Article

A Critical Re-examination of $M^8 - H$ Duality: Part I

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Abstract

This article is the first part of an article representing a critical re-examination of $M^8 - H$ duality, which is one of the cornerstones of Topological Geometrodynamics (TGD). The original version of $M^8 - H$ duality assumed that space-time surfaces in M^8 can be identified as associative or co-associative surfaces. If the surface has associative tangent or normal space and contains a complex or co-complex surface, it can be mapped to a 4-surface in $H = M^4 \times CP_2$. Later emerged the idea that octonionic analyticity realized in terms of real polynomials P algebraically continued to polynomials of complexified octonion could fulfill the dream. The vanishing of the real part $Re_Q(P)$ (imaginary part $Im_Q(P)$) in the quaternionic sense would give rise to an associative (co-associative) space-time surface. The realization of the general coordinate invariance motivated the notion of strong form of holography (SH) in H allowing realization of a weaker form of $M^8 - H$ duality by assuming that associativity/co-associativity conditions are needed only at 2-D string world sheet and partonic 2-surfaces and possibly also at their light-like 3-orbits.

The outcome of the re-examination was a positive surprise. Although no interesting associative space-time surfaces are possible, every distribution of normal associative planes (co-associativity) is integrable. Another positive surprise was that Minkowski signature is the only possible option. Equivalently, the image of M^4 as real co-associative sub-space of O_c (complex valued octonion norm squared is real valued for them) by an element of local $G_{2,c}$ or its subgroup $SU(3, c)$ gives a real co-associative space-time surface. The conjecture is that the polynomials P determine these surfaces as roots of $Re_Q(P)$. These surfaces also possess co-complex 2-D sub-manifolds allowing the mapping to H to H by $M^8 - H$ duality as a whole. SH would not be needed and would be replaced with number theoretic holography determining space-time surface from its roots and selection of real sub-space of O_c characterizing the state of motion of a particle. The equations for $Re_Q(P) = 0$ reduce to simultaneous roots of ordinary real polynomials defined by the odd and even parts of P having interpretation as complex values of mass squared mapped to light-cone proper time constant surfaces in H. Octonionic Dirac equation as analog of momentum space variant of ordinary Dirac equation forces the interpretation of M^8 as an analog of momentum space and Uncertainty Principle forces to modify the map $M^4 \subset M^8 \to M^4 \subset H$ from identification to inversion. Contrary to the earlier expectations the space-time surface in M^8 would be analogous to Fermi ball and mass squared sections would correspond to Fermi spheres. This leads to the idea that the formulation of scattering amplitudes at M^8 levels provides the counterpart of momentum space description of scattering whereas the formulation at the level of H provides the counterpart of space-time description.

1 Introduction

$M^8 - H$ duality [43, 41, 42, 49] has become a cornerstone of quantum TGD but several aspects of this duality are still poorly understood.

1.1 Development of the idea about $M^8 - H$ duality

A brief summary about the development of the idea is in order.

1. The original version of $M^8 - H$ duality assumed that space-time surfaces in M^8 can be identified as associative or co-associative surfaces. If the surface has associative tangent/normal space and contains a complex co-complex surface, it can be mapped to a 4-surface in $M^4 \times CP_2$.

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2. Later emerged the idea that octonionic analyticity realized in terms of a real polynomials P algebraically continued to polynomials of complexified octonion might realize the dream [36, 37, 38]. The original idea was that the vanishing condition for the real/imaginary part of P in quaternion sense could give rise to co-assocative/associative sense.

   $M^8 - H$ duality concretizes number theoretic vision [39, 40] summarized as adelic physics fusing ordinary real number based physics for the correlates of sensory experience and various p-adic physics ($p = 2, 3, ...$) as physics for the correlates of cognition. The polynomials of real variable restricted to be rational valued defines an extension or rationals via the roots of the polynomials and one obtains an evolutionary hierachy associated with these extensions increasing in algebraic complexity. These extensions induce extensions of p-adic numbers and the points of space-time surface in M^8 with coordinates in the extension of rationals define cognitive representations as unique discretizations of the space-time surface.

3. The realization of the general coordinate invariance in TGD framework [21, 20, 24, 54] [51] motivated the idea that strong form of holography (SH) in H could allow realizing $M^8 - H$ duality by assuming associativity/co-associativity conditions only at 2-D string world sheet and partonic 2-surfaces and possibly also at their light-like 3-orbits at which the signature of the induced metric changes from Minkowskian to Euclidian.

1.2 Critical re-examination of the notion

In this article $M^8 - H$ duality is reconsidered critically.

1. The healthy cold shower was the learning that quaternion (associative) sub-spaces of quaternionic spaces are geodesic manifolds [6]. The distributions of quaternionic normal spaces are however always integrable. Hence, co-associativity remains the only interesting option. Also the existence of co-commutative sub-manifolds of space-time surface demanding the existence of a 2-D integrable sub-distribution of subspaces is possible. This learning experience motivated a critical examination of the $M^8 - H$ duality hypothesis.

2. The basic objection is that for the conjectured associative option, one must assign to each state of motion of a particle its own octonionic structure since the time axis would correspond to the octonionic real axis. It was however clear from the beginning that there is an infinite number of different 4-D planes O_c in which the number theoretical complex valued octonion inner product reduces to real - the number theoretic counterpart for Riemann metric. In the co-associative case this is the only option. Also the Minkowski signature for the real projection turns out to be the only physically acceptable option. The mistake was to assume that Euclidian regions are co-associative and Minkowskian regions associative: both must be co-associative.

3. The concrete calculation of the octonion polynomial was the most recent step - carried already earlier [36, 37, 38] but without realizing the implications of the extremely simple outcome. The imaginary part of the polynomial is proportional to the imaginary part of octonion itself. It turned out that the roots $P = 0$ of the octonion polynomial P are 12-D complex surfaces in O_c rather than being discrete set of points defined as zeros $X = 0, Y = 0$ of two complex functions of 2 complex arguments. The analogs of branes are in question. Already earlier 6-D real branes assignable to the roots of the real polynomial P at the light-like boundary of 8-D light-cone were discovered: also their complex continuations are 12-D [41, 44].

4. P has quaternionic de-composition $P = Re_Q(P) + I_4 Im_Q(P)$ to real and imaginary parts in a quaternionic sense. The condition $Re_Q(P) = 0$ implies that the resulting surface is a 4-D complex surface X^4_c a with a 4-D real projection X^4_r, which could be co-associative. Note that the condition $X = 0$ is satisfied but not $Y = 0$ condition. The naïve expectation is that adding the condition

$Y = 0$ one obtains 3-D surface X_c^3 having a 3-D real projection X_r^3 with the analog of 6-D brane defined as a root of P. $Im_Q(P) = 0$ gives a complex surface X_c^4 with real projection X_r^4 as a candidate for an associative surface: only geodesic sub-manifolds are however possible.

The expectation is wrong! The equations $X = 0$ and $Y = 0$ involve the same complex argument o_c^2 as a complex analog for the Lorentz invariant distance squared from the tip of the light-cone. There are solutions only if the two polynomials considered have a common a_c^2 root! When the solution exists X_r^4 belongs to X_r^6 rather than having a 3-D intersection with it! This has strong consequences concerning the physical interpretation.

Co-associative X_r^4 could be also realized by assuming X_r^4 is obtained by acting with a local $G_{2,c}$ or possibly $SU_{3,c} \subset G_{2,c}$ transformation to an co-associative real plane of O_c, which can be selected in very many manners related to G_2 transformation. The co-associativity of this plane is preserved in the map because $G_{2,c}$ acts as an automorphism group of the octonions. The conjecture is that the surface obtained by giving up $Y = 0$ condition is the same surface as obtained by $G_{2,c}$ holography.

5. Octonionic Dirac equation, which is purely algebraic equation and the counterpart for ordinary Dirac equation in momentum space, serves as a second source of information. The first implication is that O_c has interpretation as an analog of momentum space for quarks: this has profound implications concerning the interpretation. The space-time surface in M^8 would be analogs of Fermi ball.

   The roots of $Re_Q(P)$ come in two types. By Lorentz invariance, the equation $X = 0$ for the co-associative 4-surfaces reduces to that for the roots of a real polynomial defined by the odd part P_{odd} of P. The simultaneous roots of X and Y give the intersection $X_r^4 \cap X_2^6$. If the condition $Y = 0$ cannot be satisfied, the intersection is empty and the momentum squared represented by O_c point must be light-like and belong to the light-like boundary of CD: one would have only massless quarks arriving at CD. If X and Y have a common root, X_r^4 belongs to X_r^6 and quarks can be massive. The massivation of quarks has an interpretation as a number theoretic phase transition occurring when even and odd parts of P have a common root.

This would fulfil the original dream about solving classical TGD exactly in terms of roots for real/imaginary parts of octonionic polynomials in M^8 and by mapping the resulting space-time surfaces to H by $M^8 - H$ duality. In particular, SH would not be needed at the level of H, and would be replaced to a dramatically stronger number theoretic holography in which the roots of a real polynomial would fix the space-time surface completely one the real projection characterizing the state of motion of the particle is selected. Fundamental physics would be ridiculously simple.

2 The situation before the cold shower

The view about $M^8 - H$ duality before the cold shower - leading to what I dare to call a breakthrough - helps to gain idea about the phenomenological side of $M^8 - H$ duality. Most of the phenomenology survives the transition to a more precise picture. This section is however not absolutely necessary for what follows it.

2.1 Can one deduce the partonic picture from $M^8 - H$ duality?

The M^8 counterparts for partons and their light like orbits in H can be understood in terms of octonionic Dirac equation in M^8 as an analog for the algebraic variant of ordinary Dirac equation at the level of momentum space [49, 48] but what about the identification of partonic 2-surfaces as interaction vertices at which several partonic orbits meet? Can one deduce the phenomenological view about elementary particles as pairs of wormhole contacts connected by magnetic flux tubes from $M^8 - H$ duality? Why should the partonic vertices reside at $t = r_n$ branes? There is also the question whether partonic orbits correspond to their own sub-CDs as the fact that their rest systems correspond to different octonionic real axes suggests.

$M^8 - H$ duality indeed conforms with the phenomenological picture about scattering diagrams in terms of partonic orbits [54, 53] [53, 54] [54], and leads to the view about elementary particles as pairs of Euclidian wormhole contacts associated with flux tubes carrying monopole flux.

2.2 What happens at the "very special moments in the life of self"?

Consider first what happens at the "very special moments in the life of self" [41, 44] corresponding to $t = r_n$, where r_n is a root of the real polynomial defining octonionic polynomial as its algebraic continuation. The moment $t = r_n$ corresponds to a 6-sphere S^6 as an analog of brane and is a special solution to the algebraic equations stating the vanishing of either imaginary or real part of octonion valued polynomial: imaginary and real parts are quaternion valued now. These branes are localized to the boundary of M^8 lightcone.

1. In the generic case there are 4 complexified conditions giving rise to a surface with complex dimension $D_c = 4$ in M^8_c. The 4 complex conditions in M^8_c correspond to 8 real conditions. There are 4 complexified polynomials in the imaginary/real part of the octonionic polynomial and one can solve the complexified M^4 time coordinate from them getting 4 complex solutions, which must be identical. This gives a complex surface with dimension $D_c = 4$ projected from M^8_c to M^8 by taking its "real part". The outcome is a space-time surface X^4 with $D_R = 4$ identified as "reality".

2. At $t = r_n$ branes the 4 complex conditions reduce to 2 complex conditions stating the location to 8-D light-cone and the condition that one has root of the polynomial. Instead of 4 complexified octonionic conditions one has only 2 and one obtains complexified S^6 as a solution having real S^6 as real projection.

2.3 What does SH mean and its it really needed?

SH has been assumed hitherto but what is its precise meaning?

1. Hitherto, SH at the level of H is believed to be needed: it assumes that partonic 2-surfaces and/or string world sheets serve as causal determinants determining X^4 via boundary conditions.

   (a) The normal or tangent space of X^4 at partonic 2-surfaces and possibly also at string world sheets has been assumed to be associative that is quaternionic. This condition should be true at the entire X^4.

   (b) Tangent or normal space has been assumed to contain preferred M^2 which could be replaced by an integrable distribution of $M^2(x) \subset M^4$. At string world sheets only the tangent space can be associative. At partonic 2-surfaces also normal space could be associative. This condition would be true only at string world sheets and partonic 2-surfaces so that only these can be mapped to H by $M^8 - H$ duality and continued to space-time surfaces as preferred extremals satisfying SH.

   The current work demonstrates that although SH could be used at the level of SH, this is not necessary. Co-associativity together with co-commutativity for string world sheets allows the mapping of the real space-time surfaces in M^8 to H implying exact solvability of the classical TGD.

2.4 Questions related to partonic 2-surfaces

There are several questions related to partonic 2-surfaces.

Q1: What are the M^8 pre-images of partons and their light-like partonic orbits in H?

It will be found that the octonionic Dirac equation in M^8 implies that octo-spinors are located to 3-D light-like surfaces Y^3_r - actually light-cone boundary and its 3-D analogs at which number theoretic norm squared is real and vanishes - or to the intersections of X^3_r with the 6-D roots of P in which case Dirac equation trivializes and massive states are allowed. They are mapped to H by $M^8 - H$ duality.

Remark: One can ask whether the same is true in H in the sense that modified Dirac action would be localized to 3-D light-like orbits and 3-D ends of the space-time surfaces at the light-like boundaries of CD having space-like induced metric. Modified Dirac action would be defined by Cherm-Simons term and would force the classical field equations for the bosonic Chern-Simons term. If the interior part of the modified Dirac action is absent, the bosonic action is needed to define the space-time surfaces as extremals. They would be minimal surfaces and universal by their holomorphy and would not depend on coupling parameters so that very general actions can have them as preferred extremals. This issue remains still open.

The naïve - and as it turned out, wrong - guess was that the images of the light-like surfaces should be light-like surfaces in H at the boundaries of Minkowskian and Euclidian regions (wormhole contacts). In the light-like case Y^3_r corresponds to the light-cone boundary so that this would be the case. X^3_r however turns out to correspond to a hyperboloid in M^4 as an analog of a mass shell and is not identifiable as a partonic orbit.

It turned out that the complex surface X^4_c allows real sections in the sense that the number theoretic complex valued metric defined as a complex continuation of Minkowski norm is real at 4-D surfaces: call them Z^4_r. They are bounded by a 3-D region at Z^3_r at which the value of norm squared vanishes. This surface is an excellent candidate for the pre-image of the light-like orbit of partonic 2-surface serving as a topological vertex. One has therefore strings worlds sheets, partonic 2-surfaces and their light-like orbits and they would connect the "mass shells" at X^4_r. All ingredients for SH would be present.

The intersections of Z^3_r with X^3_r identifiable as the section of X^4_r $a = constant$ hyperboloid would give rise to partonic 2-surfaces appearing as topological reaction vertices.

The assumption that the 4-D tangent space at these light-like 3-surfaces is co-associative, would give an additional condition determining the image of this surface in H, so that the boundary conditions for SH would become stronger. One would have boundary conditions at light-like partonic orbits. Note that string world sheets are assumed to have light-like boundaries at partonic orbits.

Q2: Why should partonic 2-surfaces appear as throats of wormhole contact in H? Wormhole contacts do not appear in M^8.

1. In M^8 light-like orbits are places where the Minkowskian signature changes to Euclidian. Does $M^8 - H$ duality map the images of these coinciding roots for Euclidian and Minkowskian branches to different throats of the wormhole contact in H so that the intersection would disappear?

2. This is indeed the case. The intersection of Euclidian and Minkowskian branches defines a single 3-surface but the tangent and normal spaces of branches are different. Therefore their H images under $M^8 - H$ duality for the partonic 2-surface are different since normal spaces correspond to different CP_2 coordinates. These images would correspond to the two throats of wormhole contact so that the H-image by SH is 2-sheeted. One would have wormhole contacts in H whereas in M^8 the wormhole contact would reduce to a single partonic 2-surface.

3. The wormhole contact in H can have only Euclidian signature of the induced metric. The reason is that the M^4 projections of the partonic surfaces in H are identical so that the points with same M^4 coordinates have different CP_2 coordinates and their distance is space-like.

Q3: In H picture the interpretation of space-time surfaces as analogs of Feynman graphs assumes that several partonic orbits intersect at partonic 2-surfaces. This assumption could be of course wrong.This raises questions.

What the pre-images of partonic 2-surfaces are in M^8? Why should several partonic orbits meet at a given partonic 2-surface? Is this needed at all?

The space-time surface X^4_r associated intersects the surface X^6_r associated with different particle - say with different value of mass along 2-D surface. Could this surface be identified as partonic 2-surface X^2_r? This occurs symmetrically so that one has a pair of 2-surfaces X^2_r. What does this mean? Could these surface map to the throats of wormhole contact in H?

Why several partonic surfaces would co-incide in topological reaction vertex at the level of H? At this moment is is not clear whether this is forced by M^8 picture.

Octonionic Dirac equation implies that M^8 has interpretation as analog of momentum space so that interaction vertices are replaced by multilocal vertices representing momenta and propagators become local being in this sense analogous to vertices of QFT. One could of course argue that without the gluing along ends there would be no interactions since the interactions in X^6_r for two 3-surfaces consist in the generic case of a discrete set of points. One could also ask whether the surfaces Y^3_r associated with the space-time surfaces X^4_r associated with incoming particles must intersect along partonic 2-surface rather than at discrete set of points.

The meeting along ends need not be true at the level of M^8 since the momentum space interpretation would imply that momenta do not differ much so that particles should have identical masses: for this to make sense one should assume that the exchanged virtual particles are massless. One other hand, if momenta are light-like for Y^3_r, this might be the case.

Q4: Why two wormhole contacts and monopole flux tubes connecting them at the level of H? Why monopole flux?

1. The tangent spaces of the light-like orbits have different light-like direction. Intuitively, this corresponds to different directions of light-like momenta. Momentum conservation requires more than one partonic orbit changing its direction meeting at partonic 2-surface. By light-likeness, the minimum is 2 incoming and two outgoing lines giving a 4-vertex. This allows the basic vertices involving Ψ and $\overline{Psi}$ at opposite throats of wormhole contacts. Also a higher number of partonic orbits is possible.

2. A two-sheeted closed monopole flux tube having wormhole contacts as its "ends" is suggested by elementary particle phenomenology. Since M^8 homology is trivial, there is no monopole field in M^8. If $M^8 - H$ duality is continuous it maps homologically trivial partonic 2-surfaces to homologically trivial 2-surfaces in H. This allows the wormhole throats in H to have opposite homology charges. Since the throats cannot correspond to boundaries there must be second wormhole contact and closed flux tube.

3. What does the monopole flux for a partonic 2-surface mean at the level of M^8? The distribution of quaternionic 4-D tangent/normal planes containing preferred M^2 and associated with partonic 2-surface in M^8 would define a homologically on-trivial 2-surface in CP_2. The situation is analogous to a distribution of tangent planes or equivalently normal vectors in S^2.

Q4: What is the precise form of $M^8 - H$ duality: does it apply only to partonic 2-surfaces and string world sheets or to the entire space-time surfaces?

$M^8 - H$ duality is possible if the X^4 in M^8 contains also integrable distribution of complex tangent or normal 2-planes at which 4-D tangent space is quaternionic/associative. String world sheets and partonic 2-surfaces define these distributions.

The minimum condition allowed by SH in H is that string world sheets and there is a finite number of partonic 2-surfaces and string world sheets. In this case only these 2-surfaces can be mapped to H and SH assigns to them a 4-D space-time surface. The original hypothesis was that these surfaces define global orthogonal slicings of the X^4 so that $M^8 - H$ duality could be applied to the entire X^4. This condition is probably too strong.

3 Challenging $M^8 - H$ duality

$M^8 - H$ duality involves several alternative options and in the following arguments possibly leading to a unique choice are discuses.

1. Are both associativity and co-associativity possible or is only either of these options allowed? Is it also possible to pose the condition guaranteeing the existence of 2-D complex sub-manifolds identifiable as string world sheets necessary to map the entire space-time surface from M^8 to H? In other words, is the strong form of holography (SH) needed in M^8 and/or H or is it needed at all?

2. The assignment of the space-time surface at the level of M^8 to the roots of real or imaginary part (in quaternionic sense) of octonionic polynomial P defined as an algebraic continuation of real polynomial is an extremely powerful hypothesis in adelic physics [40, 39] and would mean a revolution in biology and consciousness theory.

   Does P fix the space-time surface with the properties needed to realize $M^8 - H$ duality or is something more needed? Does the polynomial fix the space-time surface uniquely - one would have extremely strong number theoretic holography - so that one would have number theoretic holography with coefficients of a real polynomial determining the space-time surface?

3. $M^8 - H$ duality involves mapping of $M^4 \subset M^8$ to $M^4 \subset H$. Hitherto it has been assumed that this map is direct identification. The form of map should however depend on the interpretation of M^8. In octonionic Dirac equation M^8 coordinates are in the role of momenta [49]. This suggests the interpretation of M^8 as an analog of 8-D momentum space. If this interpretation is correct, Uncertainty Principles demands that the map $M^4 \subset M^8 \to M^4 \subset H$ is analogous to inversion mapping large momenta to small distances.

4. Twistor lift of TGD [30] is an essential part of the TGD picture. Ttwistors and momentum twistors provide dual approaches to twistor Grassmann amplitudes. Octonionic Dirac equation suggests that M^8 and H are in a similar dual relation. Could $M^8 - H$ duality allow a generalization of twistorial duality to TGD framework?

3.1 Explicit form of the octonionic polynomial

What does the identification of the octonionic polynomial P as an octonionic continuation of a polynomial with real or complexified coefficients imply? In the following I regard M^8_c as O^8_c and consider products for complexified octonions.

Remark: In adelic vision the coefficients of P must be rationals (or at most algebraic numbers in some extension of rationals).

One interesting situation corresponds to the real subspace of O_c spanned by $\{I_0, iI_k\}, = 1, ..7$, with a number theoretic metric signature $(1, -1, -1..., -1)$ of M^8 which is complex valued except at in various reals subspaces.This subspace is associative. The original proposal was that Minkowskian space-time regions as projections to this signature are associative whereas Euclidian regions are co-associative. It however turned out that associative space-time surfaces are physically uninteresting.

The canonical choice $(iI_0, I_1, I_2, iI_3, I_4, iI_5, I_6, iI_7)$ defining the complexification of the tangent space represents a co-associative sub-space realizing Minkowski signature. It turns out that both Minkowskian and Euclidian space-time regions must be co-associative .

3.1.1 Surprises

The explicit calculation of the octonionic polynomial yielded a chilling result. If one poses (co-)associativity conditions as vanishing of the imaginary or real part in quaterionic sense: $Im_Q(P) = 0$ or $Re_Q(P) = 0$, the outcome is that the space-time surface is just M^4 or E^4. Second chilling result is that quaternionic sub-manifolds are geodesic sub-manifolds. This led to the question how to modify the (co-)associativity hypothesis.

The vision has been that space-time surfaces can be identified as roots for the imaginary (co-associative) part $Im_Q(O)$ or real part $Im_Q(O)$ of octonionic polynomial using the standard decomposition $(1, e_1, e_2, e_3)$.

1. The naïve counting of dimensions suggests that one obtains 4-D surfaces. The surprise was that also 6-D brane like entities located at the boundary of M^8 light-cone and with topology of 6-sphere S^6 are possible. They correspond to the roots of a real polynomial $P(o)$ for the choice $(1, iI_1, ..., iI_7)$. The roots correspond to the values of the real octonion coordinate interpreted as values of linear M^4 time in the proposal considered. Also for the canonical proposal one obtains a similar result. In O_c they correspond to 12-D complex surfaces X^6_c satisfying the same condition conditions $x_0^2 + r^2 = 0$ and $P(x_0) = 0$.

2. There was also another surprise. As already described, the general form for the octonionic polynomial $P(o)$ induced from a real polynomial is extremely simple and reduces to $X(t^2, r^2)I_0 + iY(t^2, r^2)Im(o)$. There are only two complex variables t and r^2 involved and the solutions of $P = 0$ are 12-D complex surfaces X^6_c in O_c. Also the special solutions have the same dimension.

3. In the case of co-associativity 8 conditions are needed for $Re_Q(P) = 0$: note that $X = 0$ is required. This gives a complex manifold X^4_c with 4-D real projection X^4_r as an excellent candidate for co-associative surface. One can consider adding the condition $Y = 0$. The naïve expectation is that this gives a 6-D solution X^3_c with 3-D real projection X^3_r. The expectation turned out to be wrong since by the Lorentz invariance the froots of both $X = 0$ and $Y = 0$ are values of complex valued a_c^2 representing complex valued light-cone proper time. Simultaneous solution corresponds to a common root of X and Y. Either the intersection is empty or X^4_r is contained by X^6_r.

   X^4_r should be co-associative and in the simplest situation would have a fixed M^2 in its normal space. Does the co-associativity of the sub-space for the projection guarantee this? If this is the case, one can apply $M^8 - H$ duality and map the space-time surfaces to H.

4. One can also pose the associativity condition $Im_Q(P) = 0$ giving $4 \times 2 = 8$ conditions giving a complex manifold X^5_c having 5-D real projection X^5_r This certainly fails to give associative. The additional conditions $X = 0, Y = 0$ are naïvely expected to give a complex surface X^2_c with 2-D real projection X^2_r. $X = 0 and Y = 0$ however have either no solution or X and Y have a common root. In the latter case the dimension of the solution reduces to $D = 4$. X^4_r would represent the boundary of X^5_r and brane interpretation would be appropriate. One could also have either $X = 0$ or $Y = 0$: this would give X^4_c with 4-D real projection but we know that it cannot be associative so that $M^8 - CP_2$ duality fails.

   The conclusion is that $Re_Q(P) = 0$ gives 4-D possibly co-associative surface whereas $Im_Q(P) = 0$ gives a 5-D surface X^5_r.

3.1.2 General form of P and of the solutions to $P = 0$, $Re_Q(P) = 0$, and $Im_Q(P) = 0$

It is convenient to introduce complex coordinates for O_c since the formulas obtained allow projections to various real sections of O_c.

1. To see what happens, one can calculate o_c^2. Denote o_c by $o_c = tI_0 + \overline{o_c}$ and the norm squared of $\overline{o}$ by r^2, where $r^2 = \sum o_k^2$ where o_k are the complex coordinates of octonion. Number theoretic norm

squared for o_c is $t^2 + r^2$ and reduces to a real number in the real sections of O_c. For instance in the section $(I_1, iI3, iI_5, iI_7)$ the norm squared is $-x_1^2 + x_3^2 + x_5^2 + x_7^2$ and defines Minkowskian norm squared.

For o^2 one has:

$$o^2 = t^2 - r^2 + 2t\overline{o} \equiv X_2 + \overline{Y}_2 \ \ .$$

For o^3 one obtains

$$o^3 = tX - \overline{o} \cdot \overline{Y} + t\overline{Y} + X\overline{o} \ \ .$$

Clearly, $Im_Q(o^n)$ has always the same direction as $Im_Q(o)$. Hence one can write in the general case

$$o^n = X + Y\overline{o} \ \ . \tag{3.1}$$

This trivial result was obtained years ago but its full implications became evident only while preparing the current article. The point is that the solutions to associativity/co-associativity conditions by putting $Re(Q(P) = 0$ or $Im_Q(P) = 0$ are trivial: just M^4 or E^4. What goes wrong with basic assumptions, will be discussed later.

Remark: In M^8 sub-space one has imaginary $\overline{o}$ is proportional to the commuting imaginary unit.

2. It is easy to deduce a recursion formula for the coefficients for X and Y for n:th power of o_c. Denote by t the coordinate associated with the real octonion unit (not time coordinate). One obtains

$$\begin{aligned} o_c^n &= X_nI_0 + Y_n\overline{o} \ \ , \\ X_n &= tX_{n-1} - Y_{n-1} \ \ , \\ Y_n &= tY_{n-1} + X_{n-1} \ \ . \end{aligned} \tag{3.2}$$

In the co-associative case one has $t = 0$ or possibly constant $t = T$ (note that in the recent interpretation t does not have interpretation as time coordinate). The reason is that the choice of octonionic coordinates is unique apart from translation along the real axis from the condition that the coefficients of P remain complex numbers in powers of the new variable.

3. For $t = 0$ the recursion formula gives for the polynomial $P(o_c)$ the expression

$$P(o_c) = \sum (-1)^n r^{2n} (p_{2n-1}I_0 + p_{2n}\overline{o}) \ \ . \tag{3.3}$$

Denoting the even and of odd parts of P by P_{even} and P_{odd}, the roots r_k of $X = Re(P(o_c))$ are roots P_{odd} and roots r^2 of $Y = Im(P(o_c))$ are roots of P_{even}. Co-associativity gives roots of X and the roots of P as simultaneous roots of P_{odd} and P_{even}.

4. In the generic situation the solutions of $X = 0$ *resp.* $X = 0, Y = 0$ would be 4-D *resp.* 3-D complex surfaces. One does not have a generic situation now! Basically due to Lorentz invariance, one has two ordinary polynomial equations giving ordinary complex numbers as roots! The roots of a_c^2 for $X = 0$ and $Y = 0$ must be the same. This requires that the corresponding polynomials have a common root. Otherwise the intersection of X_r^4 with X_r^6 is empty!

5. For the co-associative option corresponding to the canonical choice with Minkowski signature the condition $X = 0$ gives the Lorentz invariant square of the complexified Minkowski norm as complex root $r_{k,odd}$ of P_{odd}:

$$a_c^2 = Re(m^2) - Im(m^2) + 2iRe(m) \cdot Im(m) = r_{k,odd} \quad . \tag{3.4}$$

   For $Y = 0$ the solution exists only if X and Y have a common root! In this case however one does not have 3-D surface X_r^3 but X_r^4 belongs to X_r^6. The idea about X_r^4 as brane-like entity connecting two 3-D regions of X_r^6 fails.

A couple of remarks are in order.

1. If all roots are identical for P_{even} and P_{odd}, one has $P_{n-1} = P_n$ and $p_{n-1} = p_n$. If P_{even} vanishes it poses no conditions: the surface $X = 0, Y = 0$ is 14-D whereas the space-time surface is 4-D.

2. Non vanishing mass squared values for octonionic spinors correspond to common roots of P_{even} and P_{odd}. When P_{even} vanishes all roots of $P = P_{odd}$ are allowed. The special solutions to $P = 0$ discovered already earlier are restricted to the boundary of CD_8 and correspond to the values of mass (rather than mass squared) coming as roots of the real polynomial P. These mass values are mapped by inversion to "very special moments in the life of self" at the level of H as special values of light-cone proper time rather than linear Minkowski time as in the earlier interpretation [41]. The new picture is Lorenz invariant.

The following summarizes the situation.

1. For the roots of P one has $X = 0, Y = 0$. The two complex conditions give a 6-D complex surface X_c^6 having real projection X_r^6. The roots of X *resp.* Y are permuted by Galois groups with n elements. The condition $Im_Q(P) = 0$ associated with the possibly associative space-time surface gives 4 complex conditions reducing giving X_c^4 possibly identifiable as 4-D space-time surface. Associativity however fails. Galois group of n-elements is involved. This means that the surface is n-sheeted.

   The condition $Re_Q(P) = 0$ gives the condition $X = 0$ plus 3 complex conditions for the remaining components of $Re_Q(P)$. This gives X_c^4 having 4-D real section X_r^4 serving as a candidate for co-associative space-time surface. The condition $X = 0, Y = 0$ has no solutions or implies that X_r^4 belongs to X_r^6.

   The experience with the octonionic Dirac equation [49] reducing to mass shell condition - to be discussed in the sequel in detail - forces the interpretation $m \cdot m$ as mass squared. M^8 would be the analog of momentum space. For the common root X_r^4 allows massive quarks. If the common root does not exist, only massless momenta arriving at the boundary of CD are possible. The emergence of a common root clearly represents a phase transition from a massless to massive phase. For given P there are both massive and massless phases and in the generic situation only massless phase.

2. Could X_r^4 be (co-)associative?: the answer is affirmative [6].

3.1.3 What about string world sheets and partonic 2-surfaces?

One can apply the above arguments also to the identification of 2-D string world sheets and partonic 2-surfaces.

1. One has two kinds of solutions: M^2 and 3-D surfaces of X^4 as analogs of 6-brane. The interpretation for 3-D *resp.* 2-D branes as a light-like 3-surface associated with the octonionic Dirac equation representing mass shell condition *resp.* string world sheet is attractive.

2. M^2 would be replaced with an integrable distribution of $M^2(x)$ in local tangent space $M^4(x)$. The space for the choices of $M^2(x)$ would be S^3 corresponding to the selection of a preferred quaternion imaginary unit equal to the choices of preferred octonion imaginary unit.

   The choices of the preferred complex subspace $M^2(x)$ at a given point would be characterized by its normal vector and parameterized by sphere S^2: the interpretation as a quantization axis of angular momentum is suggestive. One would have space $S^3 \times S^2$. Also now the integrability conditions $de_A = 0$ would hold true.

3. String world sheets could be regarded as analogs of superstrings connecting 3-D brane like entities defined by the light-like partonic orbits. The partonic 2-surfaces at the ends of light-like orbits defining also vertices could correspond to the 3-surfaces at which quaternionic 4-surfaces intersect 6-branes.

3.2 Is (co-)associativity possible?

The number theoretic vision relying on the assumption that space-time surfaces are 8-D complex 4-surfaces in o_c^8 determined as algebraic surfaces for octonionic continuations of real polynomials, which for adelic physics would have coefficients which are rational or belong to an extension of rationals. The projections to subspaces Re^8 of o_c^8 defined as space for which given coordinate is purely real or imaginary so that complexified octonionic norm is real would give rise to real 4-D space-time surfaces. $M^8 - H$ duality would map these surfaces to geometric objects in $M^4 \times CP_2$. This vision involves several poorly understood aspects and it is good to start by analyzing them.

3.2.1 Challenging the notions of associativity and co-associativity

Consider first the notions of associativity *resp.* co-associativity equivalent with quaternionicity *resp.* co-quaternionicity. The original hope was that both options are possible for surfaces of real sub-spaces of O_c ("real" means here that complexified octonionic metric is real).

1. The original idea was that the associativity of the tangent space or normal space of a real space-time surface X^4 reduces the classical physics at the level of M^8 to associativity. Associativity/co-associativity of the space-time surface states that at each point of the tangent-/normal space of the real space-time surface in O is quaternionic. The notion generalizes also to $X_c^4 \subset O_c^8$. (Co-)associativity makes sense also for the real subspaces space of O with Minkowskian signature.

2. It has been however unclear whether (co-)associativity is possible. The cold shower came as I learned that associativity allows only for geodesic sub-manifolds of quaternionic spaces about which octonions provide an example [6]. The good news was that the distribution of co-associative tangent spaces always defines an integrable distribution in the sense that one can find sub-manifold for which the associative normal space at a given point has tangent space as an orthogonal complement. Should the number theoretic dynamics rely on co-associativity rather than associativity?

3. Minkowskian space-time regions have been assumed to be associative and to correspond to the projection to the standard choice for basis as $\{1, iI_1, iI_2, iI_3\}$. The octonionic units $\{1, I_1, I_2, I_3\}$ define quaternionic units and associative subspace and their products with unit I_4 define the orthogonal co-associative subspace as $\{I_4, I_5 = I4I_1, I_6 = I_4I_2, I_7 = I_4I_3\}$. This result forces either to weaken the notion of associativity or to consider alternative identifications of Minkowskian regions, which can be co-associative: fortunately, there exists a large number of candidates.

The article [6] indeed kills the idea about the associativity of the space-time surface. The article starts from a rather disappointing observation that associative sub-manifolds are geodesic sub-manifolds and therefore trivial. Co-associative quaternion sub-manifolds are however possible. With a motivation coming from this observation, the article discusses what the author calls RC quaternionic sub-manifolds of quaternion manifolds. For a quaternion manifold the tangent space allows a realization of quaternionic units as antisymmetric tensors. These manifolds are constant curvature spaces and typically homogeneous spaces.

1. Quaternion sub-manifold allows a 4-D integrable distribution of quaternion units. The normal complement of this distribution is expressible in terms of the second fundamental form and the condition that it is trivial implies that the second fundamental form is vanishing so that one has a geodesic submanifold. Quaternionic sub-manifolds are thus too trivial to be interesting. As a diametric opposite, one can also define totally real submanifolds for which the normal space contains a distribution of quaternion units. In this case the distribution is always integrable. This case is much more interesting from the TGD point of view.

2. Author introduces the notion of CR quaternion sub-manifold $N \subset M$, where M is quaternion manifold with constant sectional curvatures. N has quaternion distribution D in its tangent spaces if the action of quaternion units takes D to itself. $D^\perp$ is the co-quaternionic orthogonal complement D in the normal space N. D would take also $D^\perp$ to itself. $D^\perp$ can be expressed in terms of the components of the second fundamental form and vanishes for quaternion sub-manifolds.

3. Author deduces results about CR quaternion sub-manifolds, which are very interesting from the TGD point of view.

   (a) Sub-manifold is CR quaternion sub-manifold only if the curvature tensor of R_M of the imbedding space satisfies $R_M(D, D, D^\perp, D\ \ ) = 0$. The condition is trivial if the quaternion manifold is flat. In the case of octonions this would be the case.

   (b) D is integrable only if the second fundamental form restricted to it vanishes meaning that one has a geodesic manifold. Totally real distribution $D^\perp$ is always integrable to a co-associative surface.

   (c) If $D^\perp$ integrates to a minimal surface then N itself is a minimal surface.

Could one consider RC quaternion sub-manifolds in TGD framework? Both octonions and their complexifixation can be regarded as quaternionic spaces. Consider the real case.

1. If the entire D is quaternionic then N is a geodesic sub-manifold. This would leave only E^4 and its Minkowskian variants with various signatures. One could have however 4-D totally real (co-associative) space-time surfaces. Simple arguments will show that the intersections of the conjectured quaternionic and co-quaternionic 4-surfaces have 2- and 3-D intersections with 6-branes.

   Should one replace associative space-time surfaces with CR sub-manifolds with $d \leq 3$ integrable distribution D whereas the co-quaternionic surfaces would be completely real having 4-D integrable $D^\perp$? Could one have 4-D co-associative surfaces for which $D^\perp$ integrates to $n \geq 1$-dimensional minimal surface (geodesic line) and the X^4 itself is a minimal surface?

   Partially associative CR manifold do not allow M^8H duality. Only co-associative surfaces allow it and also their signature must be Minkowskian: the original idea [43, 36, 37, 38] about Euclidian (Minkowskian) signature for co-associative (associative) surfaces was wrong.

2. The integrable 2-D sub-distributions D defining a distribution of normal planes could define foliations of the X^4 by 2-D surfaces. What springs to mind is foliations by string world sheets and partonic 2 surfaces orthogonal to them and light-like 3-surfaces and strings transversal to them. This expectation is realized.

3.2.2 How to identify the Minkowskian sub-space of O_c?

There are several identifications of subspaces of O_c with Minkowskian signature. What is the correct choice has been far from obvious. Here symmetries come in rescue.

1. Any subspace of O^c with 3 (1) imaginary coordinates and 1 (3) real coordinates has Minkowskian signature in octonionic norm algebraically continued to O_c (complex valued continuation of real octonion norm instead of real valued Hilbert space norm for O_c). Minkowskian regions should have local tangent space basis consisting of octonion units which in the canonical case would be $\{I_1, iI_3, iI_5, iI_7\}$, where i is commutative imaginary unit. This particular basis is co-associative having whereas its complement $\{iI_0, I_2, I_4, I_6\}$ is associative and has also Minkowskian signature.

2. The size of the isometry group of the subspace of M^8_c depends on whether the tangent basis contains real octonion unit 1 or not. The isometry group for the basis containing I_0 is $SO(3)$ acting as automorphisms of quaternions and $SO(k, 3-k)$ when $3-k$ units are proportional to i. The reason is that G_2 (and its complexification $G_{2,c}$) and its subgroups do not affect I_0. For the tangent spaces built from 4 imaginary units I_k and iI_l the isometry group is $SO(k, 4-k) \subset G_{2,c}$.

   The choice therefore allows larger isometry groups and also co-associativity is possible for a suitable choice of the basis. The choice $\{I_1, iI_3, iI_5, iI_7\}$ is a representative example, which will be called canonical basis. For these options the isometry group is the desired $SO(1, 3)$ as an algebraic continuation of $SO(4) \subset G_2$ acting in $\{I_1, I_3, I_5, I_7\}$, to $SO(1, 3) \subset G_{2,c}$.

   Also Minkowskian signature - for instance for the original canonical choice $\{I_0, iI_1, iI_2, iI_3\}$ - can have only $SO(k, 3-k)$ as isometries. This is the basic objection against the original choice $\{I_0, iI_1, iI_2, iI_3\}$. This identification would force the realization of $SO(1, 3)$ as a subgroup of $SO(1, 7)$. Different states of motion for a particle require different octonion structure with different direction of the octonion real axis in M^8. The introduction of the notion of moduli space for octonion structures does not look elegant. For the option $\{I_1, iI_3, iI_5, iI_7\}$ only a single octonion structure is needed and $G_{2,c}$ contains $SO(1, 3)$.

   Note that also the signatures $(4, 0)$, $(0, 4)$ and $(2, 2)$ are possible and the challenge is to understand why only the signature (1,3) is realized physically.

Co-associative option is definitely the only physical alternative. The original proposal for the interpretation of the Minkowski space in terms of an associative real sub-space of M^4 had a serious problem. Since time axis was identified as octonionic real axis, one had to assign different octonion structure to particles with non-parallel moment: $SO(1, 7)$ would relate these structures: how to glue the space-time surfaces with different octonion structures together was the problem. This problem disappears now. One can simply assign to particles with different state of motion real space-time surface defined related to each other by a transformation in $SO(1, 3) \subset G_{2,c}$.

3.2.3 The condition that $M^8 - H$ duality makes sense

The condition that $M^8 - H$ duality makes sense poses strong conditions on the choice of the real sub-space of M^8_c.

1. The condition that tangent space of O_c has a complexified basis allowing a decomposition to representations of $SU(3) \subset G_2$ is essential for the map to $M^8 \to H$ although it is not enough. The standard representation of this kind has basis $\{\pm iI_0 + I_1\}$ behaving like $SU(3)$ singlets $\{I_2 + \epsilon iI_3, I_4 + \epsilon iI_5, \epsilon I_6 \pm iI_7\}$ behaves like SU(3) triplet 3 for $\epsilon = 1$ and its conjugate $\overline{3}$ for $\epsilon = -1$. $G_{2,c}$ provides new choices of the tangent space basis consistent with this choice. $SU(3)$ leaves the direction I_1 unaffected but more general transformations act as Lorentz transformation changing its direction but not leaving the M^4 plane. Even more general $G_{2,c}$ transformations changing M^4 itself are in principle possible.

Interestingly, for the canonical choice the co-associative choice has $SO(1,3)$ as isometry group whereas the complementary choice failing to be associative correspond to a smaller isometry group $SO(3)$. The choice with M^4 signature and co-associativity would provide the highest symmetries. For the real projections with signature $(2,2)$ neither consistent with color structure, neither full associativity nor co-associativity is possible.

2. The second essential prerequisite of $M^8 - H$ duality is that the tangent space is not only (co-)associative but contains also (co-)complex - and thus (co-)commutative - plane. A more general assumption would be that a co-associative space-time surface contains an integrable distribution of planes $M^2(x)$, which could as a special case reduce to M^2.

   The proposal has been that this integrable distribution of $M^2(x)$ could correspond to string sheets and possibly also integrable orthogonal distribution of their co-complex orthogonal complements as tangent spaces of partonic 2-surfaces defining a slicings of the space-time surface. It is now clear that this dream cannot be realized since the space-time surface cannot be even associative unless it is just E^4 or its Minkowskian variants.

3. As already noticed, any distribution of the associative normal spaces integrates to a co-associative space-time surface. Could the normal spaces also contain an integrable distribution of co-complex planes defined by octonionic real unit 1 and real unit $I_k(x)$, most naturally I_1 in the canonical example? This would give co-commutative string world sheet. Commutativity would be realized at the 2-D level and associativity at space-time level. The signature of this plane could be Minkowskian or Euclidian. For the canonical example $\{I_1, iI_3, iI_5, iI_7\}$ the 2-D complex plane in quaternionic sense would correspond to $(a \times 1, +n_2I_2 + n_4I_6 + n_6I_6$, where the unit vector n_i has real components and one has $a = 1$ or $a = i$ is forced by the complexification as in the canonical example.

   Since the distribution of normal planes integrates to a 4-surface, one expects that its sub-distribution consting of commutative planes integrates to 2-D surface inside space-time surface and defines the counterpart of string worlds sheet. Also its normal complement could integrate to a counterpart of partonic 2-surface and a slicing of space-time surface by these surfaces would be obtained.

4. The simplest option is that the commutative space does not depend on position at X^4. This means a choice of a fixed octonionic imaginary unit, most naturally I_1 for the canonical option. This would make $SU(3)$ and its sub-group $U(2)$ independent of position. In this case the identification of the point of $CP_2 = SU(3)/U(2)$ labelling the normal space at a given point is unique.

   For a position dependent choice $SU(3)(x)$ it is not clear how to make the specification of $U(2)(x)$ unique: it would seem that one must specify a unique element of $G_2(x)$ relating $SU(3)(x)$ to a choice at special point x_0 and defining the conjugation of both $SU(3)(x)$ and $U(2)(x)$. Otherwise one can have problems. This would also mean a unique choice for the direction of time axis in O and fixing of $SO(1,3)$ as a subgroup of $G_{2,c}$. Also this distribution of associative normal spaces is integrable. Physically this option is attractive but an open question is whether it is consistent with the identification of space-time surfaces as roots $Re_Q(P) = 0$ of P.

3.2.4 Co-associativity from octonion analyticity or/and from G_2 holography?

Candidates for co-associative space-time surfaces X^4_r are defined as restrictions X^4_r for the roots X^4_c of the octonionic polynomials such that the O_c coordinates in the complement of a real co-associative sub-space of O_c vanish or are constant. Could the surfaces X^4_r or even X^4_c be co-associative?

1. X^4_r is analogous to the image of real or imaginary axis under a holomorphic map and defines a curve in complex plane preserving angles. The tangent vectors of X^4_r and X^4_c involve gradients of all coordinates of O_c and are expressible in terms of all octonionic unit vectors. It is not obvious that their products would belong to the normal space of X^4_r a strong condition would be that this is the case for X^4_c.

2. Could octonion analyticity in the proposed sense guarantee this? The products of octonion units also in the tangent space of the image would be orthogonal to the tangent space. Ordinary complex functions preserve angles, in particular, the angle between x- and y-axis is preserved since the images of coordinate curves are orthogonal. Octonion analyticity would preserve the orthogonality between tangent space vectors and their products.

3. This idea could be killed if one could apply the same approach to associative case but this is not possible! The point is that when the real tangent space of O_c contains the real octonion unit, the candidate for the 4-D space-time surface is a complex surface X_c^2. The number theoretic metric is real only for 2-D X_r^2 so that one obtains string theory with co-associativity replaced with co-commutativity and $M^4 \times CP_2$ with $M^2 \times S^2$. One could of course ask whether this option could be regarded as a "sub-theory" of the full theory.

My luck was that I did not realize the meaning of the difference between the two cases first and realized that one can imagine an alternative approach.

1. G_2 as an automorphism group of octonions preserves co-associativity. Could the image of a co-associative sub-space of O_c defined by an octonion analytic map be regarded as an image under a local G_2 gauge transformation. $SU(3) \subset G_2$ is an especially interesting subgroup since it could have a physical interpretation as a color gauge group. This would also give a direct connection with $M^8 - H$ duality since $SU(3)$ corresponds to the gauge group of the color gauge field in H.

2. One can counter-argue that an analog of pure gauge field configuration is in question at the level of M^8. But is a pure gauge configuration for $G_{2,c}$ a pure gauge configuration for G_2? The point is that the $G_{2,c}$ connection $g^{-1}\partial_\mu g$ trivial for $G_{2,c}$ contains by non-linearity cross terms from $g_2 g, c = g_{2,1} + i g_{2,2}$, which are of type $Re = X[g_{2,1}, g_{2,1}] - X[g_{2,2}, g_{2,2}] = 0$ and $Im = iZ[g_{2,1}, g_{2,2}] = 0$. If one puts $g_{2,2}$ contributions to zero, one obtains $Re = X[g_{2,1}, g_{2,1}]$, which does not vanish so that $SU(3)$ gauge field is non-trivial.

3. X_r^4 could be also obtained as a map of the co-associative M^4 plane by a local $G_{2,c}$ element. It will turn out that $G_{2,c}$ could give rise to the speculated Yangian symmetry [35] at string world sheets analogous to Kac-Moody symmetry and gauge symmetry and crucial for the construction of scattering amplitudes in $M8$.

4. The decomposition of the co-associative real plane of O_c should contain a preferred complex plane for $M^8 - H$ duality to make sense. $G_{2,c}$ transformation should trivially preserve this property so that SH would not be necessary at H side anymore.

There is a strong motivation to guess that the two options are equivalent so that $G_{2,c}$ holography would be equivalent with octonion analyticity. The original dream was that octonion analyticity would realize both associative and co-associative dynamics but was exaggeration!

3.2.5 Does one obtain partonic 2-surfaces and strings at boundaries of ΔCD_8?

It is interesting to look for the dimensions of the intersections of the light-like branes at the boundary of CD_8 giving rise to the boundary of CD_4 in M^4 to see whether it gives justification for the existing phenomenological picture involving light-like orbits of partonic 2-surfaces connected by string world sheets.

1. Complex light-cone boundary has dimension $D = 14$. $P = 0$ as an additional condition at δCD_8 gives 2 complex conditions and defines a 10-D surface having 5-D real projections.

2. The condition $Im_Q(P) = 0$ gives 8 conditions and gives a 2-D complex surface with 1-D real projection. The condition $Re_Q(P) = 0$ gives 3 complex conditions since $X = 0$ is already satisfied and the solution is a 4-D surface having 2-D real projection. Could the interpretation be in terms of the intersection of the orbit of a light-like partonic surface with the boundary of CD_8?

3. Associativity is however not a working option. If only co-associative Minkowskian surfaces allowing mapping to H without SH are present then only 4-D space-time surfaces with Minkowskian signature, only partonic 2-surfaces and their light-like orbits would emerge from co-associativity.

   This option would not allow string world sheets for which there is a strong intuitive support. What could a co-complex 2-surface of a co-associative manifold mean? In the co-associative case the products of octonion imaginary units are in the normal space of space-time surface. Could co-complex surface $X^2_c \subset X^4_c$ be defined by an integrable co-complex sub-distribution of co-associative distribution. The 4-D distribution of normal planes is always integrable.

   Could the 2-D sub-distributions of co-associative distribution integrate trivially and define slicings by string world sheets or partonic 2-surfaces. Could the distribution of string distributions and its orthogonal complement be both integrable and provide orthogonal slicings by string world sheets and partonic 2-surfaces? String world sheets with Minkowskian signature should intersect the partonic orbits with Euclidian signature along light-like lines. This brings in mind the orthogonal grid of flow lines defined by the $Re(f) = 0$ and $Im(f) = 0$ lines of an analytic function in plane.

4. In this picture the partonic 2-surfaces associated with light-like 3-surface would be physically unique and could serve as boundary values for the distributions of partonic 2-surfaces. But what about string world sheets connecting them? Why would some string world sheets be exceptional? String world sheets would have a light-like curve as an intersection with the partonic orbit but this is not enough.

   Could the physically special string world sheets connect two partonic surfaces? Could the string associated with a generic string world sheet be like a flow line in a hydrodynamic flow past an obstacle - the partonic 2-surface? The string as a flowline would go around the obstacle along either side but there would be one line which ends up to the object.

Interactions would correspond geometrically to the intersections of co-associative space-time surfaces X^4_r associated with particles and corresponding to different real sub-spaces of O_c related by Lorentz boost in $SO(1,3) \subset G_{2,c}$. In the generic case the intersection would be discrete. In the case that X and Y have a common root the real surfaces $X^4_r \subset X^6_r$ associated with quarks and depending on their state of motion would reside inside the same 6-D surface X^6_r and have a 2-D surface X^2_r as intersection. Could this surface be interpreted as a partonic 2-surface? One must however bear in mind that partonic 2-surfaces as topological vertices are assumed to be non-generic in the sense that the light-like partonic orbits meet at them. At the level of H, the intersections would be partonic 2-surfaces X^2 at which the four 3-D partonic orbits would meet along their ends. Does this hold true at the level of M^8? Or can it hold true even at the level H?

The simplest situation corresponds to 4 external quarks. There are 6 different intersections. Not all of them are realized since a given quark can belong only to a single intersection. One must have two disjoint pairs -say 12 and 34. Most naturally positive *resp.* negative energy quarks form a pair. These pairs are located in different half-cones. The intersections would give two partonic 2-surfaces and this situation would be generic. This suggests a modification of the description of particle reaction in M^8 . $M^8 - H$ duality suggests a similar description in H.

3.2.6 What could be the counterparts of wormhole contacts at the level of M^8?

The experience with H, in particular the presence of extremals with Euclidian signature of the induced metric and identified as building bricks of elementary particles, suggest that also the light-like 3-surfaces in M^8_c could have a continuation with an Euclidian signature of the number theoretic metric with norm having real values only for the projections to planes allowing real coordinates.

The earlier picture has been that the wormhole contacts as CP_2 type extremals correspond to co-associative regions and their exteriors to associative regions. If one wants $M^8 - H$ duality in strong form and thus without need for SH, one should assume that both these regions are co-associative.

1. The simplest option is that the real Minkowskian time coordinate becomes imaginary. Instead of the canonical (I_1, iI_3, iI_5, iI_7) the basis would be (iI_1, iI_3, iI_5, iI_7) having Euclidian signature and SO(4) as isometry group. The signature would naturally change at light-like 3-surface the time coordinate along light-like curves becomes zero - proper time for photon vanishes - and can ransforms continuously from real to imaginary.

2. Wormhole contacts in H behave like pairs of magnetic monopoles with monopole charges at throats. If one does not allow point-like singularity, the monopole flux must go to a parallel Minkowskian space-time sheet through the opposite wormhole throat. Wormhole contact with effective magnetic charge would correspond in M^8_c to a distribution of normal 4-planes at the partonic 2-surfaces analogous to the radial magnetic field of monopole at a sphere surrounding it. To avoid singularity of the distribution, there must be another light-like 3-surface M^8 such that its partonic throat has a topologically similar distribution of normal planes.

In the case of X^3_c dimension does not allow co-quaternion structure: could they allow 4-D co-associative sub-manifolds? It will be found that this option is not included since co-associative tangent space distributions in a quaternion manifold (now O) are always integrable.

3.3 Octonionic Dirac equation and co-associativity

Also the role of associativity concerning octonionic Dirac equation in M^8 must be understood. It is found that co-associativity allows very elegant formulation and suggests the identification of the points appearing as the ends of quark propagator lines in H as points of boundary of CD representing light-like momenta of quarks. Partonic vertices would involve sub-CDs and momentum conservation would have purely geometric meaning bringing strongly in mind twistor Grassmannian approach [13, 12, 14]. I have discussed the twistor lift of TGD replacing twistors as fields with surfaces in twistor space having induced twistor structure in [30, 29, 31] [46, 47].

3.3.1 Octonionic Dirac equation

The following arguments lead to the understanding of co-associativity in the case of octonion spinors. The constant spinor basis includes all spinors but the gamma matrices appearing in the octonionic Dirac equation correspond to co-associative octonion units.

1. At the level of O_c the idea about massless Dirac equation as partial differential equation does not make sense. Dirac equation must be algebraic and the obvious idea is that it corresponds to the on mass shell condition for a mode of ordinary Dirac equation with well-define momentum: $p^k\gamma_k\Psi = 0$ satisfying $p^kp_k = 0$. This suggests that octonionic polynomial P defines the counterpart of $p^k\gamma_k$ so that gamma matrices γ_k would be represented as octonion components. Does this make sense?

2. Can one construct octonionic counterparts of gamma matrices? The imaginary octonion units I_k indeed define the analogs of gamma matrices as $\gamma_k \equiv iI_k$ satisfying the conditions $\{\gamma_k, \gamma_l\} = 2\delta_{kl}$ defining Euclidian gamma matrices. The problem is that one has $I_0I_lk + I_kI_0 = 2I_k$. One manner to solve the problem would be to consider tensor products $I_0\sigma_3$ and $I_k\sigma_2$ where σ_3 and $sigma_2$ are Pauli's sigma matrices with anti-commutation relations $\{\sigma_i, \sigma_j\} = \delta_{i,j}$. Note that I_k do not allow a matrix representation.

   Co-associativity condition suggests an alternative solution. The restriction of momenta to be co-associative and therefore vanishing component p^0 as octonion, would selects a sub-space spanned by say the canonical choice $\{I_2, iI3, iI_5, iI_7\}$ satisfying the anticommutation relations of Minkowskian gamma matrices. Octonion units do not allow a matrix representation because they are not associative. The products for a co-associative subset of octonion units are however associative $(a(bc) = (ab)c$ so that they can be mapped to standard gamma matrices in Minkowski space. Co-associativity would allow the representation of 4-D gamma matrices as a maximal associative subset of octonion units.

3. What about octonionic spinors. The modes of the ordinary Dirac equation with a well-defined momentum are obtained by applying the Dirac operator to an orthogonal basis of constant spinors u_i to give $\Psi = p^k\gamma_k u_i$. Now the counterparts of constant spinors u_i would naturally be octonion units $\{I_0, I_k\}$: this would give the needed number 8 of real spinor components as one has for quark spinors.

   Dirac equation reduces to light-likeness conditions $p^k p_k = 0$ and p_k must be chosen to be real - if pk are complex, the real and imaginary parts of momentum are parallel. One would obtain an entire 3-D mass shell of solution and a single mode of Dirac equation would correspond to a point of this mass shell.

   Remark: Octonionic Dirac equation is associative since one has a product of form $(p_k\gamma_k)^2 u_i$ and octonion products of type x^2y are associative.

4. p^k would correspond to the restriction of $P(o_c)$ to M^4 as sub-space of octonions. Since co-associativity implies $P(o_c) = Y(o_c)o_c$ restricted to counterpart of M^4 (say subspace spanned by $\{I_2, iI3, iI_5, iI_7\}$), Dirac equation reduces to the condition $o^k o_k = 0$ in M^4 defining a light-cone of M^4. This light-cone is mapped to a curved light-like 3-surface X^3 in o_c as $o_c \rightarrow P(o_c) = Yo_c$. M^8-H duality maps points of space-time surface on M^8 H and therefore the light-cone of M^4 corresponds to either light-like boundary of CD. It seems that the image of X^3 in H has M^4 projection to the light-like boundary of CD.

   Co-associative space-time surfaces have 3-D intersections X^3 with the surface $P = 0$: the conjecture is that X^3 corresponds to a light-like orbit of partonic 2-surfaces in H at which the induced metric signature changes. At X^3 one has besides $X = 0$ also $Y = 0$ so that octonionic Dirac equation $P(o_c)\Psi = P^k I_k\Psi = Yp^k I_k\Psi = 0$ is trivially satisfied for all momenta $p^k = o^k$ defined by the M^4 projections of points of X^3 and one would have $P^k = Yp^k = 0$ so that the identification of P^k as 4-momentum would not allow to assign non-vanishing momenta to X^3. The direction of p^k is constrained only by the condition of belonging to X^3 and the momentum would be in general time-like since X^3 is inside future light-cone.

   $Y = 0$ condition conforms with the proposal that X^3 defines a boundary of Minkowskian and Euclidian region: Euclidian mass shell condition for real P^k requires $P^k = 0$. The general complex solution to $P^2 = 0$ condition is $P = P_1 + iP_2$ with $P_1^2 = P_2^2$.

3.3.2 Challenging the form of M^8-H duality for the map $M^4 \subset M^8$ to $M^4 \subset H$

The assumption that the map $M^4 \subset M^8$ to $M^4 \subset H$ in M^8-H duality is a simple identification map has not been challenged hitherto.

1. Octonionic Dirac equation forces the identification of M^8 as analog of 8-D momentum space and the earlier simple identification is in conflict with Uncertainty Principle. Inversion allowed by conformal invariance is highly suggestive: what comes first in mind is a map $m^k \rightarrow \hbar_{eff} m^k/m^k m_k$.

   At the light-cone boundary the map is ill-defined. Here on must take as coordinate the linear time coordinate m^0 or equivalently radial coordinate $r_M = m^0$. In this case the map would be of fporm $t \rightarrow \hbar_{eff}/m^0$: m^0 has interpretation as energy of massless particle.

   The map would give a surprisingly precise mathematical realization for the intuitive arguments assigning to mass a length scale by Uncertainty Principle.

2. Additional constraints on M^8-H duality in M^4 degrees of freedom comes from the following argument. The two half-cones of CD contain space-time surfaces in M^8 as roots of polynomials

$P_1(o)$ and $P_2(2T-o)$ which need not be identical. The simplest solution is $P_2(o) = P_1(2T-o)$: the space-time surfaces at half-cones would be mirror images of each other. This gives $P_1(T, Im_R(o)) = P_1(T - Im_R(o))$ Since P_1 depends on $t^2 - \overline{o}^2$ only, the condition is identically satisfied for both options.

There are two options for the identification of the coordinate t.

Option a): t is identified as octonionic real coordinate o_R identified and also time coordinate as in the original option. In the recent option octonion o_R would correspond to the Euclidian analog of time coordinate. The breaking of symmetry from $SO(4)$ to $SO(3)$ would distinguish t as a Newtonian time.

At the level of M^8, The M^4 projection of CD_8 is a union of future and past directed light-cones with a common tip rather than CD_4. Both incoming and outgoing momenta have the same origin automatically. This identification is the natural one at the level of M^8.

Option b): t is identified as a Minkowski time coordinate associated with the imaginary unit I_1 in the canonical decomposition $\{I_1, iI_3, iI_5, iI_7\}$. The half-cone at $o = 0$ would be shifted to $O = (0, 2T, 0...0)$ and reverted. M^4 projection would give CD_4 so that this option is consistent with ZEO. This option is natural at the level of Hbut not at the level of M^8.

If **Option a)** is realized at the level of M^8 and **Option b)** at the level of H, as seems natural, a time translation $m^0 \rightarrow m^0 + 2T$ of the past directed light-cone in $M^4 \subset H$ is required in order to to give upper half-cone of CD_4.

3. The map of the momenta to imbedding space points does not prevent the interpretation of the points of M^8 as momenta also at the level of H since this information is not lost. One cannot identify p^k as such as four-momentum neither at the level of M^8 nor H as suggested by the naïve identification of the Cartesian factors M^4 for M^8 and H. This problem is circumvented by a conjugation in M^8_c changing the sign of 3-momentum. The light-like momenta along the light-cone boundary are non-physical but transform to light-like momenta arriving into light-cone as the physical intuition requires.

   Therefore the map would have in the interior of light-cone roughly the above form but there is still a question about the precise form of the map. Does one perform inversion for the M^4 projection or does one take M^4 projection for the inversion of complex octonion. The inversion of M^4 projection seems to be the more plausible option. Denoting by $P(o_c)$ the real M^4 projection of X^4 point one therefore has:

$$P(o_c) \rightarrow \hbar_{eff} \frac{\overline{P(o_c)}}{P(o_c) \cdot P(o_c)} \quad . \tag{3.5}$$

   Note that the conjugation changes the direction of 3-momentum.

   At the light-cone boundary the inversion is ill-defined but Uncertainty Principle comes in rescue, and one can invert the M^4 time coordinate:

$$Re(m^0) = t \rightarrow \hbar_{eff} \frac{1}{t} \quad . \tag{3.6}$$

A couple of remarks are in order.

1. The presence of $\hbar_{eff}$ instead of $\hbar$ is required by the vision about dark matter. The value of $\hbar_{eff}/h_0$ is given by the dimension of extension of rationals identifiable as the degree of P.

2. The image points $\overline{p}^k$ in H would naturally correspond to the ends of the propagator lines in the space-time representation of scattering amplitudes.

The information about momenta is not lost in the map. What could be the interpretation of the momenta $\overline{p}^k$ at the level of H?

1. Super-symplectic generators at the partonic vertices in H do not involve momenta as labels. The modes of the imbedding space spinor field assignable to the ground states of super-symplectic representations at the boundaries of CD have 4-momentum and color as labels. The identification of $\overline{p}^k$ as this momentum label would provide a connection with the classical picture about scattering events.

   At the partonic 2-surfaces appearing as vertices, one would have a sum over the ground states (spinor harmonics). This would give integral over momenta but $M^8 - H$ duality and number theoretic discretization would select a finite subset and the momentum integral would reduce to a discrete sum. The number of M^8 points with coordinates in a given extension of rationals is indeed finite.

2. $M^4 \subset M^8$ could be interpreted as the space of 4-momenta labeling the spinor harmonics of M^8. Same would apply at the level of H: spinor harmonics would correspond to the ground states of super-symplectic representations.

3. The interpretation of the points of M^4_c as complex 4-momenta inspires the question whether the interpretation of the imaginary part of the momentum squared in terms of decay decay width so that M^8 picture would code even information about the dynamics of the particles.

4 Conclusions

$M^8 - H$ duality plays a crucial role in quantum TGD and this motivated a critical study of the basic assumptions involved leading to a surprisingly precise view realizing the most optimistic original vision.

4.1 Co-associativity is the only viable option

The notion of associativity of the tangent or normal space as a number theoretical counterpart of a variational principle is restricted to associativity of the normal space - co-associativity.

1. $Re_Q(o) = 0$ and $Im_Q(P) = 0$ allow M^4 and its complement as associative/co-associative subspaces of O_c. The roots $P = 0$ for the complexified octonionic polynomials satisfy two conditions $X = 0$ and $Y = 0$. They are 6-D brane-like entities X^6_c having real projection X^6_r ("real" means that the number theoretic complex valued octonion norm squared is real valued). The condition $Re_Q(P) = 0$ gives as a candidate for co-associative surface a complex surface X^4_c which has 4-D real projection X^4_r. Contrary to naïve expectations the intersection of X^4_r and X^6_r is not 3-dimensional.

   The reason is that the equations $X = 0$ and $Y = 0$ are reduced by Lorentz invariance to equations for the ordinary roots of polynomials for the complexified mass squared type variable. The intersection is empty unless X and Y have a common root and X^4_r belongs to X^6_r for a common root.

   Posing the condition $Im_Q(P) = 0$ gives a complex surface X^5_c. Real space-time surface would be 5-D. Associativity is impossible already due to the wrong dimension.

2. The cold shower came as I learned that 4-D associative sub-manifolds of quaternion spaces are geodesic manifolds and thus trivial. Co-associativity is however possible since any distribution of associative normal spaces integrates to a sub-manifold. Typically these sub-manifolds are minimal surfaces, which conforms with the physical intuitions. Therefore the surface X^4_r given by holography

should be co-associative. By the same argument space-time surface contains string world sheets and partonic 2-surfaces as co-complex surfaces.

3. The key observation is that G_2 as the automorphism group of octonions respects the co-associativity of the 4-D real sub-basis of octonions. Therefore a local G_2 (or even $G_{2,c}$) gauge transformation applied to a 4-D co-associative sub-space O_c gives a co-associative four-surface as a real projection.

   An open question is whether this approach is equivalent with $Y = 0$ conditions so that octonion analyticity would correspond to G_2 gauge transformation: this would realize the original idea about octonion analyticity. If this surface contains a co-complex 2-surface as a string world sheet, the conditions making possible to map X^4_r to H by $M^8 - H$ duality are satisfied and there is no need for a separate holography in H. There is no objection against this option and it would replace SH with much stronger number theoretic holography fixing the space-time region from the roots of a real polynomial. One could say that classical TGD is an exactly solvable theory.

4. Remarkably, the group $SU(3)_c \subset G_{2,c}$ has interpretation as a complexified color group and the map defining space-time surface defines a trivial gauge field in $SU(3)_c$ whereas the connection in $SU(3)$ is non-trivial. Color confinement could mean geometrically that $SU(3)_c$ reduces to $SU(3)$ at large distances. This picture conforms with the H-picture in which gluon gauge potentials are identified as color gauge potentials. Note that at QFT limit the gauge potentials are replaced by their sums over parallel space-time sheets to give gauge fields as the space-time sheets are approximated with a single region of Minkowski space.

5. Minkowski signature turns out to be the only possible option for X^4_r. Also the phenomenological picture based on co-assiative space-time sheets, light-like 3-surfaces, string world sheets and partonic 2-surfaces, and wormhole contacts carrying monopole flux emerges.

4.2 The input from octonionic Dirac equation

Octonionic Dirac equation allows a second perspective on associativity. For the co-associative option the co-associate octonions can represent gamma matrix algebra and it also allows a matrix representation. The octonionic Dirac equation is an analog of the momentum space variant of ordinary Dirac equation and forces the interpretation of M^8 as momentum space. The original wrong belief was that mass shell condition implies a localization of the octonionic spinor to a light-like 3 surface, which actually corresponds to light-cone boundary.

In the intersection of the space-time surface with 6-D brane-like surface Dirac equation is trivially satisfied and does not pose a condition on the mass of the quark. This intersection is either empty or the space-time surface is in the interior of this 6-D surface so that quarks can propagate in the entire X^4_r. This conforms with the fact that in H picture quark spinors can exist both in the interior of X^4 and at light-like 3-D partonic orbits and 2-D string world sheets. In the first case only massless quarks arriving at the boundary of CD are possible. The interpretation is as a number theoretic counterpart for a transitions from massless phase to massive phase. This applies at all levels of dark matter hierarchy. It seems also that the cognitive representations for both light-like boundary and X^4_r are not generic consisting of a finite set of points but infinite due to the Lorentz symmetry: a kind of cognitive explosion would happen when massivation occurs.

4.3 How the new picture differs from the earlier one?

The new view about $M^8 - H$ duality differs from the earlier one rather dramatically so that an explicit summary of the differences is in order to minimize confusions.

1. Octonionic Dirac equation as counterpart of Dirac equation in momentum space forced the interpretation of M^8 as the analog of momentum space so that space-time surfaces in M^8 would be the

analogs of Fermi ball and mass shells would correspond to Fermi surfaces. The earlier solutions of the octonionic Dirac equation consisted of only massless solutions located to light-like surfaces (actually the boundary CD rather than inverse images of light-like partonic orbits). Co-associativity also allows massive quarks for which this localization does not occur: this conforms with the view that in H the induced spinor fields are possible also in the space-time interior. The transition from massless to massive phase for quarks has a number theoretic interpretation as the appearance of a common root of P_{odd} and P_{even}.

2. M^4 must be identified as co-associative rather than associative sub-space of octonions - earlier Minkowskian *resp.* Euclidian regions were proposed to be associative *resp.* associative. All space-time surfaces in M^8 would be co-associative surfaces and would contain string world sheets as co-complex sub-manifolds. Slicing by partonic 2-surfaces and string world sheets is suggestive.

   The earlier view was that $M^8 - H$ duality allows to map only string world sheets, partonic 2-surfaces, and possibly also their light-like orbits to H so that SH would be needed at the level of H. In the new picture one can map the entire co-associative space-time 4-surfaces in M^8 to 4-surfaces in H by $M^8 - H$ duality.

3. The reduction of the equations for the 4-D roots of $Re_Q P$ by Lorentz invariance to the roots of ordinary real polynomial for the odd part of P led to a detailed understanding of the 4-surfaces in M^8.

4. Uncertainty Principle forces to modify the identification map $M^4 \subset M^8 \rightarrow M^4 \subset H$ appearing in $M^8 - H$ duality to inversion. The "very special moments moments in the life of self" discovered earlier would correspond to proper time constant hyperboloids of H as images of quantized mass shells in M^8.

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