Article

Some Results on the Klein Quadric Representation for a General Quantum Spacetime

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Abstract

The Klein representation is a general quantum space-time. The aggregate consisting of all $(n-1)$-planes in the complex projective space P^{2n-1} forms a manifold of dimension n^2. This space is complexified, compactified version of the quantum space-time.

The quadric of $PG(5,q)$, over its prime field $GF(p)$ (the integers modulo p).defined by equation,

$$X_0X_3 + X_1X_4 + X_2X_5 = 0$$

is called the Klein Quadric.

In this paper, we give an embedding of Alpha and Beta Planes of the Klein Quadric.

Key Words: Klein mapping, embedding, Alpha and Beta Planes
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1 Introduction and Preliminaries

The space P^{2n-1} of projective hypertwistors is a good starting point for the analysis of the conformal geometry of complex quantum space-time, which can be regarded as the Grassmannian variety of projective $(n-1)$-planes in P^{2n-1}. More precisely, the aggregate of all projective $(n-1)$-planes in P^{2n-1}constitutes a compact manifold of dimension n^2, which we identify as the complexified, compactified quantum space-time. The finite points of compactified quantum space-time correspond to those $(n-1)$-planes of P^{2n-1}that are determined by a linear relation of the form

$$w^A = ix^{AA'}\pi_{A'}$$

for some fixed $x^{AA'}$, see [6].

We now give the $n-$dimensional projective space $PG(n,K)$ for $n > 0$ and K, any (skew) field. Let V be any vector space of dimension $(n+1)$ over K. Then $PG(n,K)$, the $n-$dimensional projective space over K, is the set of all subspaces of V distinct from the trivial subspaces $\{0\}$ and V. The $1-$dimensional subspaces are called the points of $PG(n,K)$, the $2-$dimensional subspaces are called the (projective) lines and the $3-$dimensional ones are called (projective) planes. We remark that by going from a vector space to the associated projective space, the dimension drops by one unit. Hence an $(n+1)-$dimensional vector space gives rise to an $n-$dimensional projective space $[3-9]$.

Let $PG(3,q)$ be a 3-dimensional projective space over a field $GF(q)$ where q is prime, such that the points and planes are represented by homogeneous coordinates. The method is due to Julius Plücker (1801-1868). The homogeneity of Plücker coordinates suggest to view the the coordinates of a line as homogenous coordinates of points in five dimensional space $PG(5,q)$, [8]. This is a particular case of construction of the Grassmannian of lines. Homogenous coordinates in $PG(5,q)$ are written as in the form $(l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12})$ where

$$l_{ij} = x_iy_j - x_jy_i.$$

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We denote $(X_0, X_1, X_2, X_3, X_4, X_5) = (l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12})$.

(See 10)The Klein mapping,

$$\gamma : \mathcal{L} \to PG(5, q)$$

assigns to a line of L of $P(3, q)$ the point $(l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12})$ of $PG(5, q)$ where $(l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12})$ are the line's Plücker coordinates (F.Klein, 1868).

(See 6) The lines of projective three-space are, via the Klein mapping, in one-to-one correspondence with points of a hyperbolic quadric of the projective $5-$space.

The quadric of $PG(5, q)$ defined by equation,

$$X_0X_3 + X_1X_4 + X_2X_5 = 0$$

is called the Klein Quadric and is denoted by the symbol $\mathcal{H}^5$.

The Klein Quadric contains two families of 2-planes. $\alpha-$planes are lines through a point in $PG(3, q)$ and $\beta-$planes are lines in a 2-plane in $PG(3, q)$ [8].Two different $\alpha-$planes meet in a single point. This point represents the line joining the two points in $PG(3, q)$. Similarly two different $\beta-$planes meet in a point, while this time the point represents the line in $PG(3, q)$ that is the intersection of the 2-planes determined by the $\beta-$planes. In general an $\alpha-$plane and $\beta-$plane do not meet. If an $\alpha-$plane and $\beta-$plane meet then they meet in a line. In $PG(3, q)$ this line in the $\mathcal{H}^5$ corresponds to the set of lines in a plane passing through a point. The Klein quadric, being a hyperbolic quadric in five dimensions, contains points, lines and planes (but no 3-spaces). The planes can be partitioned into two classes (called Greek and Latin) using the relation:

$$\pi_1 \sim \pi_2 \Leftrightarrow \pi_1 = \pi_2 \text{ or } \pi_1 \cap \pi_2 \text{ is a point}$$

The set of lines through a given a point p in $PG(3, q)$ defines a Latin plane. The set of lines contained in a given plane $\mathcal{P}$ in $PG(3, q)$ defines a Greek plane [5-6].Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two equivalence classes of planes of $\mathcal{H}^5$. We define a geometry S as follows.

The points of S are the planes of $\mathcal{P}_1$;

The lines of S are the points of $\mathcal{H}^5$;

The planes of S are the planes of $\mathcal{P}_2$,

The incidence between a line of S and a point or a plane of S is induced by the incidence of $PG(5, q)$; a point π_1 of S and a plane π_2 of S are incident if the planes π_1 and π_2 of $PG(5, q)$ are not disjoint. (They intersect each other in a line of $PG(5, q)$.) The geometry S is a 3-dimensional projective space; more precisely, S is isomorphic to a 3-dimensional subspace of the projective space $PG(5, q)$ [10].

The space of lines in FP^3 is represented as a projective quadric, known as the Klein quadric $\mathcal{K}$ in FP^5, with projective coordinates $(P_{01} : P_{02} : P_{03} : P_{23} : P_{31} : P_{12})$, known as Plücker coordinates.

The line through two points $(q_0 : q_1 : q_2 : q_3)$ and $(u_0 : u_1 : u_2 : u_3)$ in FP^3 has Plücker coordinates, defined as follows [5]:

$$P_{ij} = q_iu_j - q_ju_i.$$

Hence, for a line in F^3, obtained by setting $q_0 = u_0 = 1$, the Plücker coordinates acquire the meaning of a projective pair of three-vectors $(w : v)$, where w is a vector in the direction of the line and for any point $q = (q_1, q_2, q_3)$ on the line,$v = q \times w$ is the line's moment vector, with respect to some fixed origin. We use the boldface notation for three-vectors throughout. Conversely, one can denote $w = (P_{01}, P_{02}, P_{03})$, $v = (P_{23}, P_{31}, P_{12})$, the Plücker coordinates then become $(w : v)$, and treat w and v as vectors in F^3, bearing in mind that, in fact, as a pair they are projective quantities. The lines in the plane at infinity in FP^3 are represented by Plücker vectors $(0 : v)$. The equation of the Klein quadric $\mathcal{K}$ in FP^5 is

$$P_{01}P_{23} + P_{02}P_{31} + P_{03}P_{12} = 0$$

The Klein quadric contains a three-dimensional family of projective two-planes, called α-planes. Elements of a single -plane are lines, concurrent at some point $(q_0 : q_1 : q_2 : q_3) \in FP^3$. If the concurrency point is

$(1:q)$, which is identified with $q \in F^3$, the plane is a graph $v = q \times\ w$. Otherwise, an ideal concurrency point $(0:w)$ gets identified with some fixed w, viewed as a projective vector. The corresponding α-plane is the union of the set of parallel lines in F^3 in the direction of w, with Plücker coordinates $(w:v)$, so $v\cdot\ w = 0$, and the set of lines in the plane at infinity incident to the ideal point $(0:w)$. The latter lines have Plücker coordinates $(0:v)$, with once again $v\cdot\ w = 0$. Similarly, the Klein quadric contains another three-dimensional family of two-planes, called β- planes, which represent co-planar lines in FP^3. A "generic" β-plane is a graph $w = u \times\ v$, for some $u \in F^3$. The case $u = 0$ corresponds to the plane at infinity, otherwise the equation of the co-planarity plane in F^3 becomes $u\cdot\ q = -1$, [4].

We use maps $\theta : PG(3,2) \to PG(5,2)$, and $\theta : PG(5,2) \to PG(19,2)$, such that θ maps the set of points and planes of $PG(3,2)$ to the set of planes of $PG(5,2)$ and the set of lines of $PG(3,2)$ to the set of points of $PG(5,F)$. These embeddings give Klein embedding $PG(3,2)$ into $PG(19,2)$.

2 Klein Mapping of PG(3,2)

Let $PG(3,2)$ be a 3-dimensional projective space over a field $GF(2)$ such that the points and planes are represented by homogeneous coordinates. $PG(3,2)$ has 15 points, 35 lines and 15 planes. We will give the full list as supplementary material.

3 Embedding of α and β Planes of Kelin Quadric

The points of the projective space $PG(3,2)$ is as follows:

$$\begin{aligned}
N_1 &= (0,0,0,1), N_6 = (0,1,1,0), N_{11} = (1,0,1,1)\\
N_2 &= (0,1,0,0), N_7 = (0,1,1,1), N_{12} = (1,1,0,0)\\
N_3 &= (0,1,0,1), N_8 = (1,1,1,0), N_{13} = (1,1,0,1)\\
N_4 &= (0,0,1,0), N_9 = (1,1,1,1), N_{14} = (1,0,0,0)\\
N_5 &= (0,0,1,1), N_{10} = (1,0,1,0), N_{15} = (1,0,0,1)
\end{aligned}$$

Consider the points $P = (x_0, x_1, x_2, x_3, x_4, x_5)$, $Q = (y_0, y_1, y_2, y_3, y_4, y_5)$, $R = (z_0, z_1, z_2, z_3, z_4, z_5)$ spanning the planes of Kelin Quadric. For the points P, Q, R, matrix $\mathcal{M}$ is defined as follows:

$$\mathcal{M} = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \\ z_0 & z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix}.$$

We define determinant P_0:

$$P_0 = p_{012} \begin{vmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & {\scriptstyle y2} & {\scriptstyle z2} \end{vmatrix}$$

and $P_1 = p_{013}$, $P_2 = p_{014}$, $P_3 = p_{015}$, $P_4 = p_{023}$, $P_5 = p_{024}$, $P_6 = p_{025}$, $P_7 = p_{034}$, $P_8 = p_{035}$, $P_9 = p_{045}$, $P_{10} = p_{123}$, $P_{11} = p_{124}$, $P_{12} = p_{125}$, $P_{13} = p_{134}$, $P_{14} = p_{135}$, $P_{15} = p_{145}$, $P_{1_{234},P6} = p_{17} = p_{235}$, $P_{18} = p_{245}$, $P_{19} = p_{345}$ cyclically.

$(P_0, P_1, P_2, ..., P_{19})$ is Grasmanian coordinate the plane of a projective space of $PG(5,2)$. It is a point of the projective space $PG(19,2)$.

Grassmanian coordinates of $\alpha-$planes of Klein Quadric are 15-cap in the projective space $PG(19,2)$.

Let the $\alpha-$plane be the image of the point $N_1(0,0,0,1)$ in the projective space $PG(3,2)$. This plane is spanned by points $(0,0,0,1,0,0)$, $(0,0,0,0,1,0)$ and $(0,0,1,0,0,0)$. It has Grassmann coordinate:

$$N_1(0,0,0,1) \to P_{\alpha 1} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0)$$

Similarly other planes have following Grassman coordinates:

$$
\begin{array}{rcl}
N_2(0,1,0,0) & \rightarrow & P_{\alpha 2} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0) \\
N_3(0,1,0,1) & \rightarrow & P_{\alpha 3} = (0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,1,1,0,0) \\
N_4(0,0,1,0) & \rightarrow & P_{\alpha 4} = (0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0) \\
N_5(0,0,1,1) & \rightarrow & P_{\alpha 5} = (0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,1,0,1,0) \\
N_6(0,1,1,0) & \rightarrow & P_{\alpha 6} = (0,0,0,0,0,0,0,0,1,1,0,0,0,0,1,1,0,0,0,0) \\
N_7(0,1,1,1) & \rightarrow & P_{\alpha 7} = (0,0,0,0,0,0,0,1,1,1,0,0,0,1,1,1,1,1,1,0) \\
N_8(1,1,1,0) & \rightarrow & P_{\alpha 8} = (1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
N_9(1,1,1,1) & \rightarrow & P_{\alpha 9} = (1,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0) \\
N_{10}(1,0,1,0) & \rightarrow & P_{\alpha 10} = (1,1,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0) \\
N_{11}(1,0,1,1) & \rightarrow & P_{\alpha 11} = (1,1,0,0,1,0,0,0,0,0,1,1,1,1,0,1,1,0,0) \\
N_{12}(1,1,0,0) & \rightarrow & P_{\alpha 12} = (1,0,1,0,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0) \\
N_{13}(1,1,0,1) & \rightarrow & P_{\alpha 13} = (1,0,1,0,1,0,1,1,0,1,0,1,0,0,0,0,1,0,0,0) \\
N_{14}(1,0,0,0) & \rightarrow & P_{\alpha 14} = (1,1,1,0,0,0,1,0,1,1,0,0,1,0,1,1,0,0,1,1) \\
N_{15}(1,0,0,1) & \rightarrow & P_{\alpha 15} = (1,1,1,0,1,0,1,1,1,1,0,1,1,1,1,1,1,1,1,0)
\end{array}
$$

Any three of these points in $PG(19,2)$ are non-collinear, hence they are 15-cap in $PG(19,2)$.

Grassmanian coordinates of β-planes of Klein Quadric is 15-cap in projective space $PG(19,2)$.

Let the $\beta-$plane be the image of the projective plane $D_1[0,0,0,1]$). This plane is spanned by points $(1,1,0,0,0,0)$, $(0,0,0,0,0,1)$ and $(0,1,0,0,0,0)$. It has Grassmann coordinates:

$$D_1[0,0,0,1] \rightarrow P_{\beta 1} = (0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$$

Similarly other planes have following Grassman coordinates:

$$
\begin{array}{rcl}
D_2[0,0,1,0] & \rightarrow & P_{\beta 2} = (0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \\
D_3[0,0,1,1] & \rightarrow & P_{\beta 3} = (0,0,1,1,0,1,1,0,0,0,0,1,1,0,0,0,0,0,0,0) \\
D_4[0,1,0,0] & \rightarrow & P_{\beta 4} = (0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0) \\
D_5[0,1,0,1] & \rightarrow & P_{\beta 5} = (0,1,0,1,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0) \\
D_6[0,1,1,0] & \rightarrow & P_{\beta 6} = (0,0,0,0,1,1,0,0,0,0,1,1,0,0,0,0,0,0,0,0) \\
D_7[0,1,1,1] & \rightarrow & P_{\beta 7} = (0,1,1,1,1,1,1,0,0,0,1,1,1,0,0,0,0,0,0,0) \\
D_8[1,0,0,0] & \rightarrow & P_{\beta 8} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1) \\
D_9[1,0,0,1] & \rightarrow & P_{\beta 9} = (0,0,0,1,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0) \\
D_{10}[1,0,1,0] & \rightarrow & P_{\beta 10} = (0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1,1) \\
D1_1[1,0,1,1] & \rightarrow & P_{\beta 11} = (0,0,1,1,0,1,1,1,1,0,0,0,0,0,0,1,0,0,0,0) \\
D_{12}[1,1,0,0] & \rightarrow & P_{\beta 12} = (0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,1,0,1) \\
D_{13}[1,1,0,1] & \rightarrow & P_{\beta 13} = (0,1,0,1,0,0,0,0,1,0,1,0,1,1,0,1,0,1,1,1) \\
D_{14}[1,1,1,0] & \rightarrow & P_{\beta 14} = (0,0,0,0,1,1,0,1,0,0,1,1,0,1,0,0,0,1,1,1) \\
D_{15}[1,1,1,1] & \rightarrow & P_{\beta 15} = (0,1,1,1,1,1,1,1,1,0,1,1,1,1,0,1,0,1,0,1)
\end{array}
$$

Any three of these points in $PG(19,2)$ are non-collinear, hence they are 15-cap in P$G(19,2)$.

Application: Each time-like geodesic in a flat four-dimensional space-time determines a quadric in projective twistor space. A quadric in P^3 has two distinct systems of generators. In the case of a time-like geodesic, the line L belongs to one of the two systems. The corresponding geodesic is a conic curve in the Klein quadric, and has the property that it passes through the point. The second system of generators then corresponds to a polar conic that lies on the null cone infinity and does not go through the point.

Description The line L of $PG(3,2)$ corresponds to the point of the projective space PG(5,2). In this case these lines are the plane arcs in PG(5,2). We use coordinates which are determined up to right multiples. Let $\{N_1, N_2, ..., N_{15}\}$ and $\{D_1, D_2, ..., D_{15}\}$ be the set of points and planes of the projective space of $PG(3,2)$. Without loss of generality, we can assign the following coordinates:

$$\begin{aligned}
N_1 &= (0,0,0,1), N_6 = (0,1,1,0), N_{11} = (1,0,1,1)\\
N_2 &= (0,1,0,0), N_7 = (0,1,1,1), N_{12} = (1,1,0,0)\\
N_3 &= (0,1,0,1), N_8 = (1,1,1,0), N_{13} = (1,1,0,1)\\
N_4 &= (0,0,1,0), N_9 = (1,1,1,1), N_{14} = (1,0,0,0)\\
N_5 &= (0,0,1,1), N_{10} = (1,0,1,0), N_{15} = (1,0,0,1)
\end{aligned}$$

and

$$\begin{aligned}
D_1 &= [0,0,0,1], D_6 = [0,1,1,0], D_{11} = [1,0,1,1]\\
D_2 &= [0,0,1,0], D_7 = [0,1,1,1], D_{12} = [1,1,0,0]\\
D_3 &= [0,0,1,1], D_8 = [1,0,0,0], D_{13} = [1,1,0,1]\\
D_4 &= [0,1,0,0], D_9 = [1,0,0,1], D_{14} = [1,1,1,0]\\
D_5 &= [0,1,0,1], D_{10} = [1,0,1,0], D_{15} = [1,1,1,1].
\end{aligned}$$

The incidence between a line of S and a point of S is induced by the incidence of $PG(5,2)$. Hence S satisfies projective plane axioms.

The quadric contains two distinct systems of projective planes, called α-planes and β-planes. Any two distinct planes of the same type in intersect at a point. Two planes of the opposite type will in general not intersect, but if they do, they intersect in a line. The points of $PG(3,2)$ correspond to $\alpha-$planes and the $2-$planes of $PG(3,2)$ correspond to $\beta-$planes.

Let $P_{\alpha i}, i = 1,2,...,15$ be $\alpha-$plane of $PG(5,2)$ and $P_{\beta i}, i = 1,2,...,15$ be $\beta-$ be a plane of $PG(5,2)$. We define a geometry S' as follows.

The projective points of S' correspond to the $\alpha-$planes of $PG(5,2)$ provide Fano axiom in the projective space $\mathcal{P}G(19,2)$;

The projective points of S' correspond to the $\beta-$planes of $PG(5,2)$ provide Fano axiom in the projective space $\mathcal{P}G(19,2)$;

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References

[1] Adamson, I. T. (1964). *Introduction to field theory.* Edinburgh: Oliver and Boyd.

[2] Akça, Z., Bayar, A., Ekmekçi, S., Kaya, R., 2010, *Bazı Geometrilerin Sonlu Projektif Uzaylara Gömülmeleri Üzerine*, Tübitak Proje No: 108T340.

[3] Akça, Z., Bayar, A., Ekmekçi, S., 4. *Mertebeden Projektif Düzlemin* 4*-boyutlu Projektif Uzaya Gömülmesi Üzerine*, Ogü Bap Proje No: 201619D38, 2017.

[4] Artin, E. (1948). *Galois theory.* Notre Dame: University of Notre Dame Press

[5] Cullinane, H., 2007, *The Klein Correspondence, Penrose Space-Time, and a Finite Model,* http://finitegeometry.org/sc/64/KleinCorr.html

[6] Dorje, C., Brody, L.P., 2014, *Twistor cosmology and quantum space-time*, arXiv:hep-th/0502218v1.

[7] Ekmekçi, S., Bayar, A., Akça, Z., 2017, $PG(4,4)$ *Projektif Uzayındaki Projektif Düzlemler Üzerine*, Ogü Bap Proje No: 201619D37 .

[8] Gokhan, S., Demirci, M., Ikikardes, N. Y., & Cangul, I. N., 2007. *Rational Points on Elliptic Curves* $y^2 = x^3 + a^3$ in F_p, where $p \equiv 5$ $(mod 6)$ is Prime. International Journal of Mathematics Sciences, 1(4), 247 – 250.

[9] Hirschfeld, J. A. Thas., 2016, *General Galois Geometries*, Springer Monongraphs in Mathematics.

[10] Klein, F.,1868, Über die *Transformation der allgemeinen Gleichung des zweiten Grades zwischen Linien-Koordinaten auf eine kanonische* Form, Math., p. 539-578

[11] Penttila, T., Siciliano A., 2015 *On collineation groups of finite projective spaces containing a Singer cycle*, Journal of Geometry 107(3).

[12] Plücker, J., 1865, *On a New Geometry of Space*, Philosophical Transactions of the Royal Society of London, Vol. 155, pp. 725-791.