Article

Power Narayana Sequences

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Abstract

In this paper, firstly, we investigated the periods of power Jacobsthal sequences and formulated the periods of power Jacobsthal sequences, based on the period of Jacobsthal sequence modulo u. Then, we descibed new infinite integer sequences providing the recurrence relation of Narayana sequence that also be power sequence for some modulus u. We investigated those modulus u for which these power sequences exist. And, we formulated the periods of these power sequences for some prime integers p, based on the period of Narayana sequence modulo p. Finally, we compared that periodic properties of Narayana, Fibonacci, Jacobsthal sequences and the power sequences of these sequences.

Keywords: Fibonacci sequence, Power Fibonacci sequence, Jacobsthal sequence, Narayana sequence, period.

1. Introduction

The Fibonacci sequence, $\{F_k\}_0^\infty$, is a sequence of numbers, beginning with the integers 0 and 1, in which the value of any element is computed by taking the summation of the two antecedent numbers. If so, for $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$ [1]. This number sequence, which was previously found by Indian mathematicians in the sixth century. But the sequence was introduced by Fibonacci as a result of calculating the problem related to the reproduction of rabbits in the book named Liber Abaci in 1202. The first seven terms of this sequence are $1, 1, 2, 3, 5, 8, 13$.

The Jacobsthal sequence $\{J_k\}_0^\infty$, is a sequence of numbers, beginning with the integers 1 and 1, in which the value of any element is computed by taking the summation of the previous term and twice of the term of two places before. If so, for $k \geq 2$, $J_k = J_{k-1} + 2J_{k-2}$. This number sequence, which was previously found by Horadam in 1988 [2]. The first seven terms of this sequence are $1, 1, 3, 5, 11, 21, 43$.

The Narayana sequence, $\{N_k\}_0^\infty$, is a sequence of numbers, beginning with the integers $1, 1$ and 1, in which the value of any element is computed by taking the summation of the previous term and term two places before it. If so, for $k \geq 3$, $N_{k+1} = N_k + N_{k-2}$ [1]. This number sequence derived from the was introduced by Narayana as the result of calculating the problem related to the birth cows in the book called Gaṇita Kaumudi in 1356. The first seven terms of this sequence are $1, 1, 1, 2, 3, 4, 6$.

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In 2012, authors, in their worh, described a new bi-infinite integer sequence modulo u and obtained the followings:

Definition 1.1. Let G be a bi-infinite integer sequence providing the recurrence relation $G_k = G_{k-1} + G_{k-2}$. Providing $G \equiv 1, \alpha, \alpha^2, \alpha^3, \dots (mod\, u)$ for some modulus u, then G is named a power Fibonacci sequence modulo u [3].
Example 1.2. For modulo $u = 29$, the two power Fibonacci sequences are following:
1, 6, 7, 13, 20, 4, 24, 28, 23, 22, 16, 9, 25, 5, 1, 6, 7, . . and 1, 24, 25, 20, 16, 7, 23, 1, 24, . . .
Theorem 1.3. There is precisely one power Fibonacci sequence modulo 5. For $u \neq 5$, there exist power Fibonacci sequences modulo u certainly when u has prime factorization $u = {p_1}^{e_1}{p_2}^{e_2} \dots {p_k}^{e_k}$ or $u = 5{p_1}^{e_1}{p_2}^{e_2} \dots {p_k}^{e_k}$, where each $p_i \equiv \mp 1 (mod\, 10)$; in either case there are definitely 2^k power Fibonacci sequences modulo u [3].
In this study, firstly, we aimed to find the periods of power Jacobsthal sequences and formulate the periods of power Jacobsthal sequences, based on the period of Jacobsthal sequence modulo u. Then, we described a new power sequences modulo u. We named it power Narayana sequences modulo u. We investigated those modulus u for which these power sequences exist and numbers of these sequences for given u. Then, we formulated the periods of these power sequences for some positive integers u, based on the period of Narayana sequence modulo u with a table. Finally, we compared periodic relations of Narayana, Fibonacci and Jacobsthal sequences.

2. Preliminaries

In this study, we used power Fibonacci and power Jacobsthal sequences modulo u and the periods of these sequences. These structures we used in this article are introduced in this section.

2.1. The Periods of Power Fibonacci Sequences

We know that $\pi(u)$ denote the period of the Fibonacci sequence modulo u and there is no known explicit formula for $\pi(u)$. But, providing $(u, v) = 1$ then $\pi(uv) = [\pi(u), \pi(v)]$ [4]. It is easily seen that, if S is any periodic sequence mod uv and $(u, v) = 1$, then its period is the least common multiple of the period of S taken mod u and the period of S taken mod v. For $u > 2$, $\pi(u)$ is even [4], [5].

In addition, we know that Ide and Renault establishes a relationship between $\pi(u)$ and the period of power Fibonacci sequences modulo u. And, they obtained following theorems:

Theorem 2.1.1. Let p be a prime of the form $p \equiv \mp 1 (mod\, 10)$ and let θ and σ be two roots of $t(x) = x^2 - x - 1 (mod\, p^e)$. Suppose $|\theta| \geq |\sigma|$.

- **i.** If $\pi(p^e) \equiv 0 (mod\, 4)$, then $|\theta| = |\sigma| = \pi(p^e)$.
- **ii.** If $\pi(p^e) \equiv 2 (mod\, 4)$, then $|\theta| = 2|\sigma| = \pi(p^e)$ [3].

Theorem 2.1.2. Let $u = {p_1}^{e_1}{p_2}^{e_2} \dots {p_k}^{e_k}$ be the product of the primes p_i, $p_i \equiv \mp 1 (mod\, 10)$.

i. If $\pi(u) \equiv 0(mod\ 4)$, then modulo u, each power Fibonacci sequence has period $\pi(u)$.
ii. If $\pi(u) \equiv 2(mod\ 4)$, then modulo u, one power Fibonacci sequence has (odd) period $\frac{\pi(u)}{2}$ and the others have period $\pi(u)$.
iii. If $\pi(u) \equiv 0(mod\ 4)$, then modulo $5u$, each power Fibonacci sequence has period $\pi(u)$.
iv. If $\pi(u) \equiv 2(mod\ 4)$, then modulo $5u$, each power Fibonacci sequence has period $2\pi(u)$ [3].

In 2020, Nalli and Ozyilmaz described a new bi-infinite integer sequence modulo u and obtained the followings:
Definition 2.1.3. Let J^* be a bi-infinite integer sequence providing the recurrence relation ${J^*}_k = {J^*}_{k-1} + 2{J^*}_{k-2}$. Providing $J^* \equiv 1, \alpha, \alpha^2, \alpha^3, \ldots (mod\ u)$ for some modulus u, at that rate J^* is named a power Jacobsthal sequence modulo u [6].

Example 2.1.4. For modulo $u = 23$, the two power Jacobsthal sequences are following:
1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12, 1, 2, 4, 8, ... and 1, 22, 1, 22, 1, ...

Theorem 2.1.5. For $u \geq 3$, there precisely exist power Jacobsthal sequences modulo u. In addition, for the number of the power Jacobsthal sequences modulo u, there is exactly one power Jacobsthal sequence modulo 3 and for $u \neq 3$, when u has prime factorization $u = 3^e {p_1}^{e_1} {p_2}^{e_2} \ldots {p_k}^{e_k}$, there are definitely 2^k power Jacobsthal sequences for $e = 0,1$, 3.2^k power Jacobsthal sequences for $e = 2$, 6.2^k power Jacobsthal sequences modulo u for $e > 2$ [6].
In addition, in [7], Yazlik, Yilmaz, Taskara and Uslu mainly focused on the lengths of period of Jacobsthal and Jacobsthal Lucas sequences modulo u and obtained a formula to find the lengths of period taking modulo u for some positive integer u. They determined the relations among lengths of period for these numbers.

3. Main Results

3.1.The Periods of Power Jacobsthal Sequences

In this section, we investigated the periods of power Jacobsthal sequences, based on the period of Jacobsthal sequence modulo u. We obtained the list of periods of these sequences for which they exist for first 25 prime numbers and expressed these values with a table. Let $L(J, u)$ and $L(J^*, u)$ denote the length of the period of Jacobsthal sequence modulo u and power Jacobsthal sequences modulo u, respectively.

Table 1. The formula of $L(J^*, p)$ based on $L(J, p)$.

p	$L(J, p)$	$L(J^*, p)$	The formula of $L(J^*, p)$ based on $L(J, p)$
3	6	2	$\frac{L(J,p)}{3}$

5	4	4 and 2	$L(J,p)$
7	6	3 and 2	$\frac{L(J,p)}{2}$
11	10	10 and 2	$L(J,p)$
13	12	12 and 2	$L(J,p)$
17	8	8 and 2	$L(J,p)$
19	18	18 and 2	$L(J,p)$
23	22	11 and 2	$\frac{L(J,p)}{2}$
29	28	28 and 2	$L(J,p)$
31	10	5 and 2	$\frac{L(J,p)}{2}$
37	36	36 and 2	$L(J,p)$
41	20	20 and 2	$L(J,p)$
43	14	14 and 2	$L(J,p)$
47	46	23 and 2	$\frac{L(J,p)}{2}$
53	52	52 and 2	$L(J,p)$
59	58	58 and 2	$L(J,p)$
61	60	60 and 2	$L(J,p)$
67	66	66 and 2	$L(J,p)$
71	70	35 and 2	$\frac{L(J,p)}{2}$
73	18	9 and 2	$\frac{L(J,p)}{2}$
79	78	39 and 2	$\frac{L(J,p)}{2}$
83	82	82 and 2	$L(J,p)$

89	22	11 and 2	$\frac{L(J,p)}{2}$
97	48	24 and 2	$\frac{L(J,p)}{2}$
10 1	100	100 and 2	$L(J,p)$

Corollary 3.1.1. Let $p, p > 3$ be a prime number and it is obvious that 2 and -1 are two roots of $j(x) = x^2 - x - 2(mod\, p^e)$. There are exactly two roots of $j(x)$.

i. It is easily seen that $|-1| = 2$. So, the period of one of the power Jacobsthal sequences is always 2.

ii. For the period of the other power Jacobsthal sequence there are two cases:$|2| = L(J^*, p^e) = \frac{L(J,p^e)}{2}$ or $|2| = L(J^*, p^e) = L(J, p^e)$.

iii. If $p = 3$, there are exactly one power Jacobsthal sequence. And, the period of this power Jacobsthal sequence is $|2| = L(J^*, 3) = \frac{L(J,3)}{3}$. For $p = 3$, similarly, the period of one of the power Jacobsthal sequences is always 2 and the period of the other power Jacobsthal sequence is $|2| = L(J^*, p^e) = \frac{L(J,p^e)}{3}$.

Corollary 3.1.2. Let $u = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_k}^{e_k}$, ($p_i > 3$ a prime number) for every i, $1 \leq i \leq k$.

i. It is easily seen that$|-1| = 2$. So, the period of one of the power Jacobsthal sequences is always 2.

ii. The period of one of the power Jacobsthal sequences(the order of 2) is $\frac{L(J,u)}{2}$ or $L(J, u)$.

iii. If $u = 3^e{p_1}^{e_1}{p_2}^{e_2} \ldots {p_k}^{e_k}$, for the period of one of the power Jacobsthal sequences (the order of 2) there are two cases: $|2| = L(J^*, u) = \frac{L(J,u)}{3}$ or $|2| = L(J^*, u) = L(J, u)$.

3.2. Some Relations about the periods of Narayana Sequence

We know that reducing Narayana sequences by a modulo p, it is obtained a repeating sequence and so, it is easily seen that Narayana sequences are periodic. K. Kirthi studied the period of Narayana series modulo p, where p is a prime number, and he founded to be either $p^2 + p + 1$ (or a divisor) or $p^2 - 1$ (or a divisor). And he obtained the list of periods for first 50 prime numbers and he expressed these values with tables [8].

In this section, we determined some relations among the lengths of period for Narayana sequences modulo u. Let $L(N, u)$ and $L(N^*, u)$ denote the length of the period of Narayana sequence modulo u and power Narayana sequences modulo u, respectively.

Lemma 3.2.1. The following properties are provided for the length of the period of Narayana sequence modulo u.

i. $L(N, p^i) = L(N, p)p^{i-1}$ for p is a prime number.

ii. $L(N, uv) = [L(N, u), L(N, u)]$ for $(u, v) = 1$.

3.3. Power Narayana Sequences

Definition 3.3.1. Let N^*, be a bi-infinite integer sequence providing the recurrence relation ${N^*}_k = {N^*}_{k-1} + {N^*}_{k-3}$. Providing $N^* \equiv 1, \omega, \omega^2, \omega^3, \ldots (mod\, u)$ for some modulus u, at that rate N^* is named a power Narayana sequence modulo u.

Example 3.3.2. Modulo $u = 11$, there is exactly one power Narayana sequence. This sequence is 1,5, 3, 4, 9, 1, 5, … .

In this study, we investigated those modulus p for which power Narayana sequences exist and the number of such sequences for a given p. When we examined the first fourty prime numbers, we obtained that power Narayana sequences exist for modulus 3, 11, 13, 17, 23, 29, 31, 37, 43, 47, 53, 61, 67, 73, 79, 83, 89, 127, 131, 137, 139, 149, 151, 167, 173 while these sequences don't exist for modulus 2, 5, 7, 19, 41, 59,71, 97, 101, 103, 107, 109, 113, 157, 163. Here, those prime modulus p for which power Narayana sequences exist couldn't be formulated.

In addition, while there is only one power Narayana sequence for modulus 3, 11, 13, 17, 23, 29, 37, 43, 47, 53, 61, 73, 79, 83, 89, 127, 131, 137, 139, 151, 167, there are two power Narayana sequences for the only modulo 31, there are three power Narayana sequences for modulo 67, 149, 173. And so, it is seen that the number of these sequences doesn't increase according to a certain rule.

3.4.The Periods of Power Narayana Sequences

In this section, firstly, we obtained the list of periods of these sequences for which these sequences exist for first 25 prime numbers and expressed these values with a table. And we found some results from the table.

Table 2. The formula of $L(N^*, p)$ based on $L(N, p)$.

p	$L(N, p)$	$L(N^*, p)$	The formula of $L(N^*, p)$ based on $L(N, p)$
3	8	2	$\dfrac{L(N,p)}{p+1}$
11	60	5	$\dfrac{L(N,p)}{p+1}$
13	168	12	$\dfrac{L(N,p)}{p+1}$
17	288	16	$\dfrac{L(N,p)}{p+1}$

23	528	22	$\frac{L(N,p)}{p+1}$
29	840	28	$\frac{L(N,p)}{p+1}$
31	930	30 or 15	$\frac{L(N,p)}{p}$ or $\frac{L(N,p)}{2p}$
37	342	9	$\frac{L(N,p)}{p+1}$
43	1848	42	$\frac{L(N,p)}{p+1}$
47	46	23	$L(N,p) < p$
53	468	26	$\frac{3L(N,p)}{p+1}$
61	1240	20	$\frac{L(N,p)}{p+1}$
67	33	11 or 33	$L(N,p) < p$
73	2664	36	$\frac{L(N,p)}{p+1}$
79	6240	78	$\frac{L(N,p)}{p+1}$
83	3444	41	$\frac{L(N,p)}{p+1}$
89	7920	88	$\frac{L(N,p)}{p+1}$
127	2016	63	$\frac{4L(N,p)}{p+1}$
131	130	130 or 13	$L(N,p) < p$
137	6256	136	$\frac{3L(N,p)}{p+1}$

13 9	1610	23	$\frac{2L(N,p)}{p+1}$
14 9	148	74	$L(N,p) < p$
15 1	22800	150	$\frac{L(N,p)}{p+1}$
16 7	4648	83	$\frac{3L(N,p)}{p+1}$
17 3	172	172 or 86	$L(N,p) < p$

From the tables, we obtained

- Power Narayana sequence modulo p establishes a cyclic group.
- If $L(N,p)$ is even, then there is at least one power Narayana sequence modulo p.
- If $p \neq 31$ and $L(N,p) > p$, then $L(N^*,p) = \frac{L(N,p)}{p+1}$ (or a multiplier).
- If $p = 31$, then $L(N,p) > p$ and $L(N,p)$ is even. There are two power Narayana sequences modulo p. In addition, different from the others, $L(N^*,p) = \frac{L(N,p)}{p}$ and $L(N^*,p) = \frac{L(N,p)}{2p}$ for these sequences.
- If $L(N,p) < p$, then $L(N^*,p) = L(N,p)$ (or a divisor).
- If $L(N,p) < p$, then there are multiple power Narayana sequences modulo p.
- If $p = 67$, $L(N,p)$ is odd and there are three power Narayana sequences modulo p.
- If $p \neq 67$ and $L(N,p)$ is even and there are has one or two power Narayana sequence modulo p.
- If $L(N,p) \equiv 0(mod\ 4)$ and $L(N,p) > p$, then $L(N^*,p) = \frac{L(N,p)}{p+1}$ (or a multiplier).

Corollary 3.4.1. If $L(N,p) > p, L(N^*,p) = \frac{L(N,p)}{p+1}$, (or a multiplier) or $L(N^*,p) = \frac{L(N,p)}{p}$ (or a divisor) for p is a prime number.

3.5. Comparing of Periodic Properties of Narayana, Fibonacci and Jacobsthal sequences

- We know that there is no known explicit formula for the period of Fibonacci sequence modulo u. Similarly, there is no known explicit formula for the periods of Jacobsthal and Narayana sequences, too.

- The following properties are provided for the periods of Fibonacci and Narayana sequences modulo u.

i. $L(N,p^i)=L(N,p)p^{i-1}$ for p is a prime number.
ii. $L(N,uv)=[L(N,u),L(N,u)]$ for $(u,v)=1$.

But for the periods of Jacobsthal sequences modulo p, $L(N,p^i)=L(N,p)p^{i-1}$ for p is a *odd* prime number.

- In addition, for power Fibonacci sequence, $L(G,p)$ is equal to $L(F,p)$ or $\frac{L(F,p)}{2}$; for power Jacobsthal sequence, $L(J^*,p)$ is equal to $L(J,p)$ or $\frac{L(J,p)}{2}$; for power Narayana sequence, $L(N^*,p)$ is equal to $\frac{L(N,p)}{p+1}$ (or a multiplier) or $\frac{L(N,p)}{p}$ (or a divisor).

Let $L(F,u)$ and $L(G,u)$ denote the length of the period of Fibonacci sequence modulo u and power Fibonacci sequences modulo u, respectively. We found the first ten prime modulus for which power Fibonacci, power Jacobsthal and power Narayana sequences exist as follows:

Table 3. Formulas of the periods of power Fibonacci, power Jacobsthal and power Narayana sequences based on Fibonacci, Jacobsthal and Narayana sequences for the first ten prime modulus.

p	$L(N,p)$ - $L(N^*,p)$	$L(N^*,p)$	$L(J,p)$ - $L(J^*,p)$	$L(J^*,p)$	$L(G,p)$ - $L(F,p)$	$L(G,p)$
11	60-5	$\frac{L(N,p)}{p+1}$	10-10 and 2	$L(J,p)$	10-10 and 5	$L(F,p)$ and $\frac{L(F,p)}{2}$
29	840-28	$\frac{L(N,p)}{p+1}$	28-28 and 2	$L(J,p)$	14-14 and 7	$L(F,p)$ and $\frac{L(F,p)}{2}$
31	930-30 and 15	$\frac{L(N,p)}{p}$ or $\frac{L(N,p)}{2p}$	10-5 and 2	$\frac{L(J,p)}{2}$	30-30 and 15	$L(F,p)$ and $\frac{L(F,p)}{2}$
61	1240-20	$\frac{L(N,p)}{p+1}$	60-60 and 2	$L(J,p)$	60-60	$L(F,p)$

79	6240-78	$\frac{L(N,p)}{p+1}$	78-39 and 2	$\frac{L(J,p)}{2}$	78-78 and 39	$L(F,p)$ and $\frac{L(F,p)}{2}$
89	7920-88	$\frac{L(N,p)}{p+1}$	22-11 and 2	$\frac{L(J,p)}{2}$	44-44	$L(F,p)$
131	130-130	$L(N,p) < p$	130-130 and 2	$L(J,p)$	130-130 and 65	$L(F,p)$ and $\frac{L(F,p)}{2}$
139	1610-23	$\frac{2L(N,p)}{p+1}$	138-138 and 2	$L(J,p)$	46- 46 and 23	$L(F,p)$ $\frac{L(F,p)}{2}$
149	148-74	$L(N,p) < p$	148-148 and 2	$L(J,p)$	148-148	$L(F,p)$
151	22800-150	$\frac{L(N,p)}{p+1}$	30-15 and 2	$\frac{L(J,p)}{2}$	50- 50 and 25	$L(F,p)$ and $\frac{L(F,p)}{2}$

4. Conclusion

In this study, firstly, we investigated the periods of power Jacobsthal sequences and formulated the periods of power Jacobsthal sequences, based on the period of Jacobsthal sequence modulo u. Then, we examined power Fibonacci sequence modulo u and Narayana sequence. And, we described power Narayana sequence modulo u. We investigated those modulus u for which these power sequences exist and the number of such sequences for a given u. When we examined the first fourty prime numbers, we obtained that power Narayana sequences exist for modulus 3, 11, 13, 17, 23, 29, 31, 37, 43, 47, 53, 61, 67, 73, 79, 83, 89, 127, 131, 137, 139, 149, 151, 167, 173 while these sequences don't exist for modulus 2, 5, 7, 19, 41, 59, 71, 97, 101, 103, 107, 109, 113, 157, 163. Here, those prime modulus u for which power Narayana sequences exist couldn't be formulated.

Here, we also aimed to formulate the periods of power Narayana sequences for some prime integers p, based on the period of Narayana sequence modulo p. And, we obtained the list of periods of power Narayana sequences for which these sequences exist for first 25 prime numbers and expressed these values in terms of the period of Narayana sequence modulo p with a table. Finally, we compared that periodic properties of Narayana, Fibonacci, Jacobsthal sequences and the power sequences of these sequences.

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