Article

The Transversal Intersection of Surfaces Generated by Time-like Mannheim Curve Pair

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Abstract

In this work, we study the local properties of the intersection curve of the tangent, rectifying developable and Darboux developable surfaces of a time-like Mannheim curve pair. We derive the curvature vector and curvature for the transversal intersection problem. Furthermore, we investigate some characteristic features of the intersection curve for all three cases and give some important results.

Keywords: Transversal intersection, rectifying developable, Darboux developable, time-like, Mannheim curve pair.

1. Introduction

A ruled Surface is generated by a moving straight line continously in Euclidean space E^3, [1]. Ruled surfaces are one of the simplest objects in geometric modeling. One important fact about ruled surfaces is that they can be generated by straight lines. A practical application of this type surfaces is that they are used in civil engineering and physics,[2]. Izumiya and Takeuichi introduced some new special ruled surfaces such as Darboux developable and Rectifying developable surfaces and investigate their properties, [3].

The curves are a fundamental structure of differential geometry. An increasing interest of the theory of curves makes a development of special curves to be examined. Especially, Bertrand curves are well-studied classical curves,[4]. Another special curves are Mannheim curves. In recent works, Liu and Wang are curious about the Mannheim curves in both Euclidean and Minkowski 3-space and they obtained the necessary and sufficient conditions between the curvature and the torsion for a curve to be the Mannheim partner curves[5],[6]. Kasap and Orbay studied the Mannheim partner curves in Euclidean space and obtain the relationships between the curvatures and the torsions of the Mannheim partner curves with respect to each other, [7].

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n this paper, we study the intersection problem for the tangent, rectifying devlopable and Darboux developable surfaces of (α,α^*) Mannheim curve pair. We investigate the charactherizations of the intersection curve for each case of surface-surface intersection. First, we express the curvature vector and the curvature of the intersection curve in terms of normal curvatures of both (α,α^*) Mannheim curve pair. Then for each type of ruled surfaces, we investigate the properties of the intersection curve.

2. Preliminaries

The 3-dimensional Minkowski Space $\mathbb{R}_1^3$ is the pair $(\mathbb{R}^3,\langle,\rangle)$. $\mathbb{R}^3$ is a three-dimensional real vector space equipped with a Lorentz metric (inner product), $\langle x,y\rangle=-x_1y_1+x_2y_2+x_3y_3$ where $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$. A vector $x\neq 0$ in $\mathbb{R}^3$ is called spacelike, time-like or a null (lightlike), if respectively holds $\langle x,x\rangle>0$, $\langle x,x\rangle<0$ or $\langle x,x\rangle=0$. Especially, the vector $x=0$ is spacelike. If $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ and its norm defined by

$$\|x\|=|\langle x,x\rangle|^{\frac{1}{2}}=\sqrt{|-x_1^2+x_2^2+x_3^2|}$$

Any given two vectors $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$ in $\mathbb{R}_1^3$ are said to be orthogonal if $\langle x,y\rangle=0$. A vector $x=(x_1,x_2,x_3)$ in $\mathbb{R}^3$ which satisfies $\langle x,x\rangle=\pm 1$ is called a unit vector. We also define the vector product of *x* and *y* (in that order) as

$$x\times y=\begin{vmatrix}-e_1 & e_2 & e_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}=(x_3y_2-x_2y_3,x_3y_1-x_1y_3,x_1y_2-x_2y_1)$$

where $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$. Remind that the vector product is nonassociative and that furthermore we have the following properties

$$\langle u\times v,x\times y\rangle=\det\begin{bmatrix}\langle u,y\rangle & \langle v,y\rangle\\ \langle u,x\rangle & \langle v,x\rangle\end{bmatrix}$$

where *u, v, x, y* are arbitrary vectors in $\mathbb{R}_1^3$ and

$$(u\times v)\times w=\langle v,w\rangle u-\langle u,w\rangle v\ ;\ u,v,w\in\mathbb{R}_1^3$$

An arbitrary curve $c=c(s)$ can locally be a spacelike, time-like or null (lightlike) if all of its velocity vectors $c'(s)$ are respectively spacelike, time-like or null. A non-null curve $c=c(s)$ is said to be parameterized by pseudo-arc length parameter s, if $\langle c'(s),c'(s)\rangle=\pm 1$. If there exists a corresponding relationship between the time-like space curves α and α^* such that the principal normal lines of α coincides with the binormal lines of α^* at the corresponding points of the curves, then α is called as a Time-like Mannheim curve and α^* is called as a Time-like Mannheim partner curve of α. The pair of (α,α^*) is said to be a Time-like Mannheim pair. There exists a relationship between the position vectors as

$$\begin{aligned} \alpha(s) &= \alpha^*(s^*) + \lambda B^*(s^*) \\ \alpha^*(s^*) &= \alpha(s) - \lambda N(s) \end{aligned} \qquad \textbf{1)}$$

and we can write $N = B^*$ and λ is the distance between the curves $\alpha(s)$ and $\alpha^*(s^*)$ at the corresponding points. Let (α,α^*) be a Time-like Mannheim pair, $\{T(s), N(s), B(s)\}$ and $\{T^*(s^*), N^*(s^*), B^*(s^*)\}$ be their Frenet frames, respectively, we can write the following relationship between these frames:

$$\begin{aligned} T &= \cosh\theta T^* + \sinh\theta N^* \\ B &= \sinh\theta T^* + \cosh\theta N^* \end{aligned} \qquad \textbf{2)}$$

and

$$\begin{aligned} T^* &= \cosh\theta T - \sinh\theta B \\ N^* &= -\sinh\theta T + \cosh\theta B \end{aligned} \qquad \textbf{3)}$$

where θ is the hyperbolic angle between the time-like tangent vectors T and T^*.

3. Transversal Intersection of Tangent Surfaces of Time-like Mannheim Curve Pair

Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their tangent surfaces, respectively. Let N^A be the unit surface normal of the tangent surface $X^A(s,u)$ and N^B be the unit surface normal of the tangent surface $X^B(s^*,u)$, we can compute N^A and N^B as;

$$N^A = \frac{X_s^A \times X_u^A}{\left\| X_s^A \times X_u^A \right\|} = \pm B$$

and

$$N^B = \frac{X_{s^*}^B \times X_u^B}{\left\| X_{s^*}^B \times X_u^B \right\|} = \pm B^*$$

where B is the binormal frenet vector of α and B^* is the binormal frenet vector of α^*. Let $c = c(s)$ be the transversal intersection curve of both tangent surfaces of X^A and X^B. This means that the tangent vector of the transversal intersection curve $c = c(s)$ lies on the tangent planes of both surfaces. Therefore, it can be obtained as the cross product of the unit surface normal vectors of the surfaces at $p=c(s)$

$$t=\frac{N^A\times N^B}{\left\|N^A\times N^B\right\|}=\pm\left\{\cosh\theta T^*+\sinh\theta N^*\right\}=\pm T$$

where N^A be the unit surface normal of the tangent surface $X^A(s,u)$ and N^B be the unit surface normal of the tangent surface $X^B(s^*,u)$. Since T is Time-like vector, then the transversal intersection curve $c=c(s)$ is a time-like curve.

Result 1. Let α and α^* are time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their tangent surfaces, respectively then, the curve $c=c(s)$ is parallel to the curve α.

Let investigate the angle between surfaces $X^A(s,u)$ and $X^B(s^*,u)$. The angle between the surfaces $X^A(s,u)$ and $X^B(s^*,u)$ is the angle between the unit surface normal vectors N^A and N^B. If η denote the angle between N^A and N^B, then we can write

$$\cos\eta=\left\langle N^A,N^B\right\rangle=\left\langle \pm B,\pm B^*\right\rangle=0$$

Result 2. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their tangent surfaces, respectively. The angle between the surfaces $X^A(s,u)$ and $X^B(s^*,u)$ is equal to the angle $\frac{\theta}{2}+2k\pi$, $k\in \square$, that is the surfaces $X^A(s,u)$ and $X^B(s^*,u)$ intersect orthogonally.

Theorem 1. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their tangent surfaces, respectively. Then, the curveature κ of the curve $c=c(s)$ is given by

$$\kappa=\left\{\left(\kappa_n^A\right)^2+\left(\kappa_n^B\right)^2\right\}^{\frac{1}{2}}.$$

Lemma 1. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their tangent surfaces, respectively. If κ_g^A and κ_g^B are the geodesic curvatures of $X^A(s,u)$ and $X^B(s^*,u)$ respectively, then we have

$$\kappa_g^A=\kappa_n^B$$
$$\kappa_g^B=\kappa_n^A.$$

Theorem 2. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their tangent surfaces, respectively. Then $c=c(s)$ is a geodesic curve of the surface $X^A(s,u)$ if and only if $c=c(s)$ is a asymptotic curve of the surface $X^B(s^*,u)$.

Theorem 3. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their tangent surfaces, respectively. Then $c=c(s)$ is a geodesic curve of the surface $X^B(s^*,u)$ if and only if $c=c(s)$ is a asymptotic curve of the surface $X^A(s,u)$.

Theorem 4. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their tangent surfaces, respectively. Then we have

$$\tau_g^A = -\tau_g^B .$$

where τ_g^A and τ_g^B are the geodesic torsions of the surfaces $X^A(s,u)$ and $X^B(s^*,u)$, respectively.

Result 3. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their tangent surfaces, respectively. $c(s)$ is a line of curvature of the surface $X^A(s,u)$ if and only if $c(s)$ is a line of curvature of the surface $X^B(s^*,u)$.

4. Transversal Intersection Rectifying Developable Surfaces of Time-like Mannheim Curve Pair

In this section, we compute the curvature of the transversal intersection curve of rectifying developable surfaces of Time-like Mannheim curve pairs. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their rectifying developable surfaces, respectively. Let N^A be the unit surface normal of the rectifying developable surface $X^A(s,u)$ and N^B be the unit surface normal of the rectifying developable surface $X^B(s^*,u)$, we can compute N^A and N^B as;

$$N^A = \frac{X_s^A \times X_u^A}{\left\| X_s^A \times X_u^A \right\|} = \pm N$$

and

$$N^B = \frac{X_{s^*}^B \times X_u^B}{\left\| X_{s^*}^B \times X_u^B \right\|} = \pm N^*$$

where N is the binormal frenet vector of α and N^* is the binormal frenet vector of α^*. Let $c=c(s)$ be the transversal intersection curve of both rectifying developable surfaces of X^A and X^B. This means that the tangent vector of the transversal intersection curve $c=c(s)$ lies on the tangent planes of both surfaces. Therefore, it can be obtained as the cross product of the unit surface normal vectors of the surfaces at $p=c(s)$

$$t=\frac{N^A\times N^B}{\left\|N^A\times N^B\right\|}=\pm\{\cosh\theta T-\sinh\theta B\}=\pm T^*$$

where N^A be the unit surface normal of the rectifying developable surface $X^A(s,u)$ and N^B be the unit surface normal of the rectifying developable surface $X^B(s^*,u)$.

Result 4. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their rectifying developable surfaces, respectively then, the curve $c=c(s)$ is parallel to the curve α^*.

Let investigate the angle between surfaces $X^A(s,u)$ and $X^B(s^*,u)$. The angle between the surfaces $X^A(s,u)$ and $X^B(s^*,u)$ is the angle between the unit surface normal vectors N^A and N^B. If η denote the angle between N^A and N^B, then we can write

$$\cos\eta=\left\langle N^A,N^B\right\rangle=\left\langle \pm N,\pm N^*\right\rangle=\left\langle B^*,N^*\right\rangle=0$$

Result 5. Let α and α^* areTime-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their rectifying developable surfaces, respectively. The angle between the surfaces $X^A(s,u)$ and $X^B(s^*,u)$ is equal to the angle $\frac{\theta}{2}+2k\pi$, $k\in\mathbb{Z}$, that is the surfaces $X^A(s,u)$ and $X^B(s^*,u)$ intersect orthogonally.

Theorem 5. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their rectifying developable surfaces, respectively. Then, the curvature κ of the curve $c=c(s)$ is given by

$$\kappa=\left\{\left(\kappa_n^A\right)^2+\left(\kappa_n^B\right)^2\right\}^{\frac{1}{2}}.$$

Lemma 2. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their rectifying developable surfaces, respectively. If κ_g^A and κ_g^B are the geodesic curvatures of $X^A(s,u)$ and $X^B(s^*,u)$ respectively, then we have

$$\kappa_g^A=\kappa_n^B$$

$$\kappa_g^B = \kappa_n^A .$$

Theorem 6. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their rectifying developable surfaces, respectively. Then $c=c(s)$ is a geodesic curve of the surface $X^A(s,u)$ if and only if $c=c(s)$ is a asymptotic curve of the surface $X^B(s^*,u)$.

Theorem 7. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their rectifying developable surfaces, respectively. Then $c=c(s)$ is a geodesic curve of the surface $X^B(s^*,u)$ if and only if $c=c(s)$ is a asymptotic curve of the surface $X^A(s,u)$.

Theorem 8. Let α and α^* are Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their rectifying developable surfaces, respectively. Then we have

$$\tau_g^A = -\tau_g^B .$$

where τ_g^A and τ_g^B are the geodesic torsions of the surfaces $X^A(s,u)$ and $X^B(s^*,u)$, respectively.

Result 6. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their rectifying developable surfaces, respectively. $c(s)$ is a line of curvature of the surface $X^A(s,u)$ if and only if $c(s)$ is a line of curvature of the surface $X^B(s^*,u)$.

5. Transversal Intersection of Darboux Developable Surfaces of Time-like Mannheim Curve Pair

Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their Darboux developable surfaces, respectively. Let N^A be the unit surface normal of the Darboux developable surface $X^A(s,u)$ and N^B be the unit surface normal of the Darboux developable surface $X^B(s^*,u)$, we can compute N^A and N^B as;

$$N^A = \frac{X_s^A \times X_u^A}{\left\| X_s^A \times X_u^A \right\|} = \pm B$$

and

$$N^B = \frac{X_{s^*}^B \times X_u^B}{\left\| X_{s^*}^B \times X_u^B \right\|} = \pm B^*$$

where N is the binormal frenet vector of α and N^* is the binormal frenet vector of α^*. Let $c=c(s)$ be the transversal intersection curve of both Darboux developable surfaces of X^A and

X^B. This means that the tangent vector of the transversal intersection curve $c=c(s)$ lies on the tangent planes of both surfaces. Therefore, it can be obtained as the cross product of the unit surface normal vectors of the surfaces at $p=c(s)$

$$t=\frac{N^A\times N^B}{\left\|N^A\times N^B\right\|}=\pm\left\{\cosh\theta T^*+\sinh\theta N^*\right\}=\pm T$$

where N^A be the unit surface normal of the Darboux developable surface $X^A(s,u)$ and N^B be the unit surface normal of the Darboux developable surface $X^B(s^*,u)$.

Result 7. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their Darboux developable surfaces, respectively then, the curve $c=c(s)$ is parallel to the curve α

Let investigate the angle between surfaces $X^A(s,u)$ and $X^B(s^*,u)$. The angle between the surfaces $X^A(s,u)$ and $X^B(s^*,u)$ is the angle between the unit surface normal vectors N^A and N^B. If η denote the angle between N^A and N^B, then we can write

$$\cos\eta=\left\langle N^A,N^B\right\rangle=\left\langle\pm B,\pm B^*\right\rangle=\left\langle B,N\right\rangle=0$$

Result 8. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their Darboux developable surfaces, respectively. The angle between the surfaces $X^A(s,u)$ and $X^B(s^*,u)$ is equal to the angle $\frac{\theta}{2}+2k\pi$, $k\in\mathbb{Z}$, that is the surfaces $X^A(s,u)$ and $X^B(s^*,u)$ intersect orthogonally.

Theorem 9. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their Darboux developable surfaces, respectively. Then, the curvature κ of the curve $c=c(s)$ is given by

$$\kappa=\left\{\left(\kappa_n^A\right)^2+\left(\kappa_n^B\right)^2\right\}^{\frac{1}{2}}.$$

Lemma 3. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their Darboux developable surfaces, respectively. If κ_g^A and κ_g^B are the geodesic curvatures of $X^A(s,u)$ and $X^B(s^*,u)$ respectively, then we have

$$\kappa_g^A=\kappa_n^B$$
$$\kappa_g^B=\kappa_n^A.$$

Theorem 10. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their Darboux developable surfaces, respectively. Then $c=c(s)$ is a geodesic curve of the surface $X^A(s,u)$ if and only if $c=c(s)$ is a asymptotic curve of the surface $X^B(s^*,u)$.

Theorem 11. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their Darboux developable surfaces, respectively. Then $c=c(s)$ is a geodesic curve of the surface $X^B(s^*,u)$ if and only if $c=c(s)$ is a asymptotic curve of the surface $X^A(s,u)$.

Theorem 12. Let α and α^* are Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their Darboux developable surfaces, respectively. Then we have

$$\tau_g^A = -\tau_g^B .$$

where τ_g^A and τ_g^B are the geodesic torsions of the surfaces $X^A(s,u)$ and $X^B(s^*,u)$, respectively.

Result 9. Let α and α^* are Time-like Mannheim curve pair, $X^A(s,u)$ and $X^B(s^*,u)$ are their Darboux developable surfaces, respectively. $c(s)$ is a line of curvature of the surface $X^A(s,u)$ if and only if $c(s)$ is a line of curvature of the surface $X^B(s^*,u)$.

Received March 18, 2021; Accepted April 11, 2021

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