Article

Kaluza-Klein Cosmological Models with Polytropic Equation of State in Lyra Geometry

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Abstract

The present research study attempt at to explore exact solution of the field equations by considering more general form of the law of variation of the Hubble parameter ($H = lb(V)$, where $b(V)$is the function of the volume) proposed by Berman. It employs Kaluza- Klein cosmological models with polytropic equation of state in relation to the Lyra geometry. So, following the line that the universe is accelerating and expanding at the present, it makes allowance for two physically viable ansatz are investigated. It is observed that the universe is oscillatory and decelerating in the one case whereas the universe is accelerating and expanding in the second case of investigated models.

Keywords: Kaluza-Klein, polytropic equation, Lyra geometry.

1. Introduction

General Theory of relativity describes gravitation in terms of geometry which motivates the geometrization of other physical field. Weyl[1] made the first attempt in this direction. He described gravitation and electromagnetism geometrically. However, due to the non integrability of the length transfer, this theory is not considered by the researcher. The non integrability of the length transfer is reduced by introducing a guage function in the Riemannian geometry. This new modification is suggested by Lyra[2].The field equations of Lyra geometry may be written as

$$R^i_j - \frac{1}{2} R g^i_j + \frac{3}{2}\phi^i \phi_j - \frac{3}{4}\phi^k \phi_k = \frac{8\pi G}{c^2} T^i_j, \qquad 1$$

where ϕ is the displacement filed vector defined as

$$\phi_i = (0,0,0,0,\beta(t)), \qquad 2$$

and all other symbols have their usual meaning as in the Riemannian geometry.

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The importance of the Lyra geometry are pointed out by Halford[3-4]. He showed that the Lyra geometry predicts the same effect as in the general relativity and the constant displacement vector field plays the role of a cosmological constant in the normal general relativistic treatment. Cosmological models based on the Lyra geometry have been studed by several authors [5-6]. Katore and Hatkar[7] have studied the strange quark matter attached to string cloud in the Lyra geometry. Pradhan et al.[8] have investigated the cosmological models of the universe with the variable deceleration parameter in the Lyra geometry. A higher dimensional cosmological model in the early universe in the Lyra geometry has investigated by Das and Sarma[9]. Mahanta and Murkherjee [10] have presented Bianchi type string cosmological models in the context of the Lyra geometry.

Recently, self gravitating Bose-Einstein condensates (BEC) model is used to describe the evolution of the universe as a whole [11-15]. Chavanis[16] has also investigated the BEC model with an equation of state of the form $P = k\rho^2 c^2$. He described the cosmic fluid by more general equation of state of the form $P = \left(\alpha\rho + k\rho^2\right)c^2$. This form of equation is the sum of a linear term describing a classical universe and a quadratic term due to the BEC [17]. In this model, it is found that at low density, the effect of the BEC is negligible and the classical universe will be recovered. On the other hand, when the density is high i.e. at the early evolution of the universe, the BEC is important. This motivates us to undertake a study of an equation of the state parameter which can be put as below [17]

$$P = \left(\alpha\rho + k\rho^{1+\frac{1}{n}}\right)c^2, \qquad 3$$

where $-1 \le \alpha \le 1$, k is a polytropic constant. The equation (3) describes radiation ($\alpha = \frac{1}{3}$), pressureless matter ($\alpha = 0$) and vacuum energy ($\alpha = -1$). When $k > 0$, the polytropic equation of state may correspond to the self gravitating BEC with repulsive whereas it corresponds to the BEC with attractive when $k < 0$. When $n = 1$, the model is called as standard BEC model, which may have another origin [13]. Here, we consider the polytropic equation of state $P = \left(-\rho + k\rho^{\gamma}\right)c^2$ with $\gamma = 1 + \frac{1}{n}$. We will investigate the constraints on constants n and k .

2. Metric and Field equations

We consider the Kaluza Klein universe in the form

$$ds^2 = c^2 dt^2 - A^2\left(dx^2 + dy^2 + dz^2\right) - B^2 d\psi^2, \qquad 4$$

where A and B are metric potentials and functions of time t only.

The study of higher dimensional cosmological models are important because the four dimensional stage of the universe could have been proceeded by a higher dimensional stage. The extra dimensions of the universe contract to unobserved plankian length scale due to dynamical contraction [18]. This forces us to study higher dimensional cosmological models. Mohanty et al. [19] have studied the five dimensional Kaluza-Klein FRW cosmological models in the Lyra maniforld. Katore and Hatkar[20] has investigated the Kaluza-Klein universe with anisotropic dark energy in the General Relativity and Lyra manifold.

The energy momentum tensor of the isotropic and homogeneous universe containing a uniform perfect fluid is taken in the form

$$T_0^0 = c^2\rho, T_1^1 = T_2^2 = T_3^3 = T_4^4 = P \,. \qquad 5$$

The field equations (1) using equation (2), (4) and (5) can be written as

$$2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}^2}{A^2} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2 = 8\pi GP \,, \qquad 6$$

$$3\frac{\ddot{A}}{A} + 3\frac{\dot{A}^2}{A^2} + \frac{3}{4}\beta^2 = 8\pi GP \,, \qquad 7$$

$$3\frac{\dot{A}^2}{A^2} + 3\frac{\dot{A}\dot{B}}{AB} - \frac{3}{4}\beta^2 = 8\pi Gc^2\rho \,, \qquad 8$$

where the over dot denotes differentiation with respect to time *t*.

From the equations (6) and (7), we obtain

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + 2\frac{\dot{A}^2}{A^2} - 2\frac{\dot{A}\dot{B}}{AB} = 0 \,. \qquad 9$$

The equation (9) further reduces to

$$\frac{d}{dt}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) + \left(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = 0 \,. \qquad 10$$

Let $\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = F$. 11

Making use of the equation (11) in the equation (10), we get

$$\frac{dF}{dt} + \left(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)F = 0 \,. \qquad 12$$

Again the equation (12) leads to

$$FV = \lambda \,, \tag{13}$$

where $V = A^3 B$ is the volume of the universe. The equation (13) can be written as

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{\lambda}{V}, \tag{14}$$

where λ is the constant of integration. The equation (14) further leads to

$$A = BC\exp\left(\int \frac{\lambda}{V} dt\right), \tag{15}$$

The law of variation of the Hubble parameter was initially proposed by Berman [21-24]. Later on, Singh and Kumar [25] have proposed a similar law of variation of the Hubble parameter for anisotropic space time metrics. Here, it is attempted to generalize the law of variation of the Hubble parameter to get wide range of class of solutions of the field equation and call it general law of variation of Hubble parameter. We consider the law in the following form

$$H = \frac{1}{4}\frac{\dot{V}}{V} = lb(V), \tag{16}$$

where *b(V)* is a function of volume *V*.

From the equation (16), we yield

$$\int \frac{1}{b(V)V} dV = 4lt + 2k_1, \tag{17}$$

where k_1 is the constant of integration. In order to solve the problem completely, we have to choose $b(V)$ in such a manner that the equation (17) can be integrable. It should be noted that when $b(V) = V^{-n}$, we recover the special law of the Hubble parameter. Here, we are looking for mathematically tractable and physically sound model of the universe. For that purpose, we consider the following ansatz:

Model I:- $b(V) = \frac{\sqrt{1-V^2}}{V}$

Model II:- $b(V) = \frac{\sqrt{V^2+1}}{V}$

3. Model I

The Hubble parameter and deceleration parameter (q) of the model is found to be

$$H = l\cot(4lt + 2k_1). \quad 18$$

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = 4\sec^2(4lt + 2k_1) - 1. \quad 19$$

The sign of the deceleration parameter (DP) indicate whether the model is accelerating or decelerating. A positive sign of the DP corresponds to the standard decelerating model whereas the negative sign of the DP indicates an accelerating model. The model is inflationary for $q = 0$. The variation of the DP with time is shown in the figure (1) below. It is clear from the graph that the sign of the DP is positive. Thus, the present model is standard decelerating model. Further, the Hubble parameter is periodic.

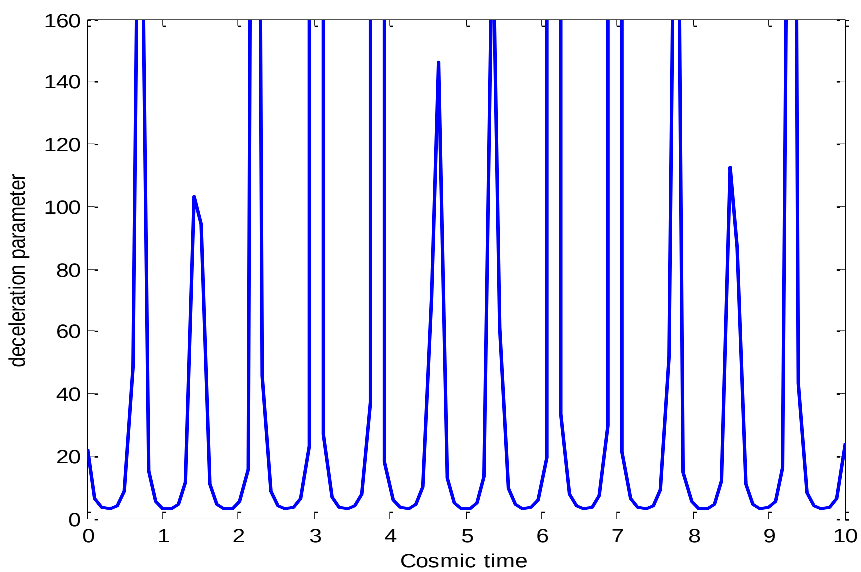

Figure 1. Plot of the deceleration parameter with cosmic time.

From the equation (17), we obtain

$$V = \sin(4lt + 2k_1). \quad 20$$

The expression of the volume shows that the universe is oscillatory. The expansion followed by contraction of the universe. The scenario of the oscillatory universe is described by many authors [26-29]. Recently, the oscillatory solutions in case of domain walls for the Bianchi type II, VIII and IX universe is obtained by Katore et al. [30].

The solution of the field equation obtained as

$$A = C^{\frac{1}{4}}\left(\sin\left(4lt + 2k_1\right)\right)^{\frac{1}{4}}\left(\tan\left(2lt + k_1\right)\right)^{\frac{\lambda}{16l}}, \quad (21)$$

$$B = C^{-\frac{3}{4}}\left(\sin\left(4lt + 2k_1\right)\right)^{\frac{1}{4}}\left(\tan\left(2lt + k_1\right)\right)^{-\frac{3\lambda}{16l}}. \quad (22)$$

The solution obtained in this model describes the oscillatory universe i.e. the universe having several origins. This is in accordance with the argument of Chavanis [17]. He believed that the polytropic equation of state may have several origins.

The density of the model becomes

$$\rho = \left[-\frac{3l^2}{2\pi Gkc^2}\right]^{\frac{1}{\gamma}}. \quad (23)$$

The displacement vector is found to be

$$\beta^2 = \frac{16l^2\cos^2\left(4lt + 2k_1\right) - \lambda^2}{2\sin^2\left(4lt + 2k_1\right)} - \frac{32\pi Gc^2}{3}\left[-\frac{3l^2}{2\pi Gc^2}\right]^{\frac{1}{\gamma}}. \quad (24)$$

From the equation (23), we observe that for $k > 0$ we get an unphysical model ($\rho < 0$). We are looking for physically viable model where the density is positive ($\rho > 0$). When $k < 0$, the density is positive and constant. It is very small and constant. The gravity is attractive since $\rho + 3P / c^2 > 0$. Moreover the polytropic pressure is negative. As the universe is oscillatory, it has past and future singularity. The result is contradictory to the result of Chavanis [17]. The term c^2 is negatively contributing to the expression of β^2 .Therefore β is imaginary.

4. Model II

The Hubble parameter and deceleration parameter of the model is found to be

$$H = l\coth\left(4lt + 2k_1\right). \quad (25)$$

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = 4\operatorname{sec}h^2\left(4lt + 2k_1\right) - 1. \quad (26)$$

The variation of the Hubble parameter and deceleration parameter with time is shown in the figure (2) and figure (3). It is clear that the Hubble parameter is decreasing function of cosmic time. It is large near $t = 0$ and as $t \to \infty, H \to 1$. Thus, the rate of expansion of the universe was large at the early stage of evolution of the universe and it is expanding with the constant rate at present. Further, the sign of the DP is negative throughout the evolution of the universe i.e. the model represents an accelerating universe.

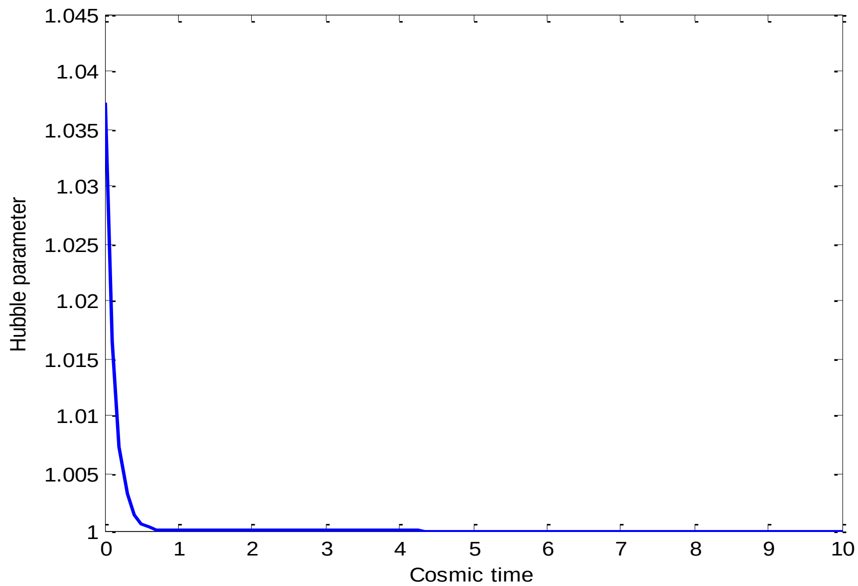

Figure 2. Plot of Hubble parameter with cosmic time.

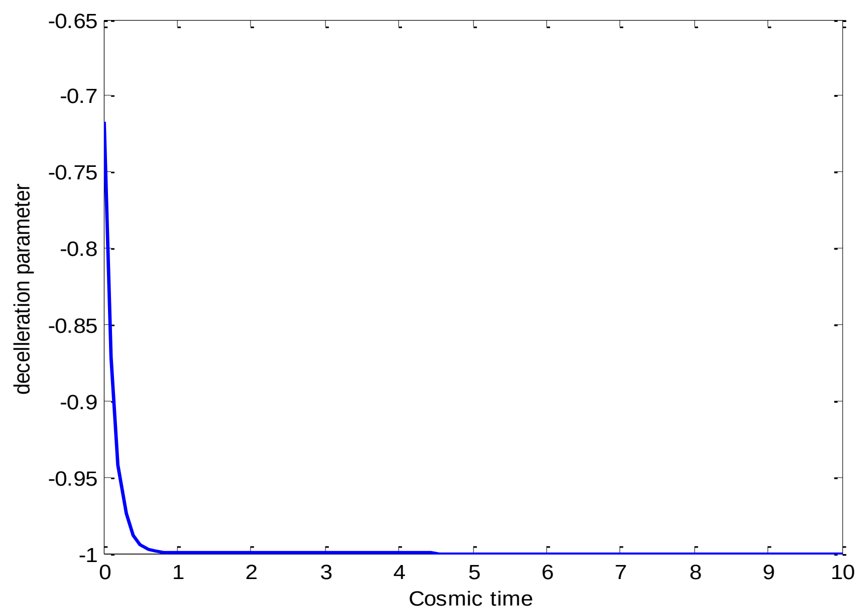

Figure 3. Plot of deceleration parameter with cosmic time.

The volume of the universe is

$$V = \sinh(4lt + 2k_1). \quad 27$$

The expansion of volume with time is shown in the figure (4) below. It is observed that the universe starts to expand from the big bang singularity. The volume of the universe increases with the increasing time.

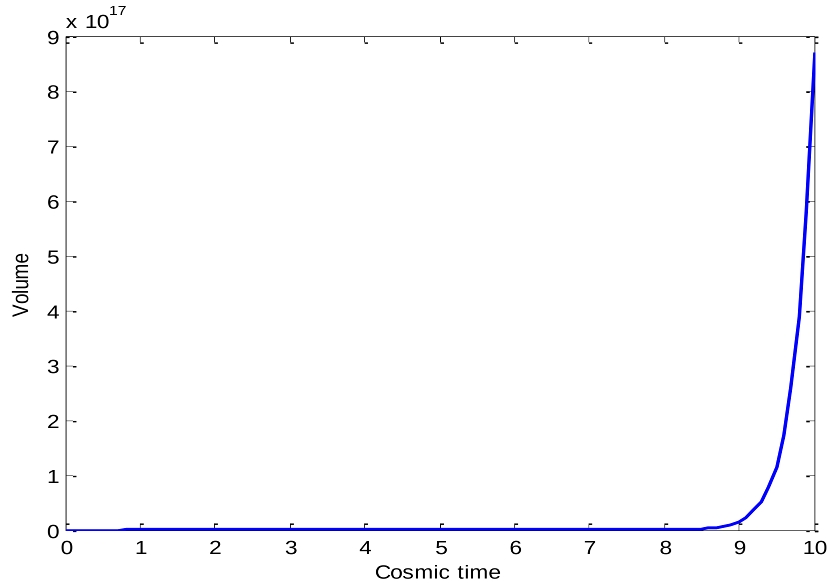

Figure 4. Plot of the volume with cosmic time.

The solution of the field equations is as follows

$$A = C^{\frac{1}{4}}(\sinh(4lt + 2k_1))^{\frac{1}{4}}(\tanh(2lt + k_1))^{\frac{\lambda}{16l}}, \quad 28$$

$$B = C^{-\frac{3}{4}}(\sinh(4lt + 2k_1))^{\frac{1}{4}}(\tanh(2lt + k_1))^{-\frac{3\lambda}{16l}}. \quad 29$$

The density of the model obtained as

$$\rho = \left[\frac{3l^2}{2\pi Gkc^2}\right]^{\frac{1}{\gamma}}. \quad 30$$

The displacement vector is found to be

$$\beta^2 = \frac{16l^2\cosh^2(4lt + 2k) - \lambda^2}{2\sinh^2(4lt + 2k)} - \frac{32\pi Gc^2}{3}\left[\frac{3l^2}{2\pi Gc^2}\right]^{\frac{1}{\gamma}}. \quad 31$$

From the equation (30), it is clear that for $k < 0$, we get an unphysical model ($\rho < 0$). We have the physically viable model ($\rho > 0$) in for $k > 0$. The density is constant. The universe starts at $t = 0$ from a singularity. Since $\rho + 3P / c^2 > 0$ the gravity is attractive. Further, for $k > 0$ the pressure is positive and leading singular model. The result corroborates Chavanis[17]. Also in this model, the term c^2 is negatively contributing to the expression of β^2 .Therefore, β is imaginary.

5. Conclusion

To sum up, we have analyzed the polytropic equation of state in the purely theoretical manner, without reference to the observations in the context of the Lyra geometry for the Kaluza-Klein space time. We have developed more general form of the law of variation of the Hubble parameter to obtain the solution of the field equations. In the first model, it is found that the universe is decelerating universe whereas in the model II, it is accelerating and expanding.

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