Article

Emergence of Quantum Mechanics from Iterated Maps

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Abstract

Iterated maps are deterministic models of dynamical systems in discrete time. A key feature of these models is the concept of *invariant density* associated with the asymptotic onset of stationarity. Drawing from the minimal fractality of spacetime near the Fermi scale, we show here that invariant density enables a step-by-step derivation of Quantum Mechanics from iterated maps.

Keywords: Quantum mechanics, iterated maps, invariant density, invariant measure, ergodicity, minimal fractal manifold.

1. Iterated maps and invariant density functions

Refs. [3-9] discuss at length the physical meaning of the *minimal fractal manifold* (MFM), a spacetime continuum characterized by arbitrarily small and scale-dependent deviations from four dimensions ($\varepsilon = 4 - D << 1$). MFM is conjectured to develop in out-of-equilibrium conditions near the ultraviolet scale Λ_{UV} and flow towards equilibrium below the Fermi scale ($M_{EW} < \Lambda_{UV}$). The main point of the MFM model is that the dimensional deviation ε runs with the energy scale as in $\varepsilon = \varepsilon(\mu)$, where $\mu = O(M_{EW}/\Lambda_{UV})$. There are reasons to believe that dimensional fluctuations driven by $\varepsilon(\mu)$ near $\varepsilon = 0$ are asymptotically compatible with the phenomenology of effective field theory, in general, and the Standard Model of particle physics in particular [3-9].

The goal of this paragraph is to outline the description of $\varepsilon(\mu)$ in terms of iterated maps and invariant density functions. It complements our earlier work on the emergence of Planck's constant from iterated maps [10].

The evolution of $\varepsilon(\mu)$ in a generic $N-$dimensional phase space X is defined by the first-order differential equation [1-2]

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$$\frac{d\varepsilon}{d\mu} = \beta(\varepsilon) \tag{1}$$

whose map analog is given by

$$\varepsilon_{n+1} = \beta\left(\varepsilon_n\right) \tag{2}$$

Here, n is the map iteration index while

$$\varepsilon_n = (\varepsilon_n^{(1)}, \varepsilon_n^{(2)}, ..., \varepsilon_n^{(N)}) \tag{3}$$

is a vector in X and

$$\beta = (\beta^{(1)}, \beta^{(2)}, ..., \beta^{(N)}) \tag{4}$$

a vector-valued function. One starts with ε_0 and iterate it step by step using (2). The sequence of iterates $\varepsilon_0, \varepsilon_1, ...$ forms a trajectory (orbit) in X. A periodic orbit of length $L = 1$ defines a fixed point of the map and satisfies the condition

$$\varepsilon^* = \beta(\varepsilon^*) = 0 \Rightarrow D = 4 \tag{5}$$

$$\beta^L(\varepsilon) = \beta(\beta(...\beta(\varepsilon))) \ , \quad L - \text{ times} \tag{6}$$

Although the iterates of (2) are deterministic events, a useful concept for the analysis of (2) is the *probability distribution of iterates*. Let us partition X into an array of disjoint cells indexed by the subscript $i = 1, 2, ..., R$, where R is the total number of cells. Let the number of iterates located in cell i be n_i. The relative frequencies (or weights) associated with a large number of iterations $n >> 1$ is given by

$$p_i = \frac{n_i}{n} = \frac{n_i}{\sum_i n_i} \tag{7}$$

Definition (7) enables bridging the gap between the theory of iterated maps and classical statistical physics [1].

Assuming unbounded precision, the numerical value of ε in the binary basis includes an infinite string of digits. A reasonable approximation is obtained by truncating the string to $M >> 1$ digits according to

$$\varepsilon = \eta_1 \eta_2 ... \eta_{M-1} \eta_M \tag{8}$$

where each unit η_j, $j=1,2,...,M$ is the quartet of binary pairs as in

$$\eta_j = (00,\quad 01,\quad 10,\quad 11) \tag{9}$$

The probability measure associated with the initial density $\rho_0(\varepsilon)$ assumes the form

$$\mu_0(A) = \int_A d\varepsilon \rho_0(\varepsilon) \tag{10}$$

It is important to note that, unlike statistical physics, the initial density $\rho_0(\varepsilon)$ *does not quantify* the statistical uncertainty of choosing an initial condition. Rather, the initial density follows from the inherent numerical approximation of ε as expressed by (8) and (9). In general, one can state that the density $\rho(\varepsilon)$ at any iteration stage reflects the distribution of *rounding errors* in the estimation of ε, a process that can be symbolically presented as

$$T \le M \Rightarrow \varepsilon \approx \eta_1, \eta_2, ..., \eta_T \Rightarrow \rho(\varepsilon) \tag{11}$$

Given the map β, one wishes to study the evolution of the ensemble of trajectories corresponding to an ensemble of initial values ε_0. Let μ_n denote the probability distribution of iterates after n iterations. The probability measure of finding an iterate ε_n in the subset of the phase space $A \subset X$ amounts to [1]

$$\mu_n(A) = \int_A d\varepsilon \rho_n(\varepsilon) \tag{12}$$

By definition, an invariant probability measure stays unchanged upon the action of the map β, which means that it satisfies the requirement

$$\mu_{n+1}(A) = \mu_n(A) \tag{13}$$

It can be shown that, based on (12) to (13), the corresponding invariant density ρ complies with the condition

$$\int_A d\varepsilon \rho(\varepsilon) = \int_{\beta^{-1}(A)} d\varepsilon \rho(\varepsilon) \tag{14}$$

where $\beta^{-1}(A)$ denotes the set of all points that are mapped onto A by one iteration step.
The ensemble expectation value of an arbitrary test function (or operator) $Q(\varepsilon)$ with respect to the invariant density ρ is given by [1]

$$\langle Q \rangle = \int_X d\varepsilon \rho(\varepsilon) Q(\varepsilon) \tag{15}$$

Ergodicity demands the identity of the ensemble average with the time average, where the latter is supplied by

$$\overline{Q} = \lim_{n \to \infty} \frac{1}{n} \sum_{n=0}^{N-1} Q(\varepsilon_n) \Rightarrow \langle Q \rangle = \overline{Q} \tag{16}$$

A remarkable property of ergodic maps is *mixing*. The map β is called "mixing" if the initial smooth density $\rho_0(\varepsilon)$ converges to the invariant density $\rho(\varepsilon)$ as in [1]

$$\lim_{n \to \infty} \rho_n(\varepsilon) = \rho(\varepsilon) \tag{17}$$

It is apparent that (17) is automatically fulfilled if the map (2) ends up on the attractor $\varepsilon^* = 0$ $(D = 4)$, where the dimensional flow $\varepsilon = \varepsilon(\mu)$ settles down.
Mixing may be also defined in terms of *correlation functions* (CF). The CF for any two integrable test functions φ_1, φ_2 takes the form

$$C(\varphi_1, \varphi_2; n) = \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} \varphi_1(\varepsilon_{k+n}) \varphi_2(\varepsilon_k) - \langle \varphi_1 \rangle \langle \varphi_2 \rangle \tag{18}$$

The map is considered mixing if

$$\lim_{n \to \infty} C(\varphi_1, \varphi_2; n) = 0 \tag{19}$$

which occurs when φ_1 and φ_2 are *statistically independent*. Mixing implies ergodicity, but the reverse is not true in general.

2. Quantum Mechanics from invariant density functions

The behavior of iterated maps previously outlined hints to an unforeseen connection between the invariant density $\rho(\varepsilon)$ in close proximity to $\varepsilon^* = 0$ and the probability density of quantum states. In particular, a straightforward conjecture is that

$$\rho(\varepsilon) \Leftrightarrow |\psi(x)|^2 \tag{20}$$

where $\psi(x)$ is the square-integrable wavefunction of Quantum Mechanics (QM). To unveil this connection, we cast the invariant density in the form

$$\rho(\varepsilon) = \rho_C(\varepsilon)\rho_C^*(\varepsilon) = \left|\sqrt{\rho(\varepsilon)}\exp[i\theta(\varepsilon)]\right|^2 \tag{21}$$

where $\rho_C(\varepsilon) = \sqrt{\rho(\varepsilon)}\exp[i\theta(\varepsilon)]$ represents the complex-valued amplitude of $\rho(\varepsilon)$. Moreover, if ε is locally defined as in $\varepsilon = \varepsilon(x)$, relations (20) and (21) imply a straightforward one-to-one correspondence written as

$$\boxed{\rho_C(\varepsilon) \Leftrightarrow \psi(x)} \tag{22a}$$

$$\boxed{\rho_C^*(\varepsilon) \Leftrightarrow \psi^*(x)} \tag{22b}$$

A critical observation is now in order. Recall that, by (11), the density $\rho(\varepsilon)$ reflects the distribution of rounding errors in the estimation of ε. In addition to the rounding error, one must consider that any measurement process involves a *finite sampling resolution* and an invariant density $\rho_\Delta(\varepsilon)$, a setting that can be symbolically presented as

$$\varepsilon \in [\varepsilon_{\min}, \varepsilon_{\max}] \subset \Delta \Rightarrow \rho_\Delta(\varepsilon) \tag{23}$$

It follows from these considerations that (22) is to be interpreted as an *infinite superposition* of complex amplitudes constrained by Δ, as embodied in the linear expansion postulate of QM

$$\psi = \sum\nolimits_s C_s \psi_s \tag{24}$$

To understand why this is the case, let us assume that the rounded value of ε can be written in the decimal base as $\varepsilon_{\min} < \varepsilon(S) = 10^{-S} < \varepsilon_{\max}$, with S being a large natural number ($1 << S \in \mathbf{N}$). Let Δ_U be the upper limit of Δ such that $\varepsilon(S) < \varepsilon_{\max} < \Delta_U$. There is an unbounded spectrum of values $\varepsilon(S,P) = 10^{-(S+P)}$ with $P \in \mathbf{N}$ and $\varepsilon(S,P) < \varepsilon(S) < \Delta_U$, as well as an unbounded spectrum of densities for $P \to \infty$

$$\rho_\Delta(\varepsilon(S)), \rho_\Delta(\varepsilon(S+1)), \rho_\Delta(\varepsilon(S+2)), ..., \rho_\Delta(\varepsilon(S+P)), ...$$

that motivates the existence of (24).

Turning next to (15), it is apparent that it represents the analogue of operator average in QM, namely

$$\left\langle Q \right\rangle = \int dx \psi^*(x) Q \psi(x) \tag{25}$$

A surprising interpretation emerges from (18) - (19). For any arbitrarily small yet non-vanishing deviation $\varepsilon > \varepsilon^* = 0$, the correlation function of two operators Q_1, Q_2 is never vanishing, which – under properly defined conditions - may lie behind the physics of *quantum entanglement*. This is to say that there are circumstances where quantum operators remain statistical dependent, regardless of the observation scale μ .

We close by showing how the quantum generators of translations and rotations, as well as the time-dependent Schrödinger equation, emerge from (20) – (22). To this end, consider a scaling transformation of (22) written in the form

$$\psi' = U\psi \tag{26}$$

Since (20) is scale-invariant for $\varepsilon \to \varepsilon^* = 0$, the operator U must be unitary

$$U^\dagger U = 1 \tag{27}$$

which means

$$\psi'^* \psi' = \psi'^* U^\dagger U \psi = \psi^* \psi \tag{28}$$

Transformation (26) is therefore norm-conserving

$$|\psi'| = |\psi| = (\psi^\dagger \psi)^{1/2} \tag{29}$$

The infinitesimal counterpart of (26) is described by ($\xi << 1$)

$$U = \exp(i\theta) = \exp(-i\xi G) \approx 1 - i\xi G \tag{30}$$

or

$$\psi' = (1 - i\xi G)\psi \tag{31}$$

where G stands for the generator of (26). It was shown in [11] that an infinitesimal transformation involving a dilation, rotation or translation is operationally equivalent to an infinitesimal scale-invariant (self-similar) transformation. Assuming that (26) refers to a translation ($x' = x - \delta x$) with $\delta x << x$ leads to

$$\psi'(x') = \psi(x-\delta x) \approx \psi(x) - \delta x \frac{\partial \psi(x)}{\partial x} \quad , \quad x = (x_\mu) \tag{32}$$

and yields the generator of translations in the form

$$G_\mu = -i \frac{\partial}{\partial x_\mu} \tag{33}$$

Including the reduced Planck's constant in (33) recovers the standard momentum operator

$$P_\mu = \hbar G_\mu \tag{34}$$

Consider now rotations about the z - axis which transform vectors $\mathbf{V} = (V_x, V_y)$ according to the matrix equation

$$\begin{pmatrix} \overline{V}_x \\ \overline{V}_y \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} \tag{35}$$

An infinitesimal rotation of angle $\delta\varphi = \xi << 1$ is equivalent to an infinitesimal scaling operation defined by

$$\begin{pmatrix} \overline{V}_x \\ \overline{V}_y \end{pmatrix} = \begin{pmatrix} 1 & -\delta\varphi \\ \delta\varphi & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} V_x - V_y \delta\varphi \\ V_y + V_x \delta\varphi \end{pmatrix} \tag{36}$$

and so

$$\delta V_x = -V_y \delta\varphi \tag{37}$$

$$\delta V_y = V_x \delta\varphi \tag{38}$$

It is known that expanding a generic function containing a two-component vector gives

$$F(V_x + \delta V_x, V_y + \delta V_y) = F(V_x, V_y) + \delta V_x \frac{\partial F}{\partial V_x} + \delta V_y \frac{\partial F}{\partial V_y} \tag{39}$$

which can be presented as

$$F(\varphi + \delta\varphi) = F(\varphi) + \delta\varphi \left(-V_y \frac{\partial}{\partial V_x} + V_x \frac{\partial}{\partial V_y}\right) \tag{40}$$

It follows from (40) that the generator of planar rotations can be written as

$$G = -i\,(V_x \frac{\partial}{\partial V_y} - V_y \frac{\partial}{\partial V_x}) \tag{41}$$

When $\mathbf{V}$ represents the position vector $\mathbf{r} = (x, y)$, (41) renders the angular momentum about the z axis in the familiar form

$$L_z = xP_y - yP_x \tag{1.1}$$

Finally, let us apply an infinitesimal time translation $t' = t - \delta t$ in (26) given by

$$\psi'(t') = U\psi(t) = (1 - \frac{i}{\hbar} H\,\delta t)\,\psi(t) \tag{42}$$

where

$$\psi'(t') = \psi(t) - \delta t \frac{\partial \psi}{\partial t} \tag{43}$$

Inspection of (42) and (43) recovers the time-dependent Schrödinger equation

$$H\psi = i\hbar \frac{\partial \psi}{\partial t} \tag{44}$$

For additional details on (26) – (44), the reader is directed to a couple of well-written introductory texts on the role of symmetry and invariance in physics [12-13].

Received August 2, 2020; Accepted September 27, 2020

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