Article

The Change of the Willmore Energy of a Curve in L^3

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Abstract In this paper, we investigate the change of the Willmore energy of curves in 3-dimensional Lorentzian space. We give the variation of Frenet vector fields, the curvature and the torsion of the curve.
Keywords: Lorentzian space, Willmore energy, Infinitesimal bending.

1. BASIC NOTIONS AND PROPERTIES

The Lorentz three dimensional space L^3 is a real vector space R^3 endowed with the standart flat metric given by

$$\langle,\rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of L^3. An arbitrary vector $v \in L^3$ is said spacelike if $\langle v, v\rangle > 0$ or $\langle v, v\rangle = 0$, timelike if $\langle v, v\rangle < 0$ and null if $\langle v, v\rangle = 0$ and $v \neq 0$. The norm of a vector v is given by $\|v\| = \sqrt{\langle v, v\rangle}$ and two vectors v and w are said to be ortogonal if $g(v, w) = 0$. Similarly, an arbitrary curve $r(t)$ can be locally spacelike, timelike or lightlike(null), if all of its velocity vectors $r'(t)$ spacelike, timelike or lightlike(null), respectively at any $t \in I$, where $r'(t) = \frac{dr}{dt}$. If r is spacelike or timelike one can say that r is a non-null curve. In such case, one can reparametrize r by the arc-length $s = s(t)$, that is, $\|r'(t)\| = 1$. One can say then that r is arc-length parametrized. Denote by $\{t, n, b\}$ the moving Serret-Frenet frame along curve $r = r(t)$ in the space L^3.
For an arbitrary timelike curve $r = r(t)$ in L^3, the following Serret-Frenet formula are given by

$$t' = k_1 n; n' = k_1 t + k_2 b; b' = -k_2 n, \tag{1}$$

where $\langle t, t\rangle = -1$, $\langle n, n\rangle = \langle b, b\rangle = 1$ and k_1 and k_2 stand for the curvature and torsion of the curve, respectively.
Let $C : r = r(s)(r : I \to L^3)$ be a regular curve of the class $C^\alpha(\alpha \geq 0)$. The Helfrich energy of the curve C is given by

$$H_\lambda(c) = \frac{1}{2}\int_I (k - c_0)^2 ds + \lambda L(C),$$

where $k = r''.n$ denotes the scalar curvature of the curve, n is the principal normal, s denotes the arc length and $L(C) = \int_I ds$ the length of C. The map $c_0 : I \to \mathbb{R}$ is called spontaneous curvature. The constant $\lambda \in \mathbb{R}$ is taken to be positive, so that the growth in length of a curve is penalized, [6].
The special case where $c_0 = 0$ and $\lambda = 0$ is known as Willmore energy, [8]

$$w(c) = \frac{1}{2}\int_I k^2 ds.$$

The Helfrich and Willmore energies are mathematically very interesting. Expecially Willmore flow is considered to be one of the most important models in which fourth order $PDEs$ appear.
Both functionals have been comprehensively studied in recent years by many scienstists [1]-[5], [7,9,11,12].

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2. INFINITESIMAL BENDING OF CURVES

Definition 1. Let us consider continuous regular curve

$$C = r = r(u), u \in J \subset \mathbb{R} \tag{2}$$

included in a family of the curves

$$C_\varepsilon : \breve{r}\ (u, \varepsilon) = \breve{r}_\varepsilon(u) + \varepsilon z(u), u \in J, \varepsilon \in (-1, 1), \tag{3}$$

where u is a real parameter and we get C for $\varepsilon = 0 (C = C_0)$. Family of curves C_ε is infinitesimal bending of a curve C if

$$ds_\varepsilon^2 - ds^2 = 0(\varepsilon), \tag{4}$$

where $z = z(u) =, z \in C^1$ is infinitesimal bending of a curve C.
Theorem 1. **[8]**, Necessary and sufficient condition for $z(u)$ to be an infinitesimal bending of a curve C is to be

$$dr.dz = 0, \tag{5}$$

the next theorem is related to determination of the infinitesimal bending of a curve C.
Theorem 2. **[10]**, infinitesimal bending of a curve C, (12), is

$$z(u) = \int \{p(u)n(u) + q(u)b(u)\}\, du, \tag{6}$$

where $p(u)$ and $q(u)$ are arbitrary integrable functions and vectors, $n(u)$ and $b(u)$ are respectively unit principal normal and binormal vector fields of the curve C.
Theorem 3. [8], Under infinitesimal bending of the curves each line element gets non-negative addition, which is the infinitesimal value of the order higher than the first with respect to ε, i.e.

$$ds_\varepsilon - ds = 0(\varepsilon) \geq 0. \tag{7}$$

Consider a regular curve

$$C : r = r(s) = r(u), s \in I, \tag{8}$$

of the class C^α, $\alpha \geq 3$, parameterized by the arc lenght s. Consider a infinitesimal bending of the curve (8),

$$C_\varepsilon : \breve{\delta}(s, \varepsilon) = r_\varepsilon(s) = r_\varepsilon(s) + \varepsilon z(s). \tag{9}$$

As the vector field z is defined in the points of the curve (18), it can be presented in the form

$$z = zt + z_1 n + z_2 b, \tag{10}$$

where zt is tangential and $z_1 n + z_2 b$ is normal component, z, z_1, z_2 are the functions of s.
Theorem 4. Necessary and sufficient condition for the field z, (20), to be infinitesimal bending of a curve C, (8), is

$$z' + k_1 z_1 = 0, \tag{11}$$

where k_1 is the curvature of C.
Proof. Using the equality (5). Necessary and sufficient condition for the field z to be infinitesimal bending of a curve C is

$$t' z' = 0,$$

i.e. $tz' = 0$. Writing (10) in the last equality and using (1), we get (11).

3. CHANGE OF GEOMETRIC MAGNITUDES UNDER INFINITESIMAL BENDING OF CURVES IN $\mathbf{L}^3$

Geometric magnitudes are changing under infinitesimal bending and that changing is described by the variation of a geometric magnitude. We prefer to [10], for the variations of the geometric magnitude.

Definition 2. [8], Let $A = A(u)$ be the magnitude that characterizes a geometric property on the curve C and $A_\varepsilon = A_\varepsilon(u)$ the corresponding magnitude on the curve C_ε being be infinitesimal bending of a curve C,

$$\Delta A = A_\varepsilon - A = \varepsilon \delta A + \varepsilon^2 \delta^2 A + \ldots + \varepsilon^n \delta^n A + \ldots \tag{12}$$

coefficients $\delta A, \delta^2 A, \ldots, \delta^n A, \ldots$ are the first, the second, ..., the nth variation of the geometric magnitude A, respectively under infinitesimal bending C_ε of the curve C. In this paper we will use the first variations under infinitesimal bending of the first order. Therefore, we can give the magnitude A_ε as

$$A_\varepsilon = A + \varepsilon \delta A$$

by disregarding the terms of order higher than 1. Apparently, for the first variation is effective

$$\delta A = \frac{d}{d\varepsilon} A_\varepsilon (u) I_{\varepsilon=0}, \tag{13}$$

i.e.

$$\delta A = \lim_{\varepsilon \to 0} \frac{\Delta A}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{A_\varepsilon (u) - A(u)}{\varepsilon}. \tag{14}$$

Furthermore, it is given following equations

$$\begin{aligned} \delta (AB) &= A \delta B + B \delta A \\ \delta \left(\frac{\partial A}{\partial u} \right) &= \frac{\delta (\partial A)}{\partial u} \\ \delta (dA) &= d (\delta A). \end{aligned}$$

Lemma 1. Under infinitesimal bending of the curve C, (8), a unit vector of the orthonormal basis and its variation are orthogonal.

Proof. The condition that the unit tangent vector remains unit after bending

$$(t + \varepsilon \delta t).(t + \varepsilon \delta t) = 1. \tag{15}$$

Shows that $t \delta t = 0$, after disregarding the terms of order higher than 1. Similarly we can get the statement for n and b.

Lemma 2. Under infinitesimal bending of the curve C, (8), variation of the line element ds is equal to zero, i.e.

$$\delta (ds) = 0. \tag{16}$$

Lemma 3. Under infinitesimal bending of the curve C, (8), variation of the unit tangent vector is

$$\delta V_1 = (z_1' - k_2 z_2 + k_1 z) V_2 + (z_2' + k_2 z_1) V_3. \tag{17}$$

Lemma 4. Under infinitesimal bending of the curve C, (8), variations of the unit principal normal and binormal vectors are respectively,

$$\delta V_2 = (z_1' - k_2 z_2 + k_1 z) V_1 + \left(z_2'' + 2k_2 z_1' + k_2' z_1 + k_1 k_2 z - k_2^2 z_2 \right) V_3 \tag{18}$$

$$\delta V_3 = (z_2' + k_2 z_1) V_1 - \frac{1}{k_1} \left(z_2'' + 2k_2 z_1' + k_2' z_1 + k_1 k_2 z - k_2^2 z_2 \right) V_2. \tag{19}$$

Lemma 5. Under infinitesimal bending of the curve C, (8), variations of the curvature is

$$\delta k_1 = k_1' z + z_1'' + \left(-k_1^2 - k_2^2\right) z_1 - 2k_2 z_2' - z_2 k_2'. \tag{20}$$

Corallary 1. Under infinitesimal bending of a plane curve, variations of the curvature is

$$\delta k_1 = k_1' z + z_1'' + \left(-k_1^2\right) z_1. \tag{21}$$

Lemma 6. Under infinitesimal bending of the curve C, (8), variation of the torsion is

$$\delta k_2 = k_2' z - k_1 \left(z_2' + k_2 z_1\right) + \left\{\frac{1}{k_1}\left(z_2'' + 2k_2 z_1' + k_2' z_1 + k_1 k_2 z - k_2^2 z_2\right)\right\}'. \tag{22}$$

4. THE RESULT AND DISCUSSION

Let a regular curve of the class C^α, $\alpha \geq 3$, be given with

$$C : r = r(s), s \in I, \left(r : I \to L^3\right), \tag{23}$$

the Willmore energy of the curve C is given with the following equation

$$w = \frac{1}{2}\int k_1^2 ds \mid_I . \tag{24}$$

The next theorem is related to determination of the Willmore energy of curve under infinitesimal bending
.

Theorem 5. Under infinitesimal bending of the curve C, (23), variation of its Willmore energy is

$$\delta w = \int \left[z_1 (k_1'' - \frac{3}{2}k_1^3 - k_1 k_2^2) + z_2 (2k_1' k_2 + k_2' k_1)\right] ds \tag{25}$$

$$+ \int \left[\frac{1}{2}k_1^2 z + k_1 z_1' - k_1 z_1' - 2k_2 k_1) z_2\right]' ds. \tag{25}$$

Proof. The Willmore energy of deformed curve will be

$$w_\varepsilon = \frac{1}{2}\int_I k_\varepsilon^2 ds_\varepsilon = \frac{1}{2}\int_I \left(k_1 + \varepsilon\delta k_1\right)^2 \left[ds + \varepsilon\delta(ds)\right] \tag{26}$$

i.e

$$w_\varepsilon = w + \varepsilon \left[\int_I k_1 \delta k_1 ds + \frac{1}{2}\int_I k_1^2 \delta(ds)\right]. \tag{27}$$

Considering Lemma 2, we get

$$w_\varepsilon = w + \varepsilon\int_I k_1 \delta k_1 ds, \tag{28}$$

$$\delta w = \varepsilon\int_I k_1 \delta k_1 ds. \tag{29}$$

Using (20), we have

$$\delta w = \int_I k_1 \left[k_1' z + z_1'' + (k_1^2 - k_2^2) z_1 - 2k_2 z_2' - z_2 k_2'\right] ds.$$

Calculating necessary operations and considering theorem 4, we have (25).

5. CONCLUSIONS

It is well-known that Willmore energy are important in the developmant of partial differential equations in mathematics. In this study, infinitesimal bending of curves in Lorentzian space are examined and change of geometric magnitudes under infinitesimal bending of curves in Lorentzian space are given.
It is hoped that this study about the change of Willmore energy of curves in 3-dimensional Lorentzian space serves researches who carry out research especially in mathematics and physics.

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