareness, just like it happened before the introduction of instruments such as microscopes and telescopes,

- fundamental physical phenomena which do not escape notice may nevertheless be thoroughly misinterpreted and misrepresented due to the coarseness of the instrument given by the usual coordinatisation employed, namely, by the field $\mathbb{R}$ of usual real numbers.

As far as mathematics is concerned, there is comparatively with physics a considerable freedom, since major criteria for relevance are not so much and so directly related to physical relevance. Consequently, the mathematics which obtains based on the usual Cartesian coordinatisation of the GSL need not lead to errors, and only risks to leave aside a vast realm of possibly relevant mathematics.

And then, above all for physics and physicists the following long disregarded critically important question arises :

- How far can one - and in fact, should one - go in more sophisticated coordinatisations of the Geometric Straight Line ?

One can recall related to this question the rather intriguing and highly relevant question of Penrose, [4] :

> “Why is it that physics never uses spaces with a cardinal larger than that of the continuum ?"

As it happens, however, there is no trace in the recent, or for that matter, earlier physics literature that this question would be considered, let alone debated ...

Now, as also illustrated by nonstandard analysis or by the reduced power algebras, the concept of "more sophisticated coordinatisations" of the GSL has at least the following two aspects, [6-14] :

- a *local* one corresponding to the richness around each coordinate point, like for instance corresponding to the nonstandard monad of infinitesimally near points to each given coordinate point, and as such, seen by Keisler's microscope, [3],
- a *global* one corresponding to the largeness of the set of all coordinate points, like for instance given by the nonstandard infinitely large coordinate points, and as such, seen by Keisler's telescope, [3].

2. Background Independence : How Far Should the Principle of Relativity Go ?

As mentioned in [6-14], beyond, and quite likely above, the previous considerations, the study of the essential difference between the abstract idea of the GSL, and on the other hand, its various specific mathematical models may have a foundational importance in theories of physics regarding the issue of *how far should the Principle of Relativity go ?*, [6,8].

A remarkable fact in this regard is that the GSL can be modelled by algebras from a very large class of reduced power algebras, and in doing so, basic results from relativity, quantum mechanics and quantum computation remain valid, [10-12]. And here one should point out that such reduced power algebras are *not* fields, since they have zero divisors. Also, these algebras are *not* one dimensional, as the GSL is supposed to be, and instead, they may be infinite dimensional. Furthermore, such algebras are in general *not* linearly ordered or Archimedean, as the GSL is assumed to be.

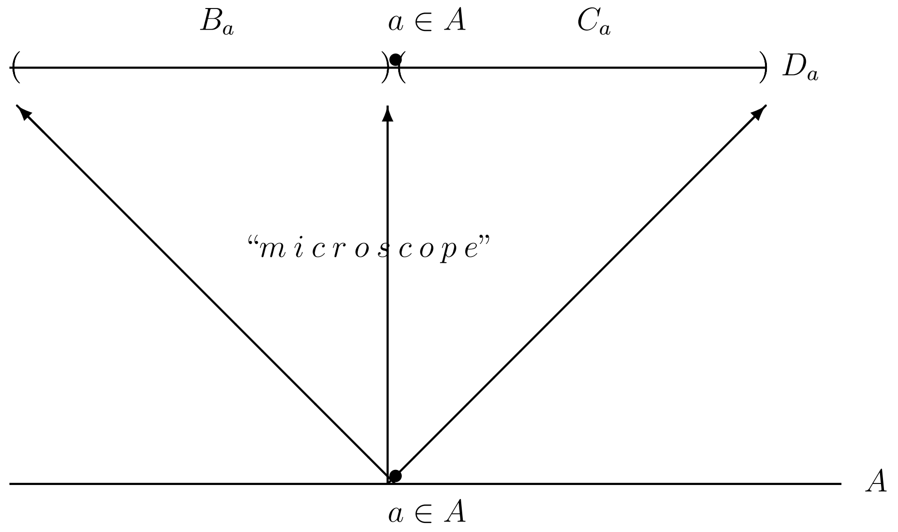
Consequently, one can note that such basic physical properties as presented in [10-12], are indeed *independent* of a very large class of mathematical models, this fact thus being an illustration to a significant extension of the Principle of Relativity. Indeed, this time one can show that certain basic physical properties do *not* depend not only on reference frames, but also on a very large class of mathematical models which replace the usual model of the GSL.

3. A General Construction for Local Richness

We shall start with a general construction which brings about *local richness*, a construction which will mostly be used in some of its particular cases, such as for instance in Example 3.1. below. However, the interest in such a generality is, among others, in the fact that it clarifies what is actually involved in the issue of local richness.

Let $(A, \leq)$ be a linearly ordered set. This set will have each of its elements $a \in A$ endowed as if with a *microscope* which will open up, both to the left and to the right, the view from each such point $a \in A$, and do so with corresponding linearly ordered sets that are the local realms not accessible within A itself.

Such a local realm at each $a \in A$, a realm which does not exist in A itself, is given by two linearly ordered sets $(B_a, \leq_a)$ and $(C_a, \leq_a)$, with the first to the left of $a \in A$, and the second to the right of $a \in A$, as illustrated below



$Fig.$ 3.1.

For convenience of notation, we suppose that the sets A, B_a, C_a are pair-wise disjoint.

Then we define the linearly ordered set

(3.1) $\quad (D, \leq)$

which is the enrichment of A with the respective local views, where

(3.2) $\quad D = \bigcup_{a \in A} D_a$

with, see Fig. 3.1. above

(3.3) $\quad D_a = B_a \cup \{a\} \cup C_a$

The linear order $\leq$ on D in (3.2) is defined as follows. For $d, d' \in D$, we have $d \leq d'$, if and only if one of the next seven conditions holds

$$
(3.4) \quad
\begin{aligned}
& i)\ d \in D_a,\ d' \in D_{a'},\ \ a \leq a',\ a \neq a' \\
& ii)\ d, d' \in B_a,\ d \leq_a d' \\
& iii)\ d \in B_a,\ d' = a \\
& iv)\ d \in B_a,\ d' \in C_a \\
& v)\ d = a,\ d' \in C_a \\
& vi)\ d, d' \in C_a,\ d \leq_a d' \\
& vii)\ d = d'
\end{aligned}
$$

Clearly, we have the *strictly increasing* mapping

(3.5) $i : A \ni a \longmapsto a \in D$

based on which, whenever there exists $a \in A$, such that $B_a \cup C_a \neq \phi$, we shall consider having the strict inclusion

(3.6) $A \subsetneqq D$

which in fact can be seen as having the linearly ordered set $(A, \leq)$ as a strict subset in $(D, \leq)$, and endowed with the induced linear order from $(D, \leq)$.

Example 3.1.

Let $(A, \leq) = \mathbb{R}$ and $(B_a, \leq_a) = \{a\} \times (-\infty, 0),\ (C_a, \leq_a) = \{a\} \times (0, \infty)$, with $a \in A = \mathbb{R}$. Then (3.3) gives

(3.7) $D_a = (\{a\} \times (-\infty, 0)) \cup \{a\} \cup (\{a\} \times (0, \infty)), \quad a \in A$

hence from (3.2) we obtain

(3.8) $D = (A \times (-\infty, 0)) \cup A \cup (A \times (0, \infty))$

therefore, the mapping

(3.9) $j : D \longrightarrow \mathbb{R}^2$

given by

$$
(3.10) \quad
\begin{aligned}
& j(a, x) = (a, x) \quad a, x \in \mathbb{R},\ x \neq 0 \\
& j(a) = (a, 0) \quad a \in \mathbb{R}
\end{aligned}
$$

is an *order isomorphism* between $(D, \leq)$ and $\mathbb{R}^2$, when the latter is considered with the *lexicographic order*, see Appendix

(3.11) $$(a, x) \preceq (b, y) \iff \begin{pmatrix} a < b \\ \text{or} \\ a = b, \;\; x \leq y \end{pmatrix}$$

which obviously is a *linear order* on $\mathbb{R}^2$.

We can, therefore, identify $(D, \leq)$ with $(\mathbb{R}^2, \preceq)$, modulo the above mapping j.

4. A Specific Method to Construct Local Richness

This method is suggested by Example 3.1. in section 3 above.

As in section 3, let $(A, \leq)$ be a linearly ordered set which will be locally enriched.

Let $(B, \leq)$ be any linearly ordered set which will produce the local enrichment at each $a \in A$. Namely, for each $a \in A$, we shall construct the linearly ordered set

(4.1) $$(B_a, \leq_a)$$

where

(4.2) $$B_a = \{a-\} \times B$$

and for $\beta = (a-, b), \beta\,' = (a-, b\,') \in B_a$, we have $\beta \leq_a \beta\,'$, if and only if

(4.3) $$b\,' \leq b$$

Similarly, we define the linearly ordered set

(4.4) $$(C_a, \leq_a)$$

where

(4.5) $$C_a = \{a+\} \times B$$

and for $\gamma = (a+, b), \gamma\,' = (a+, b\,') \in C_a$, we have $\gamma \leq_a \gamma\,'$, if and only if

(4.6) $$b \leq b\,'$$

Now we can perform the construction in section 3, and obtain the corresponding linearly ordered set in (3.1), which we shall denote by

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(4.7) $(A, \leq) \,\square\, (B, \leq)$

Clearly, we have the *injective strictly increasing* mapping, see (3.5)

(4.8) $(A, \leq) \ni a \longmapsto a \in (A, \leq) \,\square\, (B, \leq)$

based on which one can see the linearly ordered set $(A, \leq)$ as a strict subset in $(A, \leq) \,\square\, (B, \leq)$, whenever we have $A, B \neq \phi$. Furthermore, $(A, \leq)$ is endowed with the induced linear order from $(A, \leq) \,\square\, (B, \leq)$.

It is easy to see that, if the cardinals of $(A, \leq)$ and $(B, \leq)$ are infinite, then

(4.9) $car((A, \leq) \,\square\, (B, \leq)) = car(A, \leq) \,.\, car(B, \leq)$

Example 4.1.

For the linearly ordered set $(D, \leq)$ in Example 3.1., we obviously have

(4.10) $(D, \leq) = \mathbb{R} \,\square\, (0, \infty)$

5. Commutative Group Structures on the Locally Enriched Linearly Ordered Sets

Let $(A, +, \leq), (B, +, \leq)$ be linearly ordered commutative groups, and let $B_+ = \{b \in B \mid b \geq 0,\ b \neq 0\}$. Then (4.7) gives the locally enriched linearly ordered set

(5.1) $(\bar{A}, \leq) = (A, \leq) \,\square\, (B_+, \leq)$

Now let us recall that $A \times B$ is a commutative group with the binary operation

(5.2) $(a, b) + (a\,', b\,') = (a + a\,', b + b\,')$

while with the lexicographic order, denoted by $\preceq$, $A \times B$ is linearly ordered.

Furthermore, see Appendix, Proposition A1.1., $(A \times B, +, \preceq)$ is a linearly ordered commutative group.

Now, similar with (3.9), (3.10), the mapping

(5.3) $j : \bar{A} \longrightarrow A \times B$

given by

(5.4)
$$j(a, b) = (a, b) \quad a \in A,\ b \in B,\ b \neq 0$$
$$j(a) = (a, 0) \quad a \in A$$

where 0 is the neutral element in A, respectively B, is a *strictly increasing group isomorphism.*

Also, the mapping

(5.5) $\quad i : A \ni a \longmapsto (a, 0) \in \bar{A}$

is a *strictly increasing group homomorphism.*

In this way we obtain

Theorem 5.1.

Let $(A, +, \leq), (B, +, \leq)$ be linearly ordered commutative groups, and let $B_+ = \{b \in B \mid b \geq 0, \ b \neq 0\}$. Then, see (4.7)

(5.6) $\quad (\bar{A}, \leq) = (A, \leq) \,\square\, (B_+, \leq)$

is a linearly ordered commutative group, with the strictly increasing group homomorphism

(5.7) $\quad i : A \ni a \longmapsto (a, 0) \in \bar{A}$

6. Vector Space Structures on Locally Enriched Linearly Ordered Sets

Let us consider the commutative group of the locally enriched linearly ordered set $(\bar{A}, \leq) = (A, \leq) \,\square\, (B_+, \leq)$ in (5.6), and let us suppose that the respective commutative groups A and B are vector spaces over a linearly ordered filed F. Then one can define the scalar multiplication

(6.1) $\quad F \times \bar{A} \ni (c, \bar{a}) \longmapsto c\bar{a} \in \bar{A}$

by

$$c(a, b) = 0 \in A, \quad c = 0 \in F, \ a \in A, \ b \in B, \ b \neq 0$$

(6.2) $$c(a, b) = (ca, cb), \quad c \in F, \ c \neq 0, \ a \in A, \ b \in B, \ b \neq 0$$

$$ca = ca, \quad a \in A$$

Theorem 6.1.

Let $(A, +, \leq), (B, +, \leq)$ be linearly ordered commutative groups which are vector spaces over a linearly ordered field F. Then the action (6.1), (6.2) defines on $(\bar{A}, \leq) = (A, \leq) \,\square\, (B_+, \leq)$ a linearly ordered vector space over the linearly ordered field F.

Proof.

In view of (5.3), (5.4) and Theorem 5.1., we have the strictly increasing group isomorphism

(6.3) $j : \bar{A} \longrightarrow A \times B$

However, in view of Proposition A1.2., $(A \times B, +, ., \preceq)$ is a linearly ordered vector space over F. Therefore, it only remains to show that the strictly increasing group isomorphism (5.3), (5.4) makes compatible the scalar multiplications in $\bar{A}$ and $A \times B$.

Let therefore $c = 0 \in F$ and $a \in A,\ b \in B,\ b \neq 0$. Then (6.2) gives in $\bar{A}$ the relation $c(a, b) = 0 \in A$, while in $A \times B$ we have $cj(a, b) = c(a, b) = (0, 0)$. On the other hand, $j(0) = (0, 0)$.

Let now $c \in F,\ c \neq 0,\ a \in A,\ b \in B,\ b \neq 0$. Then similarly $c(a, b) = (ca, cb)$, while $cj(a, b) = c(a, b) = (ca, cb)$. On the other hand, $j(ca, cb) = (ca, cb)$.

Finally, $cj(a) = c(a, 0) = (ca, 0)$, and $j(ca) = (ca, 0)$.

7. Iterations of the Operation $\square$

Let $(A, \leq), (B, \leq)$ and $(C, \leq)$ be three linearly ordered sets.

Let $(\phi, \leq)$ denote the void linearly ordered set, then

(7.1) $\quad (A, \leq) \, \square \, (\phi, \leq) = (A, \leq), \;\; (\phi, \leq) \, \square \, (A, \leq) = (\phi, \leq)$

Further, the linearly ordered sets

(7.2) $\quad (A, \leq) \, \square \, (B, \leq), \;\; (B, \leq) \, \square \, (A, \leq)$

are in general *not* order isomorphic.

Similarly, the linearly ordered sets

(7.3) $\quad (A, \leq) \, \square \, (\, (B, \leq) \, \square \, (C, \leq) \,), \;\; (\, (A, \leq) \, \square \, (B, \leq) \,) \, \square \, (C, \leq)$

are in general *not* order isomorphic.

We note however that

(7.4) $\quad (A, \leq) \, \square \, (\, (B, \leq) \, \square \, (C, \leq) \,) = (E, \leq)$

where

(7.5) $\quad E = \bigcup_{a \in A} \{ \, [\, (\bigcup_{b \in B} (\, (\, C \times \{-b\}) \cup \{b\} \cup (\, C \times \{+b\} \,) \,) \,) \times \{-a\}] \, \cup$

$\cup \, \{a\} \, \cup$

$\cup \, [\, (\bigcup_{b \in B} (\, (\, C \times \{-b\}) \cup \{b\} \cup (\, C \times \{+b\} \,) \,) \,) \times \{+a\} \,] \, \}$

Similarly

(7.6) $\quad (\, (A, \leq) \, \square \, (B, \leq) \,) \, \square \, (C, \leq) = (F, \leq)$

where

(7.7) $\quad F = \bigcup_{a' \in \bigcup_{a \in A} (\, (\, B \times \{-a\}) \cup \{a\} \cup (\, B \times \{+a\} \,) \,)} (\, (\, C \times \{-a'\}) \, \cup$

$\cup \, \{a'\} \, \cup$

$\cup \, (\, C \times \{+a'\} \,) \,)$

By using the above defined iterations of the operation $\square$, the results in sections 5 and 6 regarding the endowment of the locally enriched linearly ordered sets with commutative group, respectively, vector space structures, obviously allow the con-

struction of *arbitrarily large* linearly ordered sets which are commutative groups or vector spaces.
Indeed, let $(A, +, \leq), (B, +, \leq)$ be linearly ordered commutative groups which are vector spaces over a linearly ordered field F. Then in view of Theorem 6.1., we obtain the following linearly ordered vector spaces over the linearly ordered field F, namely

$$\begin{array}{l} (A, +, \leq) \,\Box\, (B, +, \leq) \\ \\ (\,(A, +, \leq) \,\Box\, (B, +, \leq)\,) \,\Box\, (B, +, \leq) \\ \\ (\,(\,(A, +, \leq) \,\Box\, (B, +, \leq)\,) \,\Box\, (B, +, \leq)\,) \,\Box\, (B, +, \leq) \\ \\ \vdots \end{array} \tag{7.8}$$

And in view of (4.9), it follows that in this way one can construct linearly ordered vector spaces of *arbitrarily large cardinal* over the linearly ordered field F.

8. Infinite Iterations of $\Box$

Let $(A, \leq)$ be a nonvoid linearly ordered set, and $n \geq 1$. We define

$$(A, \leq)^{\Box n} = \left| \begin{array}{ll} (A, \leq) & \text{if } n = 1 \\ \\ ((A, \leq)^{\Box n-1})\Box(A, \leq) & \text{if } n \geq 2 \end{array} \right. \tag{8.1}$$

Clearly, that definition can be extended to all ordinal numbers α. Indeed, if α is not a limit ordinal, then we define

$$(A, \leq)^{\Box\alpha} = ((A, \leq)^{\Box\alpha-1})\Box(A, \leq) \tag{8.2}$$

while for α a limit ordinal, we define

$$(A, \leq)^{\Box\alpha} = \bigcup_{\beta<\alpha}(A, \leq)^{\Box\beta} \tag{8.3}$$

We note that, in view of (4.8), we have strict inclusions

$$\begin{array}{l} (A, \leq)^{\Box 1} \subsetneqq (A, \leq)^{\Box 2} \subsetneqq (A, \leq)^{\Box 3} \subsetneqq \ldots \subsetneqq (A, \leq)^{\Box n} \subsetneqq \ldots \\ \\ \ldots \subsetneqq (A, \leq)^{\Box\omega} \subsetneqq (A, \leq)^{\Box\omega+1} \subsetneqq \ldots \end{array} \tag{8.4}$$

and this justifies the definition in (8.3).

In view of (4.9), given any nonvoid linearly ordered set $(A, \leq)$, the cardinal of the linearly ordered set in $(A, \leq)^{\Box\alpha}$ can be arbitrary large, for sufficiently large ordinal α.

Obviously, the above construction in (8.1) - (8.4) can be extended to the case when for each ordinal α, one has given a linearly ordered set $(A_\alpha, \leq)$.

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In fact, one can let the respective iterations run over all ordinals, and thus obtain linearly ordered *proper classes* which are no longer sets.

Furthermore, iterations such as in (7.8) can be performed for arbitrary cardinals, thus obtaining arbitrarily large linearly ordered commutative groups or vector spaces over linearly ordered fields.

9. Looking through Microscopes ...

It is easy to see that the constructions in sections 3 - 7 above do the following. One takes a given nonvoid linearly ordered set $(A, \leq)$ and constructs with the help of the same, or of any other nonvoid linearly ordered set $(B, \leq)$ the corresponding extension $(\bar{A}, \leq) =$
$(A, \leq) \,\Box\, (B, \leq)$ of $(A, \leq)$, such that

- A is a strict subset of $\bar{A}$,
- the linear order on $\bar{A}$ induces the initial linear order on A,
- each point $a \in A$ obtains booth to the left and right a linearly ordered extension in $\bar{A}$.

Thus the *local* structure of A is *enriched* by $\bar{A}$. And this enrichment happens as follows

- at each point $a \in A$, a "*microscope*" given by $(B, \leq)$ is installed with the effect that instead of the single point $a \in A$, now we can see the set $(\{a-\} \times B) \cup \{a\} \cup (\{a + \times B\})$ which contains $a \in A$, and in addition, it extends to the left and to the right of $a \in A$ with a realm twice the size of B,
- for two different points $a, a' \in A$, their respective added realms do *not* intersect.

10. A General Construction for Global Largeness

Here, we present a construction similar with the above one for local enrichment which, this time, *enlarges the global* structure of any given linearly ordered set $(A, \leq)$.

Let us note first that the above general construction of local enrichment in section 3 does itself give a certain kind of global enlargement since the initial linearly ordered set $(A, \leq)$ is replaced with the larger linearly ordered set $(D, \leq)$ and one has the strictly increasing mapping (3.5), (3.6).
However, as seen next, there is also an alternative way for global enlargement of a linearly ordered set.

For that purpose, let $(B, \leq), (C, \leq)$ be two linearly ordered sets which for convenience are supposed to be disjoint. Then we define the linearly ordered set

(10.1) $(E, \leq)$

as the *global enlargement* of A, by taking

$$E = (\bigcup_{b \in B} (\{b\} \times A)) \cup A \cup (\bigcup_{c \in C} (\{c\} \times A)) = \qquad (10.2)$$
$$= (B \times A) \bigcup A \bigcup (C \times A)$$

while the linear order $e \leq e'$ is defined for $e, e' \in E$ according to the appropriate condition among the following seven

$$i)\ e \in \{b\} \times A,\ e' \in \{b'\} \times A,\ \ b \leq b',\ b \neq b'$$
$$ii)\ e = (b, a),\ e' = (b, a') \in \{b\} \times A,\ a \leq a'$$
$$iii)\ e \in \{b\} \times A,\ e' \in \cup A \cup (\bigcup_{c \in C} (\{c\} \times A))$$
$$(10.3) \qquad iv)\ e = a,\ e' = a' \in A,\ a \leq a'$$
$$v)\ e \in A,\ e' \in \bigcup_{c \in C} (\{c\} \times A)$$
$$vi)\ e \in \{c\} \times A,\ e' \in \{c'\} \times A,\ \ c \leq c',\ c \neq c'$$
$$vii)\ e = (c, a),\ e' = (c, a') \in \{c\} \times A,\ a \leq a'$$

We note a certain connection between the two constructions, namely, of local richness and global largeness. Indeed, in the particular case of section 4, we constructed the linearly ordered set

(10.4) $(\bar{A}, \leq) = (A, \leq) \square (B, \leq)$

in (4.7), (4.8), for any two linearly ordered sets $(A, \leq)$ and $(B, \leq)$. Here, $\bar{A}$ is the local enrichment of A. And it follows easily that

(10.5) $\bar{A} = ((A-) \times B) \bigcup A \bigcup ((A+) \times B)$

where $A- = \{a- \mid a \in A\}$ and $A+ = \{a+ \mid a \in A\}$.

On the other hand, the global enlargement of A in (10.1) - (10.3) is given by the linearly ordered set $(E, \leq)$, where

(10.6) $E = (B \times A) \bigcup A \bigcup (C \times A)$

Now, similar with section 4, we can particularize that construction as follows. Let $(F, \leq)$ be a linearly ordered set, and let us take $(B, \leq) = (F-, \leq)$ and $(C, \leq) = (F+, \leq)$, where the linear orders on $F-$ and $F+$ are the same with that on F. Then (10.6) gives

$$E = (\,(F-) \times A\,) \bigcup A \bigcup (\,(F+) \times A\,)$$

For convenience of notation, let us replace F with B, and thus we have

(10.7) $\quad E = (\,(B-) \times A\,) \bigcup A \bigcup (\,(B+) \times A\,)$

Let us now define the bijective mapping

(10.8) $\quad k : E \longrightarrow \bar{A}$

as follows

$$(b-, a) \longmapsto (a-, b)$$

(10.9) $\quad a \longmapsto a$

$$(b+, a) \longmapsto (a+, b)$$

Remark 10.1.

It is easy to see that the above mapping k is in general *not* an order isomorphism between $(E, \leq)$ and $(\bar{A}, \leq)$.

11. Looking through Telescopes ...

A simple interpretation of the construction of global largeness in section 10 is the following one. Both at the left and right ends of the linearly ordered set $(A, \leq)$ one respective telescope is placed. And each of these telescopes can see as many copies of $(A, \leq)$ stretching to the left and right of $(A, \leq)$, as many elements are in the linearly ordered sets $(B, \leq)$ and $(C, \leq)$, respectively.

12. Differential Calculus on the GSL

The developments in Appendix 2 below do obviously allow a Differential Calculus on any of the extensions in sections 3 and 10 above, as long as such extensions are vector spaces over a field.

Appendix 1 : Lexicographic Order and Commutative Group Structures

A1.1. Lexicographic Order
Let $(A, \leq)$ and $(B, \leq)$ be two linearly ordered sets. We recall that the *lexicographic order* $\preceq$ is defined on $A \times B$ by

(A1.1) $$(a, b) \preceq (c, d) \iff \left(\begin{array}{l} a \leq c,\ a \neq c \\ \text{or} \\ a = c,\ b \leq d \end{array} \right)$$

and it gives again a *linear order* on $A \times B$. Furthermore, for every given $b \in B$, the mapping

(A1.2) $$i : A \ni a \longmapsto (a, b) \in A \times B$$

is an *order isomorphic embedding*, that is, it is an order isomorphism between A and the image $i(A)$ of A in $A \times B$.

A1.2. Lexicographic Order and Commutative Group Structure

Let now $(A, +, \leq)$ and $(B, +, \leq)$ be two *linearly ordered commutative groups*. Namely, we assume that, for $a, a', a'' \in A$, the relations hold

(A1.3) $$a \leq a' \implies a + a'' \leq a' + a''$$

(A1.4) $$a \geq 0,\ a' \geq 0 \implies a + a' \geq 0$$

where $0 \in A$ is the neutral element in A, as well as

(A1.5) $$a,\ -a \geq 0 \implies a = 0$$

We note that (A1.3) implies (A1.4), (A1.5).
Indeed, if $a \geq 0,\ a' \geq 0$, then $0 \leq a,\ 0 \leq a'$, thus $0 \leq a' = 0 + a' \leq a + a'$.
Also, if $a, -a \geq 0$, then $0 \leq a$, hence $-a = 0 + (-a) \leq a + (-a) = 0$, thus $-a \leq 0$, which with the assumption $-a \geq 0$, gives $-a = 0$, and finally $a = 0$.

Further, assumptions similar to (A1.3) - (A1.5) are made about $(B, \leq)$.

Recalling that $A \times B$ has the *commutative group* structure given by

(A1.6) $$(a, b) + (c, d) = (a + b, c + d)$$

we obtain

Proposition A1.1.

Let $(A, +, \leq)$ and $(B, +, \leq)$ be two linearly ordered commutative groups. Then their product considered with the lexicographic order $(A \times B, +, \preceq)$ is again a linearly ordered commutative group, and the mapping

(A1.7) $$A \ni a \longmapsto (a, 0) \in A \times B$$

is a strictly increasing group homomorphism.

Proof.

As noted above, we only have to show that (A1.3) holds. Let therefore $(x,y),(u,v),(w,z) \in A \times B$. If $(x,y) \preceq (u,v)$, then

(A1.8) $\quad (x,y) + (w,z) \preceq (u,v) + (w,z)$

Here we have to consider the following situations.

Case 1. $x \leq u,\ x \neq u$. Then $x+w \leq u+w,\ x+w \neq u+w$, thus (A1.8) indeed holds.

Case 2. $x = u,\ y \leq v$. Then $x+w = u+w,\ y+z \leq v+z$, hence (A1.8) again holds.

A1.3 Lexicographic Order and Vector Space Structure

Let $(A,+,\leq)$ and $(B,+,\leq)$ be two linearly ordered commutative groups which are *vector spaces* over a field F. Then their group product $(A \times B, +)$ is naturally a vector space over F, according to the action

(A1.9) $\quad F \times (A \times B) \ni (c,(x,y)) \longmapsto (cx,cy) \in A \times B$

Proposition A1.2.

Let $(A,+,\leq)$ and $(B,+,\leq)$ be two linearly ordered commutative groups which are vector spaces over a linearly ordered field $(F,+,.,\leq)$. Then the lexicographically ordered group $(A \times B, +, ., \preceq)$ is a linearly ordered vector space over $(F,+,.,\leq)$.

Proof.

We only have to prove that, for $c \in F,\ (x,y) \in A \times B$, we have

(A1.10) $\quad c \geq 0,\ \ (x,y) \succeq (0,0) \implies (cx,cy) \succeq (0,0)$

Case 1. $c = 0$. Then $(cx,cy) = (0,0) \succeq (0,0)$.

Case 2. $c \geq 0,\ c \neq 0$. Then $x \geq 0,\ x \neq 0$ gives $cx \geq 0,\ cx \neq 0$, thus indeed $(cx,cy) \succeq (0,0)$.

If now $x = 0$, then $(x,y) \succeq (0,0)$ yields $y \geq 0$, thus $cy \geq 0$. But $cx = 0$, hence again $(cx,cy) = (0,cy) \succeq (0,0)$.

Appendix 2 : Differential Calculus on Vector Spaces

Let G, H be commutative groups which are also vector spaces over a field F.

We consider a set of sequences in F

(A2.1) $\mathcal{C}_F \subseteq F^{\mathbb{N}}$

which are called *convergent to* $0 \in F$, and which have the property to be a vector space over F.

Now we can define an operation of *limit* on certain sequences in F, as follows. Given $(c_n)_{n \in \mathbb{N}} \in F^{\mathbb{N}}$ and $c \in F$, then we have

(A2.2) $\lim_{n \to \infty} c_n = c$

if and only if

(A2.3) $(c - c_n)_{n \in \mathbb{N}} \in \mathcal{C}_F$

Similarly, we consider a set of sequences in G

(A2.4) $\mathcal{C}_G \subseteq G^{\mathbb{N}}$

which are called *convergent to* $0 \in G$, and which have the property to be a vector subspace over F.

Again, we can define an operation of *limit* on certain sequences in G, as follows. Given $(x_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}}$ and $x \in G$, then we have

(A2.5) $\lim_{n \to \infty} x_n = x$

if and only if

(A2.6) $(x - x_n)_{n \in \mathbb{N}} \in \mathcal{C}_G$

Finally, we do the same in H with a respective set of sequences

(A2.7) $\mathcal{C}_H \subseteq G^{\mathbb{N}}$

which also, are a vector subspace over F.

Now, for every given $v \in G,\ v \neq 0$, we can define a *derivation* D_v on certain functions $f : G \longrightarrow H$ as follows. Let

(A2.7) $\mathcal{D}_v \subseteq H^G$

be the set of all $f : G \longrightarrow H$, such that the limit

(A2.8) $(D_v f)(g) = \lim_{n \to \infty} (c_n)^{-1}(f(g + c_n v) - f(g)) \in H$

exists for all $g \in G$ and $(c_n)_{n \in \mathbb{N}} \in \mathcal{C}_F$, for which $c_n \neq 0$, with $n \in \mathbb{N}$, and for each $g \in G$ this limit is unique, being independent of the respective sequences $(c_n)_{n \in \mathbb{N}}$.

Proposition A2.1.

Given $v \in G,\ v \neq 0$, then $\mathcal{D}_v$ is a vector subspace in H^G and $D_v : \mathcal{D}_v \longrightarrow H^G$ is a linear mapping.

Proof.

Let $f, k \in \mathcal{D}_v$ and $(c_n)_{n \in \mathbb{N}} \in \mathcal{C}_F$, with $c_n \neq 0$, for $n \in \mathbb{N}$. Then for $g \in G, we have$

$$(D_v f)(g) = \lim_{n \to \infty} (c_n)^{-1}(f(g + c_n v) - f(g))$$

$$(D_v k)(g) = \lim_{n \to \infty} (c_n)^{-1}(k(g + c_n v) - k(g))$$

hence

$$(D_v f)(g) + (D_v k)(g) =$$

$$= \lim_{n \to \infty} (c_n)^{-1}(f(g + c_n v) - f(g)) +$$

$$+ \lim_{n \to \infty} (c_n)^{-1}(k(g + c_n v) - k(g)) =$$

$$= \lim_{n \to \infty} (c_n)^{-1}(\,(f(g + c_n v) - f(g)) + (k(g + c_n v) - k(g))\,) =$$

$$= \lim_{n \to \infty} (c_n)^{-1}(\,(f(g + c_n v) - k(g + c_n v)) + (f(g) - k(g))\,) =$$

$$= (D_v(f + k))(g)$$

Thus $f + k \in \mathcal{D}_v$ and $D_v f + D_v k = D_v(f + k)$.

Similarly, for $c \in F$, we have $cf \in \mathcal{D}_v$ and $D_v(cf) = cD_v f$.

Remark A2.1.

1) Typical further conditions required on the sequences in $\mathcal{C}_F$ are as follows, [1,16] :

(A2.9) $(c, c, c, \ldots) \in \mathcal{C}_F, \quad c \in F$

(A2.10) $\exists\, (c_n)_{n \in \mathbb{N}} \in \mathcal{C}_F \ : \ \forall\, n \in \mathbb{N} \ : \ c_n \neq 0$

Similar conditions may be required on the sequences in $\mathcal{C}_G$ or $\mathcal{C}_H$.

2) In case of certain groups G, H or fields F, it may be appropriate to replace the

convergent sequences with filters. This is a well known procedure, [1,16].

3) When the groups G, H or the fields F are linearly ordered, the respective concepts of convergence may be required to satisfy appropriate compatibility conditions.

4) When the groups G, H are not only vector spaces over the fields F but also algebras, then one can require the derivations D_v to satisfy the classical Leibniz rule of product derivative.

5) The above adaptations are to be treated elsewhere.

References

[1] Beattie R, Butzmann H P : Convergence Strutures and Applications to Functional Analysis. Kluwer, Dordrecht, 2002

[2] Conway J H : On Numbers and Games (second edition). Peters, Natick, 2001

[3] Keisler J : Elementary Calculus: An Approach Using Infinitesimals. On-line Edition.
http://www.math.wisc.edu/ keisler/calc.html

[4] Penrose R : The Road to Reality : A Complete Guide to the Laws of the Universe. Vintage Books, 2007

[5] Robinson A : Non-standard analysis (Revised edition edition of the 1966 first edition). Princeton University Press. 1996, ISBN 0-691-04490-2.

[6] Rosinger E E : From Reference Frame Relativity to Relativity of Mathematical Models : Relativity Formulas in a Variety of Non-Archimedean Setups. arXiv:physics/0701117

[7] Rosinger E E : Mathematics and "The Trouble with Physics", How Deep We Have to Go ? arXiv:0707.1163

[8] Rosinger E E : How Far Should the Principle of Relativity Go ? arXiv:0710.0226

[9] Rosinger E E : Two Essays on the Archimedean versus Non-Archimedean Debate. arXiv:0809.4509

[10] Rosinger E E : Heisenberg Uncertainty in Reduced Power Algebras. arXiv:0901.4825

[11] Rosinger E E : No-Cloning in Reduced Power Algebras.
arXiv:0902.0264

[12] Rosinger E E : Special Relativity in Reduced Power Algebras. arXiv:0903.0296

[13] Rosinger E E : Surprising Properties of Non-Archimedean Field Extensions of the Real Numbers. arXiv:0911.4824

[14] Rosinger E E : Four Departures in Mathematics and Physics. arXiv:1003.0360

[15] Rosinger E E : Two Theories of Special Relativity ? arXiv:1006.2994

[16] Rosinger E E, Van der Walt J-H : Beyond Topologies, Part I. arxiv:1001.1866

[17] Sergeyev Ya D : Arithmetic of Infinity. Edizioni Orizzonti Meridionali, CS, 2003

[18] Sergeyev Ya D : http://www.theinfinitycomputer.com, 2004

[19] Sergeyev Ya D : A new applied approach for executing computations with infinite and infinitesimal quantities. Informatica, 19(4):567ï£¡596, 2008

[20] Sergeyev Ya D : Numerical computations and mathematical modelling with infinite and infinitesimal numbers. Journal of Applied Mathematics and Computing, 29:177ï£¡195, 2009

[21] Sergeyev Ya D : Numerical point of view on Calculus for functions assuming finite, infinite, and infinitesimal values over finite, infinite, and infinitesimal domains. Nonlinear Analysis Series A: Theory, Methods & Applications, 71(12):e1688ï£¡e1707, 2009

[22] Sergeyev Ya D : Counting systems and the First Hilbert problem. Nonlinear Analysis Series A: Theory, Methods & Applications, 72(3-4):1701ï£¡1708, 2010

[23] Sergeyev Ya D : Lagrange lecture: Methodology of numerical computations with infinities and infinitesimals. Rendiconti del Seminario Matematico dellï£¡Universit‘a e del Politecnico di Torino, 68(2):95ï£¡113, 2010

[24] Sergeyev Ya D : Higher order numerical differentiation on the Infinity Computer. Optimization Letters, page (to appear), 2011

[25] Sergeyev Ya D : On accuracy of mathematical languages used to deal with the Riemann zeta function and the Dirichlet eta function (to appear)

[26] Steen L A, Seebach J A : Conterexamples in Topology. Dover, New York, 1995