Article

Anyon Physics & the Topology of Dark Matter

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Abstract

Matching current observations on non-baryonic Dark Matter (DM), Cantor Dust was recently conjectured to emerge as large-scale topological structure formed in the early stages of cosmological evolution. The mechanism underlying the formation of Cantor Dust hinges on dimensional condensation of spacetime endowed with minimal fractality. It is known that anyons are quasiparticles exhibiting anomalous statistics and fractional charges in 2+1 spacetime. This brief report is a preliminary exploration of the intriguing analogy between anyons and the Cantor Dust picture of DM.

Keywords: Dark matter, Cantor dust, anyons, topological field theory, Maxwell-Chern-Simons theory, topological phases of matter.

1. Introduction

Rooted in the Renormalization Group program of Quantum Field Theory, the *minimal fractal manifold* (MFM) describes a spacetime continuum having arbitrarily small deviations from four-dimensions ($\varepsilon = 4 - D << 1$). This fine structure of spacetime is conjectured to set in near or above the Fermi scale. The emergence of the MFM sheds light on many ongoing puzzles of the Standard Model (SM), while meeting all consistency requirements mandated by effective QFT in the conformal limit $\varepsilon = 0$. The underlying rationale, conceptual benefits and implications of the MFM for the development of Quantum Field Theory and the SM are reported elsewhere (refs. 1-12 of [1]).

Recently, a proposal was put forward according to which DM amounts to *Cantor Dust*, a large-scale topological structure formed in the early Universe. The mechanism underlying the onset of Cantor Dust hinges on dimensional condensation of spacetime endowed with minimal fractality. Condensed matter physics views anyons are quasiparticles exhibiting anomalous statistics and fractional charges in 2+1 spacetime. This brief report is a preliminary exploration of the analogy between anyons and the Cantor Dust model of DM.

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We emphasize from the outset that the ideas discussed here are in their infancy and in need for independent validation. The reader is cautioned that the sole intent of this paper is to lay the groundwork for follow-up model building and simulations.

2. Anyons in 2+1 spacetime dimensions

The concept of "*exchange statistics*" in Quantum Mechanics relates to the phase picked up by a wavefunction upon exchanging a pair of identical particles. It has been long known that the exchange statistics of identical particles in 2+1 spacetime deviates from the spin-statistics theorem of QM in 3+1 dimensions and signals the transition of quantum particles to *anyons*. For the sake of clarity and accessibility, below is a short pedagogical introduction to the physics of anyons.

Following [2] in detail, let two indistinguishable particles be located at initial positions (x_1^i, x_2^i) and let their final positions be (x_1^f, x_2^f) at some later time T. The path integral formalism requires summing up all possible paths involved in the computation of the amplitude $\left\langle x_1^f, x_2^f \left| \exp(iHT) \right| x_1^i, x_2^i \right\rangle$. In 3-dimensional space, the worldlines of the two particles *braid around each* other a number of times n. Braiding implies a partition of paths into distinct classes that cannot be topologically deformed into each other. As a result, the corresponding amplitudes do not interfere quantum mechanically, which means that to each class one can associate an additional phase factor $\exp(i\alpha_n)$, besides the usual factor determined by the action.

If one particle goes around the other through an angle $\Delta\psi_k$, the additional phase factor picked up by this path represents some function of $\Delta\psi_k$, namely $exp[\,f(\Delta\psi_k)]$. For two successive paths, the extra phase factor has to comply with the composition law $\exp[if(\Delta\psi_k + \Delta\psi_{k+1})] = \exp[if(\Delta\psi_k)]\exp[if(\Delta\psi_{k+1})]$, which means that $f(\Delta\psi)$ must be linearly dependent on $\Delta\psi$. It follows that in 2+1 spacetime and for a pair of particles (1) and (2), the additional phase factor picked up by the quantum amplitude upon *anti-clockwise braiding* is given by $\exp[i(\theta/\pi)\Delta\psi_{12}]$, where θ is an arbitrary real parameter. Likewise, the extra phase factor acquired by the quantum amplitude upon *clockwise braiding* is $\exp[-i(\theta/\pi)\Delta\psi_{12}]$. The upfront distinction between the phase sign in clockwise and anti-clockwise braiding signals a violation of parity (P) and time reversal symmetry (T) [2].

If $\psi_{12} = \pi$, the two additional phase factors reduce to the familiar expressions $\exp(-i\theta)$ and $\exp(i\theta)$, typically associated with the indistinguishability of identical particles in 3+1 spacetime. In that context, if $\Psi(1,2)$ denotes the wavefunction of the system, the exchange of (1) and (2) leaves the action unchanged up to an arbitrary phase θ, namely,

$$\Psi(1,2) = \exp(i\theta)\Psi(2,1) \tag{1}$$

By repeating the exchange, the wavefunction returns to the original state while getting multiplied by $\exp(i\theta)$. It follows that $\exp(2i\theta) = 1$, $\exp(i\theta) = \pm 1$ and thus

$$\Psi(1,2) = \pm\Psi(2,1) \tag{2}$$

The (+) sign corresponds to $\theta = 2k\pi$ ($k = 0,1,2,...$) and denotes *bosons*, while (-) corresponds to $\theta = (2k+1)\pi$ and denotes *fermions*. Now bring back again the 2+1 spacetime and consider an ensemble of $i, j = 1,2,...,N$ anyons undergoing the exchange operation. A straightforward generalization of the braiding phase factors amounts to, respectively, [3]

$$\exp(-i\frac{\theta}{\pi}\sum_{i>j}\Delta\psi_{ij}) \Leftrightarrow \text{ anti-clockwise braiding} \tag{3a}$$

$$\exp(i\frac{\theta}{\pi}\sum_{i>j}\Delta\psi_{ij}) \Leftrightarrow \text{ clockwise braiding} \tag{3b}$$

Hence, the exchange statistics of anyons is embodied in the generic relationship

$$\Psi(1,2,...i,j,...,N-1,N) = \exp(\pm i\Phi)\Psi(1,2,...j,i,...N-1,N) \tag{4}$$

in which

$$\Phi = \frac{\theta}{\pi}\sum_{i>j}\Delta\psi_{ij} \tag{5}$$

If $N \to \infty$ and taking the spectrum of phases $\Delta\psi_{ij}$ to be continuous, the pair of discrete variables ($i, j \in \mathbf{Z}_+$) map to ($\xi, \tau \in \mathbf{R}$) and the sum in the right hand side of (5) changes to

$$\Phi = \frac{\theta}{\pi}\sum_{i>j}\Delta\psi_{ij} \Rightarrow \Phi = \frac{\theta}{\pi}\eta \tag{6}$$

where

$$\eta = \iint_{\xi > \tau} \Delta \psi(\xi, \tau) d\xi d\tau \tag{7}$$

Furthermore, (6) makes possible the local definition of the statistics parameter θ in η-space according to

$$\theta(\eta) = \pi \frac{d\Phi}{d\eta} \tag{8}$$

Topological field theory provides the proper field theoretic description of anyons in 2+1 spacetime. The non-relativistic limit of this framework involves Schrödinger bosons ϕ minimally coupled to a gauge potential a_μ as in [2, 3]

$$L = i\phi^{+} D_0 \phi + \frac{1}{2m} \phi^{+} \mathbf{D}^2 \phi + \frac{1}{4\theta} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \tag{9}$$

in which $D_\mu = \partial_\mu + i a_\mu$ are covariant derivatives, and $\varepsilon^{\mu\nu\lambda}$ is the total anti-symmetric symbol in 2+1 spacetime. The equations for a^μ are given by

$$j^\mu = \frac{1}{2\theta} \varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda \tag{10}$$

where j^μ is the conserved current associated with the Schrödinger bosons in (9). In terms of the anyon density $\rho = \rho(r)$, (10) takes the form

$$\varepsilon^{ij} \partial_i a_j = 2\theta\rho \tag{11}$$

which explicitly shows that θ acts as the source of the gauge potential. If θ has a continuous spectrum defined by (8), (11) may be formally extended to

$$\varepsilon^{ij} \partial_i a_j(\eta) = 2\,\theta(\eta)\,\rho = 2\pi\rho \frac{d\Phi}{d\eta} \tag{12}$$

which highlights the local definition of a_μ in η-space.

An extension of the ordinary topological field theory is provided by the Maxwell-Chern-Simons (MCS) Lagrangian, in which the third term of (9) is supplemented by the Maxwell term [3]

$$\frac{1}{4g^2} f^{\mu\nu} f_{\mu\nu} \tag{13}$$

with $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. It is apparent that this term decouples from the Lagrangian in the limit $g \to \infty$ where one recovers the ordinary topological field theory. Considering only the gauge sector of this model ($g < \infty,\ j_\mu = 0$)

$$L_{gauge} = -\frac{1}{4g^2} f^{\mu\nu} f_{\mu\nu} + \frac{1}{4\theta} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \tag{14}$$

yields the following equation of motion for the Maxwell field

$$\partial_\mu \partial^\mu f_\rho + \frac{g^4}{4\theta^2} f_\rho = 0 \tag{15}$$

where the dual field tensor is given by

$$f^\mu = \frac{1}{2} \varepsilon^{\mu\nu\lambda} f_{\nu\lambda} \tag{16}$$

Equation (15) implies that photons in the MCS theory acquire mass on account of the field coupling g and statistics phase θ. Since the Maxwell term dominates at large distances, one expects $g << \infty$ and thus the gain in photon mass amounts to

$$m_\gamma = \frac{g^2}{2\theta}\ , \quad g << \infty,\ \theta \neq 0 \tag{17}$$

This mass generation mechanism has a clear topological underpinning and is manifestly different from the symmetry breaking process driven by the Higgs scalar. It *is not*, however, in conflict with SM as it only occurs in 2+1 spacetime.

3. Anyons embedded in minimal fractal spacetime

So far, building the anyon theory contained in (1) to (17) was carried out in 2+1 dimensions. A natural question is then: How does one extrapolate the anyon formalism on the minimal fractal manifold (MFM), which represents a spacetime background endowed with $D = 4 - \varepsilon$ ($\varepsilon << 1$) dimensions [4–9].

Posing this question is relevant in a more general context, namely in the long-term analysis of Renormalization Group flows approaching *strange attractors* and exhibiting *chaotic mixing and diffusion* [10-12]. Given that the dynamics of MFM is typically formulated using fractional operators and/or fractional functions, a reasonable "leading-order" approximation may be developed by substituting the exponential factor of (1) - (4) with its fractional counterpart. Following [13], the fractional exponential function is accordingly chosen to be

$$e_\beta(\omega,t)=\begin{cases}\exp(-i\,|\omega|^{1/\beta}\,t),\ \omega\le 0\\ \exp(i\,|\omega|^{1/\beta}\,t),\ \omega\ge 0\end{cases}\tag{18}$$

where $\beta=1-\varepsilon_s=1-(3-D_s)$, corresponding to $\varepsilon_s=3-D_s<<1$ deviations from three spatial dimensions and where parameters (ω,t) play the role of "angular velocity" and "braiding time". In light of (18), (6) and (7) are upgraded to, respectively,

$$\Phi_{\varepsilon_s}=\frac{1}{\pi}\int\theta_{\varepsilon_s}(\eta)\,d\eta_{\varepsilon_s}\ =|\omega|^{1/1-\varepsilon_s}\,t\tag{19}$$

$$\eta_{\varepsilon_s}=\iint_{\xi>\tau}\Delta\psi_{\varepsilon_s}(\xi,\tau)\,d\xi\,d\tau\tag{20}$$

in which

$$\Delta\psi_{\varepsilon_s}(\xi,\tau)=\Omega(\xi,\tau)^{1/1-\varepsilon_s}\,t(\xi,\tau)\tag{21}$$

and

$$d\eta_{\varepsilon_s}=\Delta\psi_{\varepsilon_s}(\xi,\tau)\,d\xi\,d\tau=[\Omega(\xi,\tau)^{1/1-\varepsilon_s}\,t(\xi,\tau)]\,d\xi\,d\tau\tag{22}$$

Note that (19) and (21) contain locally defined angular velocities $\Omega(\xi,\tau)$ and braiding times $t(\xi,\tau)$. Also note that, in general, angular velocities entering (19) and (21) are assumed to be different, i.e., $\omega\neq\Omega(\xi,\tau)$ for any pair (ξ,τ). By (19) and (20), the analog expression of (8) on the MFM is given by

$$\theta_{\varepsilon_s}(\eta_{\varepsilon_s})=\pi\frac{d\Phi_{\varepsilon_s}}{d\eta_{\varepsilon_s}}\tag{23}$$

Since $\beta=1-\varepsilon_s=1-(3-D_s)<<1$, the frequency entering (18) may be approximated as

$$|\omega|^{1/\beta}=|\omega|^{1/(1-\varepsilon_s)}\approx|\omega|^{(1+\varepsilon_s)}=|\omega||\omega|^{\varepsilon_s}\approx|\omega|(1+\varepsilon_s\ln|\omega|)=|\omega|+\varepsilon_s(|\omega|\ln|\omega|)$$

and enables a convenient formulation of (19) and (21), namely

$$\Phi_{\varepsilon_s} = \Phi + \varepsilon_s |\omega| \ln|\omega| t \tag{24}$$

$$\Delta\psi_{\varepsilon_s}(\xi,\tau) = \Delta\psi(\xi,\tau) + \varepsilon_s |\Omega(\xi,\tau)| \ln|\Omega(\xi,\tau)| t(\xi,\tau) \tag{25}$$

It is apparent from (19) to (25) that the standard anyon theory in 2+1 dimensions is recovered by letting $\varepsilon_s = 3 - D_s$ drop to zero. Likewise, the analog approximation of photon mass (17) on the MFM can be presented as

$$m_{\gamma,\varepsilon_s} = \frac{g^2}{2\theta_{\varepsilon_s}} = \frac{g^2}{2\pi}\left(\frac{d\Phi_{\varepsilon_s}}{d\eta_{\varepsilon_s}}\right)^{-1} \tag{26}$$

Ordinary massless photons correspond to singular slopes of the phase angle (19) in $\eta-$ space, $d\Phi_{\varepsilon_s}/d\eta_{\varepsilon_s} \to \infty$. In a broader interpretation, it is tempting to speculate that an entire hierarchy of gauge boson masses may be derived from (26) upon letting the dimensional parameter $\varepsilon_s = 3 - D_s$ sweep a progressive sequence of values ordered according to the Feigenbaum scenario [4, 24].

4. Further extensions

Among the many questions open for follow-up clarifications, we mention the following:

1) How does the spin-statistics connection change in the presence of chaotic mixing, a hallmark feature of strange attractors? In particular, how does quantum entanglement enter the picture when describing swapping of anyons in minimal fractal spacetime?

2) Can (17) and (26) supply a viable explanation for the recently claimed discovery of massive boson X17 [20]? (excluding, of course, the effect of systematic errors or faulty theoretical premises)

3) How does our approach relate to many contemporary topics of condensed matter theory such as (but not limited to) fractional quantum Hall effect, anyon condensation, topological phases of matter, string-net condensation, quantum criticality? To give a single example, the statistics parameter in the analysis of *quantum Hall fluids* is given by

$$\theta = \frac{\pi}{\nu} \ , \quad \nu = \frac{1}{2k+1}, \ k = 1, 2, ... \tag{27}$$

in which the filling factor ν takes on fractional values. Is there a meaningful connection between (23) and (27) that can be deployed in numerical simulations of DM?

4) How does our approach change upon working with the full formalism of fractional calculus and fractional vector calculus, which is entirely *non-local* (see, e.g. [23])?

5) One can make the case that the phase angle (19) contains either "fast" angular frequencies for short time spans (UV regime) or "slow" angular frequencies for long time spans (IR regime). Either way, the term describing the contribution of the MFM $\varepsilon_s |\omega| \ln|\omega| t$ may be interpreted as being *asymptotically undefined* in the limit $\varepsilon_s \to 0, \omega \to \infty, t \to 0$ or $\varepsilon_s \to 0, \omega \to 0, t \to \infty$. Likewise, it would be instructive to explore the regime where the angular frequency $|\omega|^{1/\beta}$ becomes singular and the phase $\exp[\pm i |\omega|^{1/\beta} t]$ undergoes large fluctuations for finite time intervals $t > 0$.

6) How does this approach relate to the superfluid model of DM and its anisotropy [21]?

For future reference, we close by tabulating a helpful (albeit preliminary) analogy between anyons in 2+1 spacetime and anyons on the MFM.

Anyons in 2+1 dimensions	**Anyons in minimal fractal spacetime**
Breaking P and T symmetries by braiding operations	Breaking P and T symmetries through fractional dynamics [14]
Long-range phase interactions through topological field theory	Long-range coupling through fractional dynamics [4, 14, 23]
Topological field theory is metric independent [2]	Fractional dynamics in flat spacetime is dual to conventional dynamics in curved spacetime [14-16]

Charge fractionalization in quantum Hall fluids [2]	Charge fractionalization through fractional dynamics [17]
Massive photons in the MCS theory per (17) and (26)	Mass generation through the minimal fractal geometry of spacetime [4, 11, 19, 22]
Topological condensation of quasiparticles	Higgs scalar as weakly-coupled topological condensate of gauge bosons [4, 18]

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