Article

Why Ramified Primes Are So Special Physically

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Abstract

Ramified primes are special in the sense that their expression as a product of primes P_i of extension contains higher than first powers and the number P_i is smaller than the maximal number n defined by the dimension of the extension. The proposed interpretation of ramified primes is as p-adic primes characterizing space-time sheets assignable to elementary particles and even more general systems. It is not quite clear why ramified primes appear as preferred p-adic primes and in the following Dedekind zeta functions and what I call ramified zeta functions inspired by the interpretation of zeta function as analog of partition function are used in attempt to understand why ramified primes could be physically special. The intuitive feeling is that quantum criticality is what makes ramified primes so special. In $O(p) = 0$ approximation the irreducible polynomial defining the extension of rationals indeed reduces to a polynomial in finite field F_p and has multiple roots for ramified prime, and one can deduce a concrete geometric interpretation for ramification as quantum criticality using $M^8 - H$ duality.

Keywords: Ramified primes, elementary particle, zeta function, quantum criticality, TGD framework.

1 Introduction

Ramified primes (see http://tinyurl.com/m32nvcz and http://tinyurl.com/y6yskkas) are special in the sense that their expression as a product of primes of extension contains higher than first powers and the number of primes of extension is smaller than the maximal number n defined by the dimension of the extension. The proposed interpretation of ramified primes is as p-adic primes characterizing space-time sheets assignable to elementary particles and even more general systems.

In the following Dedekind zeta functions (see http://tinyurl.com/y5grktvp) as generalization of Riemann zeta [6, 7] are studied to understand what makes them so special. Dedekind zeta function characterizes given extension of rationals and is defined by reducing the contribution from ramified reduced so that effectively powers of primes of extension are replaced with first powers.

If one uses the naive definition of zeta as analog of partition function and includes full powers $P_i^{e_i}$, the zeta function becomes a product of Dedekind zeta and a term consisting of a finite number of factors having poles at imaginary axis. This happens for zeta function and its fermionic analog having zeros along imaginary axis. The poles would naturally relate to Ramond and N-S boundary conditions of radial partial waves at light-like boundary of causal diamond CD. The additional factor could code for the physics associated with the ramified primes.

The intuitive feeling is that quantum criticality is what makes ramified primes so special. In $O(p) = 0$ approximation the irreducible polynomial defining the extension of rationals indeed reduces to a polynomial in finite field F_p and has multiple roots for ramified prime, and one can deduce a concrete geometric interpretation for ramification as quantum criticality using $M^8 - H$ duality.

This article is one in a series of articles related to the number theoretical aspects of TGD. $M^8 - H$ duality central concept in following and discussed in [5, 10, 8, 9] [3]. Also the notion of cognitive representation as a set of points of space-time surface with preferred imbedding space coordinates belonging to the extension of rationals defining the adele [2] is important and discussed in [12, 11, 14].

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2 Dedekind zeta function and ramified primes

One can take mathematics and physical intuition guided by each other as a guideline in the attempts to understand ramified primes.

1. Riemann zeta can be generalized to Dedekind zeta function ζ_K for any extension K of rationals (see `http://tinyurl.com/y5grktvp`). ζ_K characterizes the extension - maybe also physically in TGD framework since zeta functions have formal interpretation as partition function. In the recent case the complexity is not a problem since complex square roots of partition functions would define the vacuum part of quantum state: one can say that quantum TGD is complex square root of thermodynamics.

   ζ_K satisfies the same formula as ordinary zeta expect that one considers algebraic integers in the extensions K and sums over non-zero ideals a - identifiable as integers in the case of rationals - with n^{-s} replaced with $N(a)^{-s}$, where $N(a)$ denotes the norm of the non-zero ideal. The construction of ζ_K in the extension of rationals obtained by adding i serves as an illustrative example (see `http://tinyurl.com/y563wcwv`). I am not a number theorists but the construction suggests a poor man's generalization strongly based on physical intuition.

2. The rules would be analogous to those used in the construction of partition function. $log(N(a))$ is analogous to energy and s is analogous to inverse temperature so that one has Boltzmann weight $exp(-log(N(a)s)$ for each ideal. Since the formation of ideals defined by integers of extension is analogous to that for forming many particle states labelled by ordinary primes and decomposing to primes of extension, the partition function decomposes to a product over partition functions assignable to ordinary primes just like in the case of Riemann zeta. Let K be an extension of rationals Q.

3. Each rational prime p decomposes in the extension as $p = \prod_{i=1,...g} P_i^{e_i}$, where n is the dimension of extension and e_i is the ramification degree. Let f_i be so called residue degree of P_i defined as the dimension of K mod P_i interpreted as extension of rational integers $Z \ mod \ p$. Then one has $\sum_1^g e_i f_i = n$.

   Remark: For Galois extensions for which the order of Galois group equals to the dimension n of the extension so that for given prime p one has $e_i = e$ and $f_i = f$ and $efg = n$.

4. Rational (and also more general) primes can be divided into 3 classes with respect to this decomposition.

   For ramified primes dividing the discriminant D associated with the polynomial ($D = b^2 - 4c$ for $P(x) = x^2 + bx + c$) one has $e_i > 1$ at least for one i so that $f_i = 0$ is true at least for one index. A simple example is provided by rational primes (determined by roots of $P(x) = x^2 + 1$ with discriminant -4): in this case $p = 2$ corresponds to ramified prime since on has $(1 + i)(1 - i) = 2$ and $1 + i$ and $1 - i$ differ only by multiplication by unit $-i$.

5. Split primes have n factors P_i and thus have $(e_i = 1, f_i = 1, g = n)$. They give a factor $(1-p^{-s})^{-n}$. The physical analogy is n-fold degenerate state with original energy energy $nlog(p)$ split to states with energy $log(p)$.

   Inert primes are also primes of extension and there is no splitting and one has $(e_1 = e = 1, g = 1, f_1 = f = n)$. In this case one obtains factor $1/(1 - p^{-ns})$. The physical analogy is n-particle bound state with energy $nlog(p)$.

6. For ramified primes the situation is more delicate. Generalizing from the case of Gaussian primes $Q[i]$ (see `http://tinyurl.com/y563wcwv`) ramified primes p_R would give rise to a factor

$$\prod_{i=1}^{g} \frac{1}{1-p_R^{-f_i s}} \quad .$$

g is the number of *distinct* ideals P_i in the decomposition of p to the primes of extension.

For Gaussian primes $p=2$ has $g=1$ since one can write $(2)=(1+i)(1-i)\equiv(1+i)^2$. This because $1+i$ and $1-i$ differ only by multiplication with unit $-i$ and thus define same ideal in $Q[i]$. Only the number g of distinct factors P_i in the decomposition of p matters.

One could understand this as follows. For the roots of polynomials ramification means that several roots co-incide so that the number of distinct roots is reduced. $e_i>1$ is analogous to the number co-inciding roots so that number if distinct roots would be 1 instead of e_i. This would suggests $k_i=1$ always. For ramified primes the number of factors Z_p the number $\sum_{i=1}^g f_i k_i = n$ for un-ramified case would reduce from to $\sum_{i=1}^g f_i k_i = n_d$, which is the number of distinct roots.

7. Could the physical interpretation be that there are g types of bound states with energies $f_i log(p)$ appearing with degeneracy $e_i=1$ both in ramified and split case. This should relate to the fact that for ramified primes p L/p contains non-vanishing nilpotent element and is not counted. One can also say that the decomposition to primes of extension conserves energy: $\sum_{i=1,...,g} e_i f_i log(p) = n_d log(p)$.

   For instance, for Galois extensions ($e_i=e, f_i=f, g=n_d/ef$) for given p the factor is $1/(1-p^{-es})^{fg}$: $efg=n_d$. If there is a ramification then all P_i are ramified. Note that e,f an g are factors of n_d.

8. One can can extract the factor $1/(1-p^{-s})$ from each of the 3 contributions and organize these factors to give the ordinary Riemann zeta. The number of ramified primes is finite whereas the numbers of split primes and inert primes are infinite. One can therefore extract from ramified primes the finite product

   $$\zeta^1_{R,K} = \prod_{p_R}(1-p_R^{-s})\times\zeta^2_{R,K} \quad , \quad \zeta^2_{R,K} = \prod_{p_R}[\prod_{i=1}^g \frac{1}{1-p^{-f_i s}}] \quad .$$

   One can organize the remaining part involving infinite number of factors to a product of ζ and factors $(1-p^{-s})/(1-\prod p^{-s}))^n$ and $(1-p^{-s})/(1-p^{-ns})$ giving rise to zeta function -call it $\zeta_{si,K}$ - characterizing the extension. Note that $\zeta^2_{R,K}$ has interpretation as partition function and has pole of order n_d at origin.

   One therefore can write the ζ_L as

   $$\zeta_K = \zeta^1_{R,K}\times\zeta_{si,K}\times\zeta \quad .$$

   where $\zeta_{si,K}$ is the contribution of split and inert primes multiplied by $(1-p^{-s})$

ζ_L has pole only at $s=1$ and it carries in no obvious manner information about ramified primes. The naive guess for ζ_L would be that also ramified primes p_R would give rise to a factor

$$\prod_{i=1}^{g} \frac{1}{(1-p_R^{-f_i s})^{e_i}} \quad .$$

One could indeed argue that at the limit when e_i prime ideals P_i of extension co-incide, one should obtain this expression. The resulting ζ function would be product

$$\zeta_{naive,K} = \zeta_{R,K}\zeta_K \quad , \qquad \zeta_{R,K} = \prod_{p_R} X(p_R)$$
$$X(p_R) = \prod_{i=1}^g \frac{1}{(1-p_R^{-f_i s})^{e_i-1}} \quad .$$

Note that the parameters e_i, f_i, g depend on p_R and that for Galois extensions one has $e_i = d, f_i = f$ for given p_R. $\zeta_{R,L}$ would have poles at along imaginary axis at points $s = -n2\pi/log(p)$. Ramified primes would give rise to poles along imaginary axis. As far as the proposed physical interpretation of ramified primes is considered, this form looks more natural.

2.1 Fermionic counterparts of Dedekind zeta and ramified ζ

One can look the situation also for the generalization of fermionic zeta as analog of fermionic partition function, which for rationals has the expression

$$\zeta^F(s) = \prod_p (1 + p^{-s}) = \frac{\zeta(s)}{\zeta(2s)} \quad .$$

Supersymmetry of supersymmetric arithmetic QFT suggest the product of fermionic and bosonic zetas. Also the supersymmetry of infinite primes for which first level of hierarchy corresponds to irreducible polynomials suggests this. On the other hand, the appearance of only fermions as fundamental particles in TGD forces to ask whether the ramified part of fermionic zeta might be fundamental.

1. By an argument similar to used for ordinary zeta based on interpretation as partition function, one obtains the decomposition of the fermionic counterpart of ζ_K^F Dirichlet zeta to a product $\zeta_K^F = \zeta_{R,K}^F \zeta_{si,K}^F \zeta^F$ of ramified fermionic zeta $\zeta_{R,K}^F$, $\zeta_{si,K}^F$, and ordinary fermionic zeta ζ^F. The basic rule is simple: replace factors $1/(1 - p^{-ks}$ appearing in ζ_K with $(1 + p^{-ks})$ in ζ_K^F and extract ζ^F from the resulting expression. This gives

$$\zeta_{R,K}^{F,1} = \prod_{p_R} (1 - p_R^{-s}) \zeta_{R,K}^F \quad , \quad \zeta_{R,K}^F = \prod_{p_R} [\prod_{i=1}^g (1 + p_R^{-f_i s})] \quad .$$

   where p_R is ramified prime dividing the discriminant. $\zeta_{R,K}^F$ is analogous to a fermionic partition function for a finite number of modes defined by ramified primes p_R of extension.

2. Also now one can wonder whether one should define ζ_K^F as a product in which ramified primes give factor

$$\prod_{p_R} [\prod_{i=1}^{g} (1 + p_R^{-f_i s})^{e_i}]$$

   so that one would have

$$\zeta_{naive,K}^F = \zeta_{R,K}^F \zeta_K^F \quad , \qquad \zeta_R^F = \prod_{p_R} Y(p_R) \quad ,$$

$$Y(p_R) = \prod_{i=1}^g (1 + p_R^{-f_i s})^{e_i - 1}$$

   $\zeta_F(naive, K)$ would have zeros along imaginary axis serving as signature of the ramified primes.

2.2 About physical interpretation of $\zeta_{R,K}$ and $\zeta_{R,K}^F$

$\zeta_{R,K}$ and $\zeta_{R,K}^F$ are attractive from the view point of number theoretic vision and the idea that ramified primes are physically special. TGD Universe is quantum critical and in catastrophe theory the ramification for roots of polynomials is analogous to criticality. Maybe the ramification for p-adic primes makes them critical. $K/(p_R)$ has nilpotent elements, which brings in mind on mass shell massless particles.

1. $\zeta_{R,K}$ has poles at

$$s = i\frac{2n\pi}{log(p)f_i}$$

and $p_R^s = exp(in2\pi/f_i)$ is a root of unity, which conforms with the number theoretical vision. Only P_i with $e_i > 1$ contribute.

2. $Z^F_{R,K}$ has zeros

$$s = i\frac{(2n+1)\pi}{log(p)f_i}$$

and $p_R^s = exp(i(2n+1)\pi/f_i)$ is a root of unity. Zeros are distinct from the poles of $Z_{R,K}$.

3. The product $\zeta_{R,tot,K} = \zeta_{R,K}\zeta^F_{R,K}$ has the poles and zeros of $\zeta_{R,K}$ and $\zeta^F_{R,K}$. In particular, there is n:th order pole of $Z_{R,K}$ at $s = 0$. The zeros of $z_{F,K}$ along imaginary axis at $p^{iy} = -1$ also survive in $\zeta_{R,tot,K}$.

   $\zeta^F_{R,K}$ has only zeros and since fundamental fermions are primary fields in TGD framework, one could argue that only it carries physical information. On the other hand, supersymmetric arithmetic QFT [1] and the fact that TGD allows SUSY [13] suggests that the product $\zeta_{R,K} \times Z^F_{R,K}$ is more interesting.

From TGD point of view the ramified zeta functions $\zeta_{R,K}$, $\zeta^F_{R,K}$ and their product $\zeta_{R,K} \times \zeta^F_{R,K}$ look interesting.

1. $\zeta_{R,K}$ behaves like s^{-n_d}, $n_d = \sum_1^g (e_i - 1)$ near the origin. Could n_d-fold pole at $s = 0$ be interpreted in terms of a massless state propagating along light-cone boundary of CD in radial direction? This would conform with the proposal that zeros of zeta correspond to complex radial conformal weights for super-symplecti algebra. That ramified primes correspond to massless particles would conform with the identification of ramified prime as p-adic primes labelling elementary particles since in ZEO their mass would result from p-adic thermodynamics from a mixing with very massive states [9].

   Besides this there would be stringy spectrum of real conformal weights along negative real axis and those coming as non-trivial zeros and these could correspond to ordinary conformal weights.

2. The zeros of $\zeta^F_{R,K}$ along imaginary axis might have interpretation as eigenvalues of Hamiltonian in analogy with Hilbert-Polya hypothesis. Maybe also the poles of $\zeta_{R,K}$ could have similar interpretation. The real part of zero/pole would not produce troubles (on the other hand, for waves along light-cone boundary it can be however absorbed to the integration measure.

3. A possible physical interpretation of the imaginary conformal weights could be as conformal weights associated with radial waves assignable to the radial light-like coordinate r of the light-cone boundary: r indeed plays the role of complex coordinate in conformal symmetry in the case of super-symplectic algebra suggested to define the isometries of WCW. Poles and zero could correspond to radial modes satisfying periodic/anti-periodic boundary conditions.

   The radial conformal weights s defined by the zeros of $\zeta^F_{R,K}$ would be number theoretically natural since one could pose boundary condition $p^{is(r/r_0)} = -1$ at $r = r_0$ requiring $p^{is} = -1$ at the corner of cd (maximum value of r in $CD = cd \times CP_2$.

   For the poles of $\zeta_{R,K}$ the periodic boundary condition $p^{is(r/r_0)} = 1$ is natural. The two boundary conditions could relate to Ramond an N-S representations of super-conformal algebras (see `http:`

//tinyurl.com/y49y2ouj). With this interpretation $s = 0$ would correspond to a radial plane-wave constant along light-like radial direction and therefore light-like momentum propagating along the boundary of CD. Other modes would correspond to other massless modes propagating to the interior of CD.

4. I have earlier considered an analogous interpretation for a subset zeros of zeta satisfying similar condition. The idea was that for given prime p as subset of $s = 1/2 + iy_i$ of non-trivial zeros ζ $p^s = p^{1/2+iy_i}$ is an algebraic number so that p^{iy_i} would be a root of unity. Zeros would decompose to subsets labelled by primes p. Also for trivial zeros of ζ (and also poles) the same holds true as for the zeros and poles ζ_R. This encourages the conjecture that the property is true also for L-functions.

The proposed picture suggests an assignment of "energy" $E = nlog(p)$ to each prime and separation of "ramified" energy $E_d = n_d log(p)$, $n_d = \sum_1^g f_i(e_i - 1)$, to each ramified prime. The interpretation as partition function suggests that that one has g types of states of f_i identical particles and energy $E_i = f_i log(p)$ and that this state is e_i-fold degenerate with energies $E_i = f_i log(p)$. For inert primes one would have $f_i = f = n$. For split primes one would have $e_i = 1, f_i = 1$. In case of ramified primes one can separate one of these states and include it to the Dedekind zeta.

2.3 Can one find a geometric correlate for the picture based on prime ideals?

If one could find a geometric space-time correlate for the decomposition of rational prime ideals to prime ideals of extensions, it might be also possible to understand why quantum criticality makes ramified primes so special physically and wha this means.

What could be correlate for f_i fundamental fermions behaving like single unit and what degeneracy for $e_i > 1$ does mean? One can look the situation first at the level of number fields Q and K and corresponding Galois group $Gal(K/Q)$, finite fields $F = Q/p$ and $F_i = K/P_i$, and corresponding Galois group $Gal(F_i/F)$. Appendix summarizes the basic terminology.

1. Inertia degree f_i is the number of elements of F_i/F_p ($F_i = K/P_i$ is extension of finite field $F_p = Q/p$). The Galois group $Gal(F_i/F_p)$ is identifiable as factor group D_i/I_i, where the *decomposition group* D_i is the subgroup of Galois group taking P_i to itself and the *inertia group* I_i leaving P_i point-wise invariant. The orbit under $Gal(F_i/F_p)$ in F_i/F_p would behave like single particle with energy $E_i = f_i log(p)$.

   For inert primes with $f_i = n$ inertia group would be maximal. For split primes the orbits of ideals would consist of $f_i = 1$ points only and isotropy group would be trivial.

2. Ramification for primes corresponds intuitively to that for polynomials meaning multiple roots as is clear also from the expression $p = \prod P_i^{e_i}$. In accordance with the intuition about quantum criticality, ramification means that the irreducible polynomial reduced to a reducible polynomial in finite field Q/p has therefore a multiple roots with multiplicities e_i (see Appendix). For Galois extensions one has $(e_i = e, f_i = f)$ Criticality would be seen at the level of finite field $F_p = Q/p$ associated with ramified prime p.

The interpretation of roots of corresponding octonionic polynomials as n-sheeted covering space like structures encourages to ask whether the independent tensor factors labelled by i suggested by the interpretation as a partition function could be assigned with the sheets of covering so that ramification with $e_i > 1$ would correspond to singular points of cognitive representation for which e_i sheets co-incide in some sense, maybe in finite field approximation ($O(p) = 0$). Galois groups indeed act on the coordinates of point of cognitive representation belonging to the extension K. In general the action does not take the point to a point belonging to a cognitive representation but one can consider quantum superpositions of cognitive representations.

This suggests an interpretation in terms of space-time surfaces accompanied by cognitive representation under Galois group. Quantum states would be superpositions of preferred extremals at orbits of Galois group and for cognitive representations the situation would be discrete.

1. To build a concrete connection between geometric space-time picture and number theoretic picture, one should find geometric counterparts of integers, ideals, and prime ideals. The analogs of prime ideals should be associated with the discretizations of space-time surfaces/cognitive representations in $O(p) = 0$ or $O(P_i) = 0$ approximation. Could one include only points of cognitive representations differing from zero in $O(p) = 0$ approximation and form quantum states as quantum superpositions of these points of cognitive representation?

   in $O(p) = 0$ approximation and for ramified primes irreducible polynomials would have multiple roots so that e_i sheets would co-incide at these points in $O(p) = 0$ approximation. Th conjecture that elementary particles correspond to this kind of singularities has been speculated already earlier with inspiration coming from quantum criticality.

2. In M^8 picture the octonionic polynomials obtained as continuation of polynomials with rational coefficients would be reduced to polynomials in finite field F_p. One can study corresponding discrete algebraic surfaces as discrete approximations of space-time surfaces.

3. One would like to have only single imbedding space coordinate since the probability that all imbedding space coordinates correspond to the same P_i is small. $M^8 - H$ duality reduces the number of imbedding space coordinates characterizing partonic 2-surfaces containing vertices for fundamental fermions to single one identifiable as time coordinate.

   At the light-like boundary of 8-D CD in M^8 the vanishing condition for the real or imaginary part (quaternion) of octonionic polynomial $P(o)$ reduces to that for ordinary polynomial, and one obtains n roots r_n, which correspond to the values of M^4 time $t = r_n$ defining 6-spheres as analogs of branes. Partonic 2-surfaces corresponde to intersections of 4-D roots of $P(o)$ at partonic 2-surfaces. Galois group of the polynomial naturally acts on r_n labelling these partonic 2-surfaces by permuting them. One could form representations of Galois group using states identified as quantum superpositions of these partonic 2-surfaces corresponding to different values of $t = r_n$. Galois group leaves invariant the degenerate roots $t = r_n$.

4. The roots can be reduced to finite field F_p or K/P_i. Ramification would take place in this approximation and mean that e_i roots $t = r_n$ are identical in $O(p) = 0$ approximation. e_i time values $t = r_n$ would nearly co-incide. This gives more concrete contents to the statement of TGD inspired theory of consciousness that these time values correspond to very special moments in the life of self. Since this is the situation only approximately, one can argue that one must indeed count each root separately so that partition function must be defined as product of the contribution form ramified primes an Dedekind zeta.

   The assignment of fundamental fermions to the points of cognitive representations at partonic 2-surfaces assignable to the intersections of 4-D roots and universal 6-D roots of octonionic polynomials (brane like entities) conforms with this picture.

5. The analogs of 6-branes would give rise to additional degrees of freedom meaning effectively discrete non-determinism. I have speculated with this determinism with inspiration coming from the original identification of bosonic action as Kähler action having huge 4-D spin glass degeneracy. Also the number theoretic vision suggest the possibility of interpreting preferred extremals as analogs of algebraic computations such that one can have several computations connecting given states [4]. The degree n of polynomial would determine the number of steps and the degeneracy would correspond to n-fold degeneracy due to the discrete analogs of plane waves in this set.

2.4 What extensions of rationals could be winners in the fight for survival?

It would seem that the fight for survival is between extensions of rationals rather than individual primes p. Intuition suggests that survivors tend to have maximal number of ramified primes. These number theoretical speciei can live in the same extension - to "co-operate".

Before starting one must clarify some basic facts about extensions of rationals.

1. Extension of rationals are defined by an irreducible polynomial with rational coefficients. The roots give n algebraic numbers which can be used as a basis to generate the numbers of extension ast their rational linear combinations. Any number of extension can be expressed as a root of an irreducible polynomial. Physically it is is of interest, that in octonionic picture infinite number of octonionic polynomials gives rise to space-time surface corresponding to the same extension of rationals.

2. One can define the notion of integer for extension. A precise definition identifies the integers as ideals. Any integer of extension are defined as a root of a monic polynomials $P(x) = x^n + p_{n-1}x^{n-1}x^{n-1} + ... + p_0$ with integer coefficients. In octonionic monic polynomials are subset of octonionic polynomials and it is not clear whether these polynomials could be all that is needed.

3. By definition ramified primes divide the discriminant D of the extension defined as the product $D = \prod_{i\neq j}(r_i - r_j)$ of differences of the roots of (irreducible) monic polynomial with integer coefficients defining the basis for the integers of extension. Discriminant has a geometric interpretation as volume squared for the fundamental domain of the lattice of integers of the extension so that at criticality this volume interpreted as p-adic number would become small for ramified primes an vanish in $O(p)$ approximation. The extension is defined by a polynomial with rational coefficients and integers of extension are defined by monic polynomials with roots in the extension: this is not of course true for all monic polynomials polynomial (see `http://tinyurl.com/k3ujjz7`).

4. The scaling of the $n-1$-tuple of coefficients $(p_{n-1}, ..., p_1)$ to $(ap_{n-1}, a^2p_{n-1}....., a^np_0)$ scales the roots by a: $x_n \rightarrow ax_n$. If a is rational, the extension of rationals is not affected. In the case of monic polynomials this is true for integers k. This gives rational multiples of given root.

   One can decompose the parameter space for monic polynomials to subsets invariant under scalings by rational $k \neq 0$. Given subset can be labelled by a subset with vanishing coefficients $\{p_{i_k}\}$. One can get rid of this degeneracy by fixing the first non-vanishing p_{n-k} to a non-vanishing value, say 1. When the first non-vanishing p_k differs from p_0, integers label the polynomials giving rise to roots in the same extension. If only p_0 is non-vanishing, only the scaling by powers k^n give rise to new polynomials and the number of polynomials giving rise to same extension is smaller than in other cases.

   Remark: For octonionic polynomials the scaling symmetry changes the space-time surface so that for generic polynomials the number of space-time surfaces giving rise to fixed extension is larger than for the special kind polynomials.

Could one gain some understanding about ramified primes by starting from quantum criticality? The following argument is poor man's argument and I can only hope that my modest technical understanding of number theory does not spoil it.

1. The basic idea is that for ramified primes the minimal monic polynomial with integer coefficients defining the basis for the integers of extension has multiple roots in $O(p) = 0$ approximation, when p is ramified prime dividing the discriminant of the monic polynomial. Multiple roots in $O(p) = 0$ approximation occur also for the irreducible polynomial defining the extension of rationals. This would correspond approximate quantum criticality in some p-adic sectors of adelic physics.

2. When 2 roots for an irreducible rational polynomial co-incide, the criticality is exact: this is true for polynomials of rationals, reals, and all p-adic number fields. One could use this property to

construct polynomials with given primes as ramified primes. Assume that the extension allows an irreducible olynomial having decomposition into a product of monomials $= x - r_i$ associated with roots and two roots r_1 and r_2 are identical: $r_1 = r_2$ so that irreducibility is lost.

The deformation of the degenerate roots of an irreducible polynomial giving rise to the extension of rationals in an analogous manner gives rise to a degeneracy in $O(p) = 0$ approximation. The degenerate root $r_1 = r_2$ can be scaled in such a manner that the deformation $r_2 = r_1(1 + q))$, $q = m/n = O(p)$ is small also in real sense by selecting $n >> m$.

If the polynomial with rational coefficients gives rise to degenerate roots, same must happen also for monic polynomials. Deform the monic polynomial by changing $(r_1, r_2 = r_1)$ to $(r_1, r_1(1 + r))$, where integer r has decomposition $r = \prod_i p_i^{k_i}$ to powers of prime. In $O(p) = 0$ approximation the roots r_1 and r_2 of the monic polynomial are still degenerate so that p_i represent ramified primes.

If the number of p_i is large, one has high degree of ramification perhaps favored by p-adic evolution as increase of number theoretic co-operation. On the other hand, large p-adic primes are expected to correspond to high evolutionary level. Is there a competition between large ramified primes and number of ramified primes? Large $h_{eff}/h_0 = n$ in turn favors large dimension n for extension.

3. The condition that two roots of a polynomial co-incide means that both polynomial $P(x)$ and its derivative dP/dx vanish at the roots. Polynomial $P(x) = x^n + p_{n-1}x^{n-1} + ..p_0$ is parameterized by the coefficients which are rationals (integers) for irreducible (monic) polynomials. $n - 1$-tuple of coefficients $(p_{n-1}, ..., p_0)$ defines parameter space for the polynomials. The criticality condition holds true at integer points $n - 1 - D$ surface of this parameter space analogous to cognitive representation.

   The condition that critical points correspond to rational (integer) values of parameters gives an additional condition selecting from the boundary a discrete set of points allowing ramification. Therefore there are strong conditions on the occurrence of ramification and only very special monic polynomials are selected.

   This suggests octonionic polynomials with rational or even integer coefficients, define strongly critical surfaces, whose p-adic deformations define p-adically critical surfaces defining an extension with ramified primes p. The condition that the number of rational critical points is non-vanishing or even large could be one prerequisite for number theoretical fitness.

4. There is a connection to catastrophe theory, where criticality defines the boundary of the region of the parameter space in which discontinuous catastrophic change can take place as replacement of roots of $P(x)$ with different root. Catastrophe theory involves polynomials $P(x)$ and their roots as well as criticality. Cusp catastrophe is the simplest non-trivial example of catastrophe surface with $P(x) = x^4/4 - ax - bx^2/2$: in the interior of V-shaped curve in (a, b)-plane there are 3 roots to $dP(x) = 0$, at the curve 2 solutions, and outside it 1 solution. Note that now the parameterization is different from that proposed above. The reason is that in catastrophe theory diffeo-invariance is the basic motivation whereas in M^8 there are highly unique octonionic preferred coordinates.

If p-adic length scale hypothesis holds true, primes near powers of 2, prime powers, in particular Mersenne primes should be ramified primes. Unfortunately, this picture does not allow to say anything about why ramified primes near power of 2 could be interesting. Could the appearance of ramified primes somehow relate to a mechanism in which $p = 2$ as a ramified prime would precede other primes in the evolution. $p = 2$ is indeed exceptional prime and also defines the smallest p-adic length scale.

For instance, could one have two roots a and $a + 2^k$ near to each other 2-adically and could the deformation be small in the sense that it replaces 2^k with a product of primes near powers of 2: $2^k = \prod_i 2^{k_i} \rightarrow \prod_i p_i$, p_i near 2^{k_i}? For the irreducible polynomial defining the extension of rationals, the deforming could be defined by $a \rightarrow a + 2^k$ could be replaced by $a \rightarrow a + 2^k/N$ such that $2^k/N$ is small also in real sense.

3 Appendix: About the decomposition of primes of number field K to primes of its extension L/K

The followings brief summary lists some of the basic terminology related to the decomposition of primes of number field K in its extension.

1. A typical problem is the splitting of primes of K to primes of the extension L/K which has been already described. One would like to understand what happens for a given prime in terms of information about K. The splitting problem can be formulated also for the extensions of the local fields associated with K induced by L/K.

2. Consider what happens to a prime ideal p of K in L/K. In general p decomposes to product $p = \prod_{i=1}^{g} P_i^{e_i}$ of powers of prime ideals P_i of L. For $e_i > 1$ ramification is said to occur. The finite field K/p is naturally imbeddable to the finite field L/P_j defining its extension. The degree of the residue field extension $(L/P_i)/(K/p)$ is denoted by f_i and called inertia degree of P_i over p. The degree of L/K equals to $[L:K] = \sum e_i f_i$.

   If the extension is Galois extension (see `http://tinyurl.com/zu5ey96`), one has $e_i = e$ and $f_i = f$ giving $[L:K] = efg$. The subgroups of Galois group $Gal(L/K)$ known as decomposition group D_i and inertia group I_i are important. The Galois group of F_i/F equals to D_i/I_i.

   For Galois extension the Galois group $Gal(L/K)$ leaving p invariant acts transitively on the factors P_i permuting them with each other. Decomposition group D_i is defined as the subgroup of $Gal(L/K)$ taking P_i to itself.

   The subgroup of $Gal(L/K)$ inducing identity isomorphism of P_i is called inertia group I_i and is independent of i. I_i induces automorphism of $F_i = L/P_i$. $Gal(F_i/F)$ is isomorphic to D_i/I_i. The orders of I_i and D_i are e and ef respectively. The theory of Frobenius elements identifies the element of $Gal(F_i/F) = D_i/I_i$ as generator of cyclic group $Gal(F_i/F)$ for the finite field extension F_i/F. Frobenius element can be represented and defines a character.

3. Quadratic extensions $Q(\sqrt{n})$ are simplest Abelian extensions and serve as a good starting point (see `http://tinyurl.com/zofhmb8`) the discrimant $D = n$ for $p \mod 4 = 1$ and $D = 4n$ otherwise characterizes splitting and ramification. Odd prime p of the extension not dividing D splits if and only if D quadratic residue modulo p. p ramifies if D is divisible by p. Also the theorem by Kronecker and Weber stating that every Abelian extension is contained in cyclotomic extension of Q is a helpful result (cyclotonic polynomials has as it roots all n roots of unity for given n)

Even in quadratic extensions L of K the decomposition of ideal of K to a product of those of extension need not be unique so that the notion of prime generalized to that of prime ideal becomes problematic. This requires a further generalization. One ends up with the notion of ideal class group (see `http://tinyurl.com/hasyllh`): two fractional ideals I_1 and I_2 of L are equivalent if the are elements a and b such that $aI_1 = bI_2$. For instance, if given prime of K has two non-equivalent decompositions $p = \pi_1\pi_2$ and $p = \pi_3\pi_4$ of prime ideal p associated with K to prime ideals associated with L, then π_2 and π_3 are equivalent in this sense with $a = \pi_1$ and $b = \pi_4$. The classes form a group J_K with principal ideals defining the unit element with product defined in terms of the union of product of ideals in classes (some products can be identical). Factorization is non-unique if the factor J_K/P_K - ideal class group - is non-trivial group. $Q(\sqrt{-5})$ gived a representative example about non-unique factorization: $2 \times 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ (the norms are 4×9 and 6×6 for the two factorizations so that they cannot be equivalent.

This leads to class field theory (see `http://tinyurl.com/zdnw7j3` and `http://tinyurl.com/z3s4kjn`).

1. In class field theory one considers Abelian extensions with Abelian Galois group. The theory provides a one-to-one correspondence between finite abelian extensions of a fixed global field K and appropriate classes of ideals of K or open sub-groups of the idele class group of K. For example,

the Hilbert class field, which is the maximal unramified abelian extension of K, corresponds to a very special class of ideals for K.

2. Class field theory introduces the adele formed by reals and p-adic number fields Q_p or their extensions induced by algebraic extension of rationals. The motivation is that the very tough problem for global field K (algebraic extension of rationals) defines much simpler problems for the local fields Q_p and the information given by them allows to deduce information about K. This because the polynomials of order n in K reduce effectively to polynomials of order $n \ mod \ p^k$ in Q_p if the coefficients of the polynomial are smaller than p^k. One reduces monic irreducible polynomial f characterizing extension of Q to a polynomial in finite field F_p. This allows to find the extension Q_p induced by f.

   An irreducible polynomial in global field need not be irreducible in finite field and therefore can have multiple roots: this corresponds to a ramification. One identifies the primes p for which complete splitting (splitting to first ordinary monomials) occurs as unramified primes.

3. Class field theory also includes a reciprocity homomorphism, which acts from the idele class group of a global field K, i.e. the quotient of the ideles by the multiplicative group of K, to the Galois group of the maximal abelian extension of K. Wikipedia article makes the statement "*Each open subgroup of the idele class group of K is the image with respect to the norm map from the corresponding class field extension down to K*". Unfortunately, the content of this statement is difficult to comprehend with physicist's background in number theory.

Received September 16, 2019; Accepted October 26, 2019

References

[1] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdnumber.html#visionc, 2006.

[2] Pitkänen M. Philosophy of Adelic Physics. In *TGD as a Generalized Number Theory*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdnumber.html#adelephysics, 2017.

[3] Pitkänen M. TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, $M^8 - H$ Duality, SUSY, and Twistors. In *Towards M-Matrix*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdquantum.html#MCKaygeneral, 2019.

[4] Pitkänen M. Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD? Available at: http://tgdtheory.fi/public_html/articles/Yangianagain.pdf, 2017.

[5] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry? Available at: http://tgdtheory.fi/public_html/articles/ratpoints.pdf, 2017.

[6] Pitkänen M. TGD view about coupling constant evolution. Available at: http://tgdtheory.fi/public_html/articles/ccevolution.pdf, 2018.

[7] Pitkänen M. Does coupling constant evolution reduce to that of cosmological constant? Available at: http://tgdtheory.fi/public_html/articles/ccevoTGD.pdf, 2019.

[8] Pitkänen M. $M^8 - H$ duality and consciousness. Available at: http://tgdtheory.fi/public_html/articles/M8Hconsc.pdf, 2019.

[9] Pitkänen M. $M^8 - H$ duality and the two manners to describe particles. Available at: http://tgdtheory.fi/public_html/articles/susysupertwistor.pdf, 2019.

[10] Pitkänen M. New results related to $M^8 - H$ duality. Available at: http://tgdtheory.fi/public_html/articles/M8Hduality.pdf, 2019.

[11] Pitkänen M. Scattering amplitudes and orbits of cognitive representations under subgroup of symplectic group respecting the extension of rationals . Available at: http://tgdtheory.fi/public_html/articles/symplorbsm.pdf, 2019.

[12] Pitkänen M. Secret Link Uncovered Between Pure Math and Physics. Available at: http://tgdtheory.fi/public_html/articles/KimTGD.pdf, 2019.

[13] Pitkänen M. SUSY in TGD Universe. Available at: http://tgdtheory.fi/public_html/articles/susyTGD.pdf, 2019.

[14] Pitkänen M. TGD view about McKay Correspondence, ADE Hierarchy, and Inclusions of Hyperfinite Factors. Available at: http://tgdtheory.fi/public_html/articles/McKay.pdf, 2019.