

Article

Brief Lecture Notes on Self-Referential Mathematics, and Beyond

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Abstract

Recently delivered lectures on Self-Referential Mathematics, [2], at the Department of Mathematics and Applied Mathematics, University of Pretoria, are briefly presented. Comments follow on the subject, as well as on Inconsistent Mathematics.

0. Prologue

The basic idea in the Self-Referential Mathematics, [2], is to replace the Foundation Axiom, (FA), in Set Theory with a suitable Anti-Foundation Axiom, (AFA), in such a way as to :

- keep all the sets in the usual Set Theory

while at the same time, to :

- allow a large class of new sets, sets given this time by *self-referential* definitions.

In other words, one obtains a significant extension of usual Set Theory, an extension which is proved to be consistent, provided usual Set Theory is consistent.

As it happens not seldom in science, the terminology used may turnout to be rather inappropriate, if not in fact misleading. The same happens in [2], where the term "vicious circle" is used instead of "self-referential".

A likely reason for that particular terminology in [2], which has a clear negative connotation, comes from the fact that the 1903 Russell Paradox in Set Theory is based on self-reference, being but a reformulation in mathematical, in particular, set theoretic terms of the ancient Greek paradox of the liar.

On the other hand, when considered in the larger and longer perspective of human tradition and civilization, self-reference, together with infinity and change, have since the earliest known, in fact, prehistoric times been some of the fundamental

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ideas preoccupying human thought, and as such, they have not had any sort of inevitable negative connotation, see section 5. It follows, therefore, that the term "vicious" can be seen as an overstatement resulted from a partial view of what self-reference does in fact encompass and mean in its more full generality.

We can in essence clarify as follows the aims and the means of the Self-Referential Mathematics in [2]. Let us consider the following three groups of axioms of Set Theory, see section 6 for all the usual axioms, in particular, those used in [2] :

$$\text{ZFC}^- = \text{Zermello - Fraenkel} + \text{Choice}$$

$$\text{ZFC} = \text{ZFC}^- + \text{FA}$$

$$\text{ZFA} = \text{ZFC}^- + \text{AFA}$$

where the AFA axiom will be specified in section 2.

At first, it may appear that the Set Theories corresponding to ZFC and ZFA may be rather different, since their common part corresponds only to ZFC^- , while the respective additional axioms FA and AFA seem in fact to be inconsistent with one another.

However, as it turns out this is not the case. And what happens instead is that :

- The Set Theory based on the ZFA axioms contains all the sets in the Set Theory based on the ZFC, and in addition, contains a large class of other sets obtained by self-referential definitions.
- The axioms ZFA are consistent, provided that the axioms ZFC are consistent.

As for the traditional and still exclusively predominant idea of the absolute necessity of consistency, one should consider the recent emergence of Inconsistent Mathematics, see [11,12]. And in fact, as far as everyday practice is concerned, we have for more than half a century by now been basing much of our lives on a specific form of Inconsistent Mathematics. Indeed, our ever more pervasive and critically important electronic digital computers are - even when only operating on non-negative integers - functioning according to the Peano Axioms, plus the Machine Infinity Axiom, namely

$$\exists M \gg 1 : M + M = 1$$

where M is called "machine infinity", and typically is larger than 10^{1000} . And obviously, the Peano Axioms are trivially inconsistent with the Machine Infinity Axiom.

1. Sets, Ur-Elements and Classes

We denote by SET the *class* of all sets, including the unique void set ϕ . As is well known, a lot of mathematics can be built up starting alone with the void set ϕ . Indeed, as a first step, and following von Neumann, one can define the non-negative integers by

$$0 = \phi, 1 = \{\phi\}, 2 = \{\{\phi\}\}, \dots$$

and then, step by step build all the integers, the rational and real numbers, and so on. Further, one can define Cartesian products, binary relations, functions, etc., and obtain a considerable part of mathematics in this manner.

In the sequel, it will be convenient to allow, in addition to the void set ϕ , other such starting entities in the construction of mathematics. The class of such entities is denoted by \mathcal{U} , and any respective element $u \in \mathcal{U}$ is called an *ur-element*, assumed to have only one property, similar to that of the void set ϕ , namely that the relation

$$a \in u$$

does *not* hold for any entity a in the theory.

In this way, there will be three types of entities in the theory, namely

- 1) SET , which is the class of all sets,
- 2) \mathcal{U} which is the class of all ur-elements, and
- 3) $CLASS$ which denotes all the classes.

Here it is understood that any set $a \in SET$ is a "small" class, while SET itself is one of the "proper" classes, since it is not itself a set. In other words, $SET \in CLASS$, $SET \notin SET$, $CLASS \notin SET$.

As for ur-elements, it is assumed that $\mathcal{U} \in CLASS$ and it is another instance of "proper" class, thus in particular $\mathcal{U} \notin SET$.

Briefly, we have therefore

- 1) SET denotes all the sets, and it is a proper class
- 2) $CLASS$ denotes all the classes
- 3) \mathcal{U} denotes all the ur-elements, and it is a proper class
- 4) a set is a "small" class
- 5) a class which is not a set is "large", thus it is a *proper* class
- 6) an ur-element does not have any elements, either sets, ur-elements,

or classes

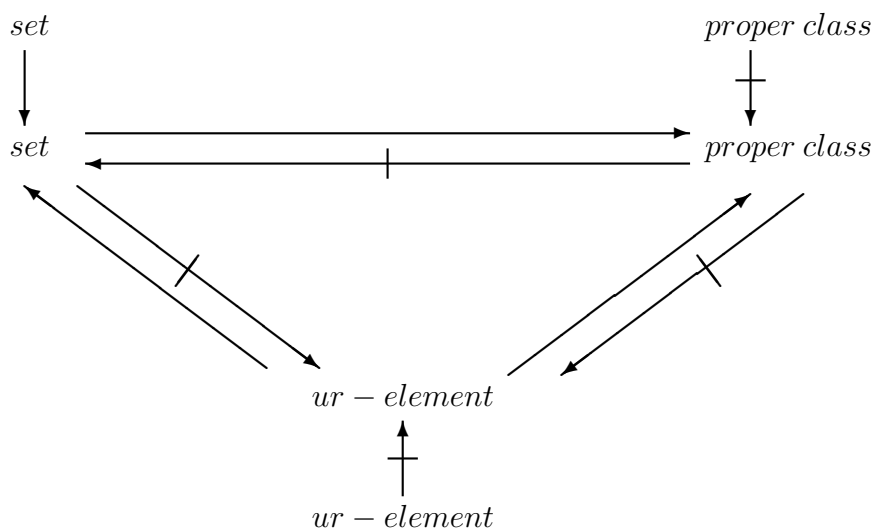
- 7) non-set = proper class \vee ur-element
- 8) non-set \wedge non-class = ur-element
- 9) every predicate determines a class
- 10) a subclass of a set is a set
- 11) sets are closed under a number of operations, among them, pairing, union, power set, see below
- 12) $a \in b \in CLASS \implies a \in SET$
- 13) the class SET of all sets is "large", thus it is a proper class

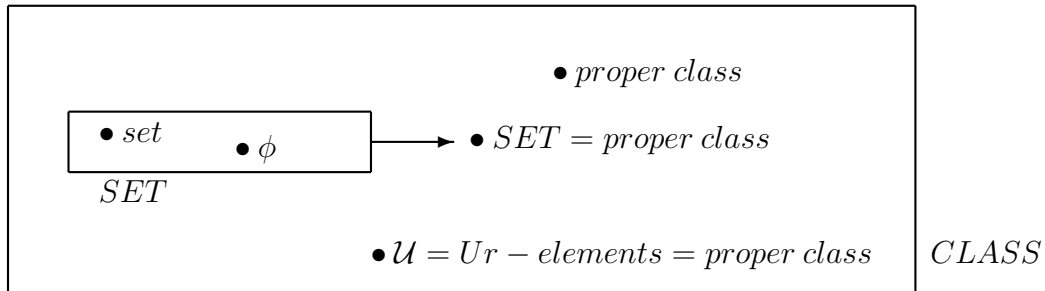
indeed, according to Russell's Paradox, let SET be a set, then $R = \{a \in SET \mid a \notin a\}$ is a set, thus $R \in SET$, and therefore $R \in R \iff R \notin R$, which is absurd there-

As for the binary relation \in , we have

- set \in set
- set \in proper class
- set \notin ur-element
- proper class \notin set
- proper class \notin proper class
- proper class \notin ur-element
- ur-element \in set
- ur-element \in proper class
- ur-element \notin ur-element

thus denoting \in by \rightarrow , while \notin by \nrightarrow , we have





Note on ur-elemets

Clearly

$$a \notin \phi$$

holds for all entities a in the theory, however, it is nevertheless considered that

$$\phi \in SET \text{ and } \phi \notin \mathcal{U}$$

Also, it is possible that

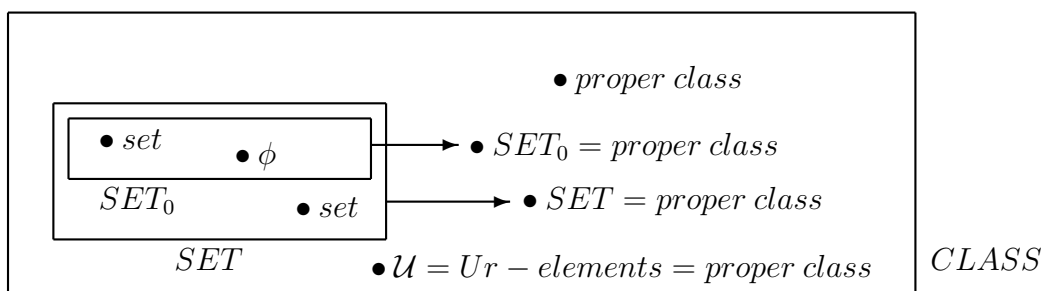
$$X \in SET \text{ and } X \cap \mathcal{U} \neq \phi, \text{ or even } X \subseteq \mathcal{U}$$

For instance, if $x \in \mathcal{U}$, then

$$X = \{x\} \in SET, \quad X \subseteq \mathcal{U}.$$

Note on SET

Here we should clarify that SET denotes, in fact, all the sets which exist in the Set Theory based on the ZFA axioms. Therefore, let us denote by SET_0 all the sets in the Set Theory based on the ZFC axioms. Then as mentioned in section 0, and seen later, we have $SET_0 \subsetneq SET$, thus the above diagram can be made more precise as follows



Examples of Sets

There are only *two* kind of sets $a \in SET$, namely

$$(1.1) \quad a = \phi, \text{ which is equivalent with } \neg(\exists b \in SET \cup \mathcal{U} : b \in a)$$

$$(1.2) \quad \exists b \in SET \cup \mathcal{U} : b \in a$$

Operations with sets

An *ordered pair* is the set $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$, with $a, b \in SET$

$$\text{thus } \langle a, b \rangle = \langle c, d \rangle \iff a = c, b = d$$

A *relation* $R \in SET$ has all its elements given by pairs $\langle a, b \rangle$, where $a \in A, b \in B$, for two suitably given sets $A, B \in SET$. Often for convenience one denotes aRb for $\langle a, b \rangle \in R$.

If $A \in SET$ is such that $\langle a, b \rangle \in R \implies a, b \in A$, then R is a relation on A .

A *relational structure* is $\langle A, R \rangle$, with R relation on A .

A *function* is a relation R such that $\langle a, b \rangle, \langle a, c \rangle \in R \implies b = c$

If f is a function, then

$$(1.3) \quad \text{dom}(f) = \{a \mid \exists b : f(a) = b\}$$

$$(1.4) \quad \text{rng}(f) = \{b \mid \exists a : f(a) = b\}$$

thus

$$(1.5) \quad f \in c \rightarrow d \iff c = \text{dom}(f), \text{rng}(f) \subseteq d$$

The *power set* of $a \in SET$ is

$$(1.6) \quad \mathcal{P}(a) = \{b \mid b \subseteq a\}$$

Example : if $a = \{\phi, p\}$, with $p \in SET \cup \mathcal{U}$, then $\mathcal{P}(a) = \{\phi, \{\phi\}, \{p\}, a\}$.

Consider the *predicate* $P(x)$ given by

$$(1.7) \quad x \text{ is an ordered pair } \langle a, b \rangle \text{ and } b = \mathcal{P}(a)$$

then this defines the *power set* function $\mathcal{P} : SET \rightarrow SET$, and

$$(1.8) \quad \bigcup \text{dom}(\mathcal{P}) = SET$$

thus it is "large", and therefore, a proper class.

The *natural numbers* are

$$(1.9) \quad 0 = \phi, 1 = \{0\} = \{\phi\}, 2 = \{0, 1\} = \{\phi, \{\phi\}\}, \dots$$

Disjoint union is $A + B = (\{0\} \times A) \cup (\{1\} \times B)$

For $a \in SET$, we define

$$(1.10) \quad \bigcup a = \{x \mid \exists y \in a : x \in y\} = \{x \in y \in a\}$$

2. Systems of Equations Which Define Sets

In usual, that is, ZFC Set Theory, one way to define a set X is by an equation

$$X = \{x \mid P(x)\}$$

where P is a suitable predicate. Within ZFC, an essential restriction on P is that it *cannot* in any way refer to the set X which it is supposed to define. This condition is meant to avoid a "vicious circle", or in more proper terms, self-referentiality, an avoidance which has until recently been universally accepted, and in fact required, since Russell's paradox.

In particular, one cannot define any set $a \in SET_0$, even by such a simple equation, like

$$(2.1) \quad a = \{a\}$$

since obviously, it is a self-referential equation. On the other hand, as seen in 1), 4), 5) in Examples 2.1. below, this equation can easily be solved in SET , that is, based on the Anti-Foundation Axiom, (AFA).

Here we can note that one cannot define any set $a \in SET_0$, or for that matter, $a \in SET$, even by the yet more simple equation

$$(2.2) \quad a = a$$

since this equation will obviously *not* give a unique set in SET_0 , or in SET .

Also, as seen in 3) in Proposition 2.3. below, one cannot define a set $a \in SET$ by the equation

$$(2.3) \quad a = \mathcal{P}(a)$$

Let us consider now the equation

$$x = \{a, x\}$$

where $a \in SET \cup \mathcal{U}$ is given. Then

$$x = \{a, x\} = \{a, \{a, x\}\} = \{a, \{a, \{a, x\}\}\} = \dots$$

thus an intuitive solution would be

$$x = \{a, \{a, \{a, \dots\}\}\}$$

which however is not possible within ZFC, since it would obviously lead to the *infinite* descending sequence

$$\dots x \in x \in x \in x \in x$$

thus contradict the Foundation Axiom, (FA), see below.

Let us now return to the general situation and enquire what should the solution given by sets be of a corresponding system of equations. Let us as an example consider for that purpose the following system of equations, where $p, q \in SET \cup \mathcal{U}$ are given, and where we want to find sets $x, y, z \in SET$, such that

$$x = \{x, y\}$$

$$y = \{p, q, y, z\}$$

$$z = \{p, x, y\}$$

Let $e : X = \{x, y, z\} \longrightarrow$ the right-hand sides of the above equations

thus

$$e_x = \{x, y\}, e_y = \{p, q, y, z\}, e_z = \{p, x, y\}$$

What is then a solution s to these equations supposed to be ?

One way is given by $s : X \longrightarrow SET$, namely $X \ni v \longmapsto s_v \in SET$, with

$$s_x = \{s_x, s_y\}, s_y = \{p, q, s_y, s_z\}, s_z = \{p, s_x, s_y\}$$

or equivalently

$\forall v \in X :$

$$\begin{aligned} s_v &= \{s_w \mid w \in e_v \cap X\} \cup \{w \mid w \in e_v \cap A\} = \\ &= s[e_v \cap X] \cup (e_v \cap A) \end{aligned}$$

where $A = \{p, q\}$

Returning now to the equation, see (2.2)

$$x = x$$

one way to avoid the inconvenience of non-unique solutions in SET , is to take, see also (2.10) below

$$X = \{x\} \subseteq \mathcal{U}$$

and then the solution s , if it exists, is a *function*

$$s : X \ni x \longrightarrow s_x \in SET$$

This liberty to *distinguish* between *indeterminates*, and on the other hand, the *solution* is in fact familiar from usual algebra. Indeed, if for instance we have the system of equations in real numbers

$$\begin{aligned} 2x + 3y &= 7 \\ 5x - 4y &= 1 \end{aligned}$$

then the set of indeterminates is $X = \{x, y\}$, while a solution s , which in this case exists, is given by a function $s : X \ni v \longrightarrow s_v \in \mathbb{R}$. This distinction is even more obvious when, as with a system of equations like

$$\begin{aligned} 3x + 2y &= 5 \\ 5x - 3y &= 2 \end{aligned}$$

the indeterminates have the same value $x = y = 1$, thus the solution cannot be identified with the single number 1, but only with the function $s : X \ni v \longrightarrow s_v \in \mathbb{R}$, for which $s_x = 1, s_y = 1$.

Finally we can note that, as seen in Definition 2.2. below, the requirement $X \subseteq \mathcal{U}$ can on occasion be done away with.

Now, the approach in [2] to Self-Referential Mathematics will be able to accept for sets in SET such definitions which are given by certain systems of equations that can be *self-referential*. There are in this regard *three* kind of systems of equations considered so far. The first kind of systems is given by

Definition 2.1.

A structure $\mathcal{E} = \langle X, A, e \rangle$ is called a *flat system of equations*, if and only if

$$(2.4) \quad X, A \in SET, \quad X \subseteq \mathcal{U}, \quad X \cap A = \phi$$

with X the set of *indeterminates* and A the set of *atoms*, while

$$(2.5) \quad e : X \longrightarrow \mathcal{P}(X \cup A)$$

defines the *right hand terms* of the equations of the system, see (2.8) below, with

$$(2.6) \quad X \ni v \longmapsto b_v = e_v \cap X$$

being the set of indeterminates on which v immediately depends, and similarly, with

$$(2.7) \quad X \ni v \longmapsto c_v = e_v \cap A$$

being the set of atoms on which v immediately depends

In this way, the flat system of equations is given by

$$(2.8) \quad x = e_x, \quad x \in X$$

which of course can in particular be one single equation, when $X = \{x\}$ is a set with one single element.

A *solution* to \mathcal{E} is a function

$$(2.9) \quad X \ni x \longmapsto s_x = \{s_y \mid y \in b_x\} \cup c_x \in SET$$

and we denote

$$\begin{aligned} \text{solution} - \text{set}(\mathcal{E}) &= \bigcup \{s_x \mid x \in X\} = \\ &= \{s_y \mid y \in e_x \cap X, x \in X\} \cup \bigcup_{x \in X} (e_x \cap A) = s[X] \in SET \end{aligned}$$

as well as

$$\begin{aligned} V[A] &= \bigcup \left\{ \text{solution} - \text{set}(\mathcal{E}) \mid \begin{array}{l} \mathcal{E} \text{ flat system of equations} \\ \text{with atoms } A \end{array} \right\} = \\ &= \left\{ c \mid \begin{array}{l} \exists \mathcal{E} = \langle X, A, e \rangle \text{ flat system of equations :} \\ c \in \text{solution} - \text{set}(\mathcal{E}) \end{array} \right\} = \end{aligned}$$

$$= \left\{ s_y \mid \begin{array}{l} \exists \mathcal{E} = \langle X, A, e \rangle \text{ flat system of equations :} \\ y \in e_x \cap X, \quad x \in X \end{array} \right\} \cup$$

$$\cup \left\{ c \mid \begin{array}{l} \exists \mathcal{E} = \langle X, A, e \rangle \text{ flat system of equations :} \\ c \in e_x \cap A, \quad x \in X \end{array} \right\} \subseteq SET$$

and clearly, $V[A]$ is always a proper class, see 2) in Note 2.1. below.

□

There are *two remarkable* facts about the concept of *flat systems of equations* given in the above Definition 2.1., namely

- the Anti-Foundation Axiom, (AFA), upon which the whole of Self-Referential Mathematics in [2] rests, has a most simple formulation in terms of *flat systems of equations*, as seen next,
- the *flat systems of equations* do in fact give *all* the additional new sets in $SET \setminus SET_0$, that is those which due to their self-referential definitions, cannot be obtained by the usual ZFC Set Theory, see the equivalence Theorem 2.1. below, see also 1) in Note 2.1. below.

ANTI-FOUNDATION AXIOM (AFA)
 $\forall \mathcal{E}$ flat system of equations : $\exists!$ s solution

Here, for the sake of further clarity, let us recall that in ZFC we have

AXIOM OF FOUNDATION (FA)
 $\forall a \in SET$: $\langle a, \in \rangle$ well-founded

where the concept of a *well-founded* relational structure is defined as follows.

A relational structure $\langle S, R \rangle$ is called *well-founded*, if and only if it has no infinite descending sequence

$$\dots a_n R a_{n-1} R \dots R a_2 R a_1 R a_0$$

with $a_0, a_1, a_2, \dots, a_{n-1}, a_n, \dots \in S$.

For a relational structure $\langle S, R \rangle$, we denote

$$\langle S, R \rangle_{wf} = \{a \in S \mid \text{no infinite descending sequence in } R \text{ starting with } a\}$$

Clearly

$$\langle S, R \rangle \text{ well - founded} \iff \langle S, R \rangle_{wf} = S$$

For simplicity, we denote

$$R_{wf} = \langle S, R \rangle_{wf}$$

Remark 2.1.

In view of the above (AFA) axiom, the question arises :

- Which flat systems of equations have solutions under the (FA) axiom ?

The answer is obtained based on the following concept. A flat system of equations $\mathcal{E} = \langle X, A, e \rangle$ is called *well-founded*, if and only if the relation $<$ defined on X by

$$x < y \iff y \in e_x$$

is well-founded. And then we have, see [1,2]

Mostowski Collapsing Lemma

In ZFC^- well-founded flat systems of equations have unique solutions.

Corollary 2.1.

In ZFC^- we have the equivalence

$$(FA) \iff \left(\begin{array}{l} \text{only well-founded flat systems} \\ \text{of equations have solutions} \end{array} \right)$$

Note 2.1.

1) Flat systems of equations can trivially recover all sets $E \in SET$. Indeed, let $A = E$. Further, let $X = \{x\}$, with $x \in \mathcal{U}$, such that $x \notin E$, which is possible since $\mathcal{U} \not\subseteq E$, given the fact that \mathcal{U} is a proper class. Then $X \subseteq \mathcal{U}$ and $X \cap A = \phi$, hence

$$e_x = E$$

is obviously a flat system of equations, thus according to (AFA), it has a unique solution s . Now in view of (2.6), (2.7), we have $b_x = e_x \cap X = \phi$, $c_x = e_x \cap A = E$, and then (2.9) gives

$$s_x = \{s_y \mid y \in e_x \cap X\} \cup (e_x \cap A) = E$$

In this way, the set E was obtained as the unique solution

$$s : X = \{x\} \ni x \mapsto s_x = E$$

of the above flat system of equations.

2) In view of the above example, each $x \in \mathcal{U}$ leads to a flat system of equations with the respective unique solution s_x . And clearly, if $x, x' \in \mathcal{U}$, $x \neq x'$, then $s_x \neq s_{x'}$. As for \mathcal{U} , it is a proper class, therefore, so is $\{s_x \mid x \in \mathcal{U}\}$.

Examples 2.1.

Let us illustrate the above in the case of the equations (2.1) - (2.3).

1) For (2.1), we can take

$$X = \{x\} \subseteq \mathcal{U}, \quad A = \{\phi\}, \quad e_x = \{x\} \in \mathcal{P}(X \cup A)$$

therefore, it is a flat system of equations, made up of a single equation. As for its unique solution $s_x \in SET$, we shall see the details in 4) and 5) below.

2) For (2.2), we can take

$$X = \{x\} \subseteq \mathcal{U}, \quad A = \{\phi\}, \quad e_x = x$$

thus

$$e_x \in X \cup A, \quad e_x \notin X \cup A, \quad e_x \notin \mathcal{P}(X \cup A)$$

therefore, it is *not* a flat system of equations.

Also, with (2.2), we can immediately note why the condition

$$(2.10) \quad X \subseteq \mathcal{U}$$

was requested in Definition 2.1. Indeed, without that condition, equation (2.2) is satisfied by *all* sets $a \in SET$, thus (2.2) does *not* have a unique solution in SET .

3) With the equation (2.3), we can take

$$X = \{x\} \cup x \subseteq \mathcal{U}, \quad A = \phi, \quad e_x = \mathcal{P}(x) \not\subseteq X \cup A, \quad e_x \notin \mathcal{P}(X \cup A)$$

which, however, does not turn (2.3) into a flat system of equations. Also, as seen in 3) in Proposition 2.3. below, equation (2.3) does *not* have any solution in SET .

4) In ZFA, the equation

$$(2.11) \quad x = \{x\}$$

has a *unique solution* $\Omega \in SET$. Indeed, as note at 1) above, if we take

$$X = \{x\} \subseteq \mathcal{U}, \quad A = \phi, \quad e_x = \{x\}$$

then (2.11) is a flat system of equations, thus in view of (AFA), it has a unique solution $s_x \in SET$, and according to (2.9), we have

$$(2.12) \quad s_x = \{s_y \mid y \in b_x\} \cup c_x$$

However, (2.6) gives $b_x = e_x \cap X = \{x\} \cap X = X = \{x\}$, while (2.7) implies $c_x = e_x \cap A = \phi$. Thus (2.12) becomes

$$s_x = \{s_x\}$$

5) The above unique solution $\Omega \in SET$ obviously has the property

$$\Omega = \{\Omega\} = \{\{\Omega\}\} = \{\{\{\Omega\}\}\} = \dots$$

however, this need *not* mean that the bracket pairs $\{ \}$ could be infinitely many, namely, that we could have for instance

$$\Omega = \dots \{\{\{\Omega\}\}\} \dots$$

let alone that the bracket pairs $\{ \}$ could reach to transfinite ordinals, or go through all the ordinals, see Remark 2.3. below.

6) In ZFA, there is a *unique set*

$$(2.13) \quad \{0, \{1, \{2, \{3, \dots\}\}\}\} \in SET$$

Indeed, we consider the flat system of equations

$$\begin{aligned} x_0 &= \{0, x_1\} \\ x_1 &= \{1, x_2\} \\ x_2 &= \{2, x_3\} \\ x_3 &= \{3, x_4\} \\ &\vdots \end{aligned}$$

where $X = \{x_0, x_1, x_2, x_3, \dots\} \subseteq \mathcal{U}$, $A = \{0, 1, 2, 3, \dots\}$, while $e_{x_n} = \{n, x_{n+1}\}$, with $n \geq 0$. Then (AFA) gives a unique solution s , and in view of (2.6), (2.7), (2.9), we obtain the relations

$$\begin{aligned} s_{x_0} &= \{s_x \mid x \in b_{x_0}\} \cup c_{x_0} = \{s_x \mid x \in e_{x_0} \cap X\} \cup (e_{x_0} \cap A) = \{0, s_{x_1}\} \\ s_{x_1} &= \{s_x \mid x \in b_{x_1}\} \cup c_{x_1} = \{s_x \mid x \in e_{x_1} \cap X\} \cup (e_{x_1} \cap A) = \{1, s_{x_2}\} \\ s_{x_2} &= \{s_x \mid x \in b_{x_2}\} \cup c_{x_2} = \{s_x \mid x \in e_{x_2} \cap X\} \cup (e_{x_2} \cap A) = \{2, s_{x_3}\} \end{aligned}$$

⋮

thus

$$s_{x_0} = \{0, s_{x_1}\} = \{0, \{1, s_{x_2}\}\} = \{0, \{1, \{2, s_{x_3}\}\}\} = \dots$$

7) Let us consider that flat system of equations *without* atoms, that is, with $A = \phi$, namely

$$\begin{aligned} x_0 &= \{y_0, x_1\} & y_0 &= \phi \\ x_1 &= \{y_1, x_2\} & y_1 &= \{y_0\} \\ x_2 &= \{y_2, x_3\} & y_2 &= \{y_0, y_1\} \\ x_3 &= \{y_3, x_4\} & y_3 &= \{y_0, y_1, y_2\} \\ & & & \vdots \end{aligned}$$

where $X = \{x_0, y_0, x_1, y_1, x_2, y_2, \dots\} \subseteq \mathcal{U}$, $A = \phi$, while $e_{x_n} = \{y_n, x_{n+1}\}$, $e_{y_{n+1}} = \{y_0, \dots, y_n\}$, for $n \geq 0$, and $e_{y_0} = \phi$. In this case, (2.6), (2.7) give

$$b_{x_n} = e_{x_n} \cap X = \{y_n, x_{n+1}\}, \quad n \geq 0$$

$$\begin{aligned} b_{y_0} &= e_{y_0} \cap X = \phi \\ b_{y_{n+1}} &= e_{y_{n+1}} \cap X = \{y_0, \dots, y_n\}, \quad n \geq 0 \end{aligned}$$

$$c_{x_n} = c_{y_n} = \phi, \quad n \geq 0$$

therefore, in view of (2.9), the unique solution s given by (AFA), is such that

$$s_{x_n} = \{s_x \mid x \in b_{x_n}\} \cup c_{x_n} = \{s_{y_n}, s_{x_{n+1}}\}, \quad n \geq 0$$

$$\begin{aligned} s_{y_0} &= \{s_y \mid y \in b_{y_0}\} \cup c_{y_0} = \phi \\ s_{y_{n+1}} &= \{s_y \mid y \in b_{y_{n+1}}\} \cup c_{y_{n+1}} = \{s_{y_0}, \dots, s_{y_n}\}, \quad n \geq 0 \end{aligned}$$

In particular

$$\begin{aligned} s_{y_0} &= \phi \\ s_{y_1} &= \{s_{y_0}\} = \{\phi\} \\ s_{y_2} &= \{s_{y_0}, s_{y_1}\} = \{\phi, \{\phi\}\} \\ s_{y_3} &= \{s_{y_0}, s_{y_1}, s_{y_2}\} = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\} \\ & \vdots \end{aligned}$$

which means that, for $n \geq 0$, we obtain that

$$s_{y_n} = n \text{ in the von Neumann representation}$$

Proposition 2.1.

Let $\mathcal{E} = \langle X, A, e \rangle$ be a flat system of equations. If $A \subseteq \mathcal{U}$, then set *solution – set*(\mathcal{E}) is *transitive*, namely

$$b, c \in SET, c \in b \in \text{solution – set}(\mathcal{E}) \implies c \in \text{solution – set}(\mathcal{E})$$

Proof.

We recall that

$$\text{solution – set}(\mathcal{E}) = \{s_y \mid y \in e_x \cap X, x \in X\} \cup \bigcup_{x \in X} (e_x \cap A)$$

Let $b \in \text{solution – set}(\mathcal{E})$, then

either $b = s_y$, for some $y \in e_x \cap X$, with suitable $x \in X$,

or $b \in e_x \cap A$, for some $x \in X$.

In the first case, if $c \in b = s_y = \{s_z \mid z \in e_y \cap X\} \cup (e_y \cap A)$, then

either $c = s_z$, thus $c \in \text{solution – set}(\mathcal{E})$,

or $c \in e_y \cap A$, thus again $c \in \text{solution – set}(\mathcal{E})$.

In the second case, if $b \in e_x \cap A$, then $b \in \mathcal{U}$, thus there cannot be $c \in b$.

Proposition 2.2.

If $A \subseteq \mathcal{U}$, then

$$V[A] \subseteq V_{afa}[A]$$

where the operation V_{afa} is defined in (3.5) in the next section.

Note : Here we make an advance use of some notations and results in section 3 below. However, placing Proposition 2.2. here in section 2 helps in the better understanding of the concept of flat system of equations, as well as of its fundamental importance seen in the equivalence Theorem 2.1. below.

Proof.

Let $c \in V[A]$, then for some $\mathcal{E} = \langle X, A, e \rangle$ flat system of equations we have either

$c = s_y$, for some $y \in e_x \cap X, x \in X$

or

$c \in e_x \cap A$, for some $x \in X$

In the first case we also have

$$c \subseteq Z = \text{solution} - \text{set}(\mathcal{E}) = \\ = \{s_y \mid y \in e_x \cap X, x \in X\} \cup \bigcup_{x \in X} (e_x \cap A)$$

since in view of (2.9)

$$c = s_y = \{s_z \mid z \in e_y \cap X\} \cup (e_y \cap A)$$

But in view of Proposition 2.1., the set $Z = \text{solution} - \text{set}(\mathcal{E})$ is transitive. Therefore, for $x \in X$, we have, see (3.3) below

$$s_x \subseteq TC(s_x) \subseteq Z$$

which gives

$$TC(s_x) \cap \mathcal{U} \subseteq Z \cap \mathcal{U} \subseteq A$$

Indeed, if $b \in Z \cap \mathcal{U}$, then in particular

either $b = s_y$, for some $y \in e_x \cap X$, with suitable $x \in X$,

or $b \in e_x \cap A$, for some $x \in X$.

In the first case, $b \notin \mathcal{U}$, since $s_y \in SET$.

In the second case obviously $b \in A$.

In conclusion, in view of (3.5), we have $c \in V_{afa}[A]$.

Remark 2.1.

The following is, of course, a fundamental question :

- How many sets $a \in SET$ can be obtained as solutions of flat systems of equations ?

In 1) in Note 2.1. above, we have seen that flat systems of equations can trivially recover as solutions all sets in SET . A more precise and rather natural answer, and as such, best possible answer will be given in Theorem 2.2. below.

Needless to say, this answer highlights the importance of flat systems of equations. However, in various contexts, other two concepts of systems of equations will prove to be useful, concepts given in Definitions 2.2. and 2.3. below.

□

Now, the second kind of systems of equations aims to eliminate the above restriction $X \subseteq \mathcal{U}$ in (2.10) on the flat systems of equations. And as we shall see in Theorem 2.1. below, this is in fact possible, in spite of the above problem with lack of uniqueness of solutions, provided that the STRONG AXIOM OF PLENITUDE is accepted.

Definition 2.2.

A structure $\mathcal{E} = \langle X, A, e \rangle$ is called a *generalized flat system of equations*, if and only if

$$(2.14) \quad X, A \in SET, \quad X \cap A = \phi$$

with X the set of *indeterminates* and A the set of *atoms*, while

$$(2.15) \quad e : X \longrightarrow \mathcal{P}(X \cup A)$$

□

There is a close connection between the solutions of flat, and on the other hand, generalized flat systems of equations, provided that the following axiom holds

STRONG AXIOM OF PLENITUDE

There is an operation $new(a, b)$, such that

- 1) $\forall a \in SET, b \subset \mathcal{U} : new(a, b) \in \mathcal{U} \setminus a$
- 2) $\forall a, a' \in SET, a \neq a', b \subset \mathcal{U} : new(a, b) \neq new(a', b)$

Theorem 2.1.

Assuming the STRONG AXIOM OF PLENITUDE, every generalized flat system of equations $\mathcal{E} = \langle X, A, e \rangle$ has a unique solution s . Furthermore, there exists an associated flat system of equations $\mathcal{E}' = \langle Y, A, e' \rangle$, such that

$$solution - set(\mathcal{E}) = solution - set(\mathcal{E}')$$

Proof.

We have to replace X by a set $Y \subset \mathcal{U}$, such that $Y \cap A = \phi$. Thus we take

$$Y = \{y_x \mid y_x = new(x, A), \quad x \in X\}$$

then

$$Y \subset \mathcal{U}, \quad Y \cap A = \phi$$

Let now

$$e'_{y_x} = \{y_z \mid z \in e_x \cap X\} \cup (e_x \cap A)$$

Clearly, $\mathcal{E}' = \langle Y, A, e' \rangle$ is a flat system of equations, and thus, it has a unique solution s' .

Now we get the solution s of $\mathcal{E} = \langle X, A, e \rangle$ given by

$$s_x = s'_{y_x}, \quad x \in X$$

The uniqueness of s follows from the fact that every solution t of $\mathcal{E} = \langle X, A, e \rangle$ gives a solution t' of $\mathcal{E}' = \langle Y, A, e' \rangle$. And we must have $t' = s'$, thus it follows that $t = s$. □

Example 2.2.

For every set $a \in SET$, we associate the *canonical generalized flat system of equations* $\mathcal{E}_a = \langle X_a, A_a, e_a \rangle$, with, see (3.1)

$$A_a = TC(a) \cap \mathcal{U}$$

$$X_a = TC(\{a\}) \setminus A_a$$

and

$$X_a \ni x \mapsto (e_a)_x = x \subseteq X_a \cup A_a$$

where we have to prove the inclusion in the last relation.

We note in this regard that, see (3.2*), (3.3*) and 1) in Examples 3.1.

$$A_a = \{x \in \mathcal{U} \mid x \in a_n \in \dots \in a_1 \in a_0 = a, \quad n \geq 0\}$$

$$\begin{aligned} X_a &= (\{a\} \cup TC(a)) \setminus (TC(a) \cap \mathcal{U}) = \\ &= (\{a\} \cup \{x \mid x \in a_n \in \dots \in a_1 \in a_0 = a, \quad n \geq 0\}) \setminus \\ &\quad \setminus \{y \in \mathcal{U} \mid y \in b_m \in \dots \in b_1 \in b_0 = a, \quad m \geq 0\} = \\ &= \{a\} \cup \{x \in SET \mid x \in a_n \in \dots \in a_1 \in a_0 = a, \quad n \geq 0\} \end{aligned}$$

thus

$$x \in X_a \implies x \in SET$$

Furthermore

$$x \in X_a \implies x \subseteq X_a \cup A_a$$

Indeed, given $x \in X_a$, then there are the two cases

$$(i.1) \quad x = a$$

$$(i.2) \quad x \in a_n \in \dots \in a_1 \in a_0 = a, \text{ for some } n \geq 0$$

Let now $y \in x$. In case (i.1), we have $y \in a$, thus $y \in X_a$, provided that $y \notin \mathcal{U}$. Otherwise obviously $y \in A_a$. In case (i.2), clearly $y \in (X_a \cup A_a)$.

Assuming now the STRONG AXIOM OF PLENITUDE, we have in view of Theorem 2.1., a unique solution s_a of \mathcal{E}_a . And in fact, we have

$$s_a : X_a \ni x \longmapsto (s_a)_x = x \in SET$$

Indeed, (2.9) gives for $x \in X_a$

$$(s_a)_x = \{(s_a)_y \mid y \in (e_a)_x \cap X_a\} \cup ((e_a)_x \cap A_a) \in SET$$

while

$$(e_a)_x \cap X_a = x \cap X_a, \quad (e_a)_x \cap A_a = x \cap A_a$$

thus

$$(s_a)_x = \{(s_a)_y \mid y \in x \cap X_a\} \cup (x \cap A_a) \in SET$$

which in our case becomes the identity

$$x = \{y \mid y \in x \cap X_a\} \cup (x \cap A_a) \in SET$$

Theorem 2.2. Equivalence

Assuming now the STRONG AXIOM OF PLENITUDE, we have for $A \subseteq \mathcal{U}$ the equivalence between sets in SET which have support in A , and sets in SET which are solutions of flat systems of equations with atoms in A , namely

$$V_{afa}[A] = V[A]$$

Proof.

In view of Proposition 2.2., we have

$$V[A] \subseteq V_{afa}[A]$$

Let now $a \in V_{afa}[A]$. Then by definition, see (3.5) below

$$TC(a) \cap \mathcal{U} \subseteq A$$

Now, in view of Example 2.2., we consider the unique solution s_a of the canonical generalized flat system of equations $\mathcal{E}_a = \langle X_a, A_a, e_a \rangle$, which is $s_a = id_{X_a}$. Thus, recalling that $a \in X_a$ and $A_a = TC(a) \cap \mathcal{U} \subseteq A$, we obtain

$$s_a = a$$

which gives

$$a \in \text{solution} - \text{set}(\mathcal{E}_a)$$

On the other hand, in view of Theorem 2.1., there is a flat system of equations $\mathcal{E} = \langle X, A_a, e \rangle$ with the same atoms A_a , such that

$$\text{solution} - \text{set}(\mathcal{E}_a) = \text{solution} - \text{set}(\mathcal{E})$$

$$\text{hence } a \in \text{solution} - \text{set}(\mathcal{E}) \subseteq V[A]$$

□

Finally, the third kind of systems of equations allows considerably more general right hand terms e_x in (2.5), (2.15), although it has to accept harder restrictions on X and A , than in Definition 2.1., that is, in (2.10). Namely

Definition 2.3.

A structure $\mathcal{E} = \langle X, A, e \rangle$ is called a *generalized system of equations*, if and only if

$$(2.16) \quad X, A \in SET, \quad X, A \subseteq \mathcal{U}, \quad X \cap A = \phi$$

with X the set of *indeterminates* and A the set of *atoms*, while

$$(2.17) \quad e : X \longrightarrow V_{afa}(X \cup A)$$

where the operation V_{afa} is defined in (3.5) in the next section.

Remark 2.2.

As we shall see in section 3, the range $V_{afa}(X \cup A)$ of the mappings e in (2.17) is considerably *larger* than $\mathcal{P}(X \cup A)$, which is the range of the corresponding mappings in (2.5) and (2.15). Therefore, the generalized systems of equations defined above contain as a rather *small* particular case the flat and the generalized flat sys-

tems of equations.

Examples 2.3.

1) For a given $a \in SET \cup \mathcal{U}$, let us consider the equation

$$(2.18) \quad x = x \longrightarrow a$$

hence it is *not* a flat or generalized flat system of equations, if we take

$$X = \{x\}, \quad A = \{a\}, \quad e_x = x \longrightarrow a \subseteq x \times a, \quad e_x \notin X \cup A$$

$$e_x \notin \mathcal{P}(X \cup A)$$

although (2.18) *has* solution in SET, see 5) in Proposition 2.3. below.

2) For given $p, q \in SET \cup \mathcal{U}$, the equation

$$(2.19) \quad x = \{\{x, q\}, p\}$$

is *not* a flat or generalized flat system of equations, if considered with

$$X = \{x\}, \quad A = \{p, q\}, \quad e_x = \{\{x, q\}, p\} \notin X \cup A, \quad e_x \notin \mathcal{P}(X \cup A)$$

although it can be written as a flat system of equations, provided that $x \in \mathcal{U}$, namely

$$x = \{y, p\}$$

$$y = \{x, q\}$$

hence

$$X = \{x, y\}, \quad A = \{p, q\}, \quad e_x = \{y, p\}, \quad e_y = \{x, q\} \subseteq X \cup A,$$

$$e_x, e_y \in \mathcal{P}(X \cup A)$$

Proposition 2.2.

Within ZFC we have

$$1) \quad \forall a \in SET : a \notin a$$

$$2) \quad \neg \exists a_1, \dots, a_n \in SET : a_1 \in \dots a_n \in a_1$$

$$3) \quad \neg \exists a, b \in SET : a \in TC(b) \in a$$

- 4) $\neg \exists a, b \in SET : a \in TC(b), b \in TC(a)$
- 5) $\forall a, b, c \in SET : c = \langle a, b \rangle \implies c \neq a, c \neq b, c \notin a, c \notin b$
- 6) $\forall A, X \in SET : X \neq \phi \implies X \neq A \times X$
- 7) $\forall X \in SET : X = X \times X \implies X = \phi$
- 8) $\neg \exists \text{ function } f = A \longrightarrow B, A, B \in SET : f \in \text{dom}(f)$
- 9) $\neg \exists \text{ functions } f_1 : A_1 \longrightarrow A_2, \dots, f_n : A_n \longrightarrow A_1,$
 $A_1, \dots, A_n \in SET, a_1 \in A_1 :$
 $f_n(\dots f_1(a_1) \dots) = f_1$
- 10) $\forall A, X \in SET : X \neq X \longrightarrow A$

It is important to note that, as seen next, even in ZFC^- , that is, without FA, one can obtain *impossibilities* of self-reference.

Proposition 2.3.

Within ZFC^- we have

- 1) $\forall \text{ function } F : A \longrightarrow B, A, B \in SET :$

$$\{x \in \text{dom}(F) \mid x \notin F(x)\} \notin \text{rng}(F)$$

- 2) $\forall \text{ function } F : A \longrightarrow \mathcal{P}(A), A \in SET :$

$$\{x \in A \mid x \in R_{wf}\} \notin \text{rng}(F)$$

where $R = \{\langle x, y \rangle \mid x \in F(y)\}$

- 3) $\forall X \in SET : X \neq \mathcal{P}(X)$
- 4) $\forall X \in SET : X = X \longrightarrow \phi \implies X = \{\phi\}$
- 5) $\forall A, X \in SET : X = X \longrightarrow A \implies$
 $\implies A = \{a\}, X = \{f\}, f(f) = a$
- 6) $\forall X \in SET : X = X \longrightarrow X \implies X = \{x\}, x = \{\langle x, x \rangle\}$

The proofs of the above two Propositions 2.2. and 2.3. are rather simple and immediate, and can be found at [1, pp. 25-27]

Remark 2.3.

Related to 5) and 6) in Examples 2.1. above, let us note the following two kind of situations encountered so far with sets which have *infinitely many brackets*, namely :

$$(2.20) \quad \Omega \stackrel{?}{=} \dots \{ \{ \{ \Omega \} \} \} \dots$$

$$(2.21) \quad \{0, \{1, \{2, \{3, \dots\}\}\}\} \in SET$$

The second one was, in 6) in Examples 2.1. above, proved to exist uniquely, and be well defined in ZFA, while the first one will be considered in more detail in 2) below.

1) Related to (2.21), we note the following immediate generalization. Let α be any infinite ordinal number and let us take

$$X = \{x_\beta \mid \beta < \alpha\} \subseteq \mathcal{U}, \quad A = \{\beta \mid \beta < \alpha\}$$

while

$$e_{x_\beta} = \{\beta, x_{\beta+1}\}, \quad \beta < \alpha$$

Then obviously, we obtain a flat system of equations, therefore (AFA) gives a unique solution s which, in view of (2.9), has the property

$$s_{x_\beta} = \{s_x \mid x \in b_{x_\beta}\} \cup c_{x_\beta} \in SET, \quad \beta < \alpha$$

where according to (2.6), (2.7), we have

$$b_{x_\beta} = e_{x_\beta} \cap X = \{x_{\beta+1}\}, \quad c_{x_\beta} = e_{x_\beta} \cap A = \phi, \quad \beta < \alpha$$

hence

$$s_{x_\beta} = \{s_{x_{\beta+1}}\} \in SET, \quad \beta < \alpha$$

which gives

$$(2.22) \quad s_{x_0} = \{0, s_{x_1}\} = \{0, \{1, s_{x_2}\}\} = \{0, \{1, \{2, s_{x_3}\}\}\} = \dots$$

where the pairs of brackets $\{ \}$ occur once for each $\beta < \alpha$.

Thus (2.21) is the particular case of the above s_{x_0} in (2.22) corresponding to $\alpha = \omega$ which is the first infinite ordinal. In the general case of an infinite ordinal α , the above s_{x_0} in (2.22) gives instead of (2.21) the set

$$(2.23) \quad \{0, \{1, \{2, \{3, \dots \{\beta, \dots\} \dots\}\}\}\} \in SET$$

which contains all $\beta < \alpha$.

2) Let us return to (2.20) and consider it as a particular case of the following general operation : given $a \in SET \cup \mathcal{U}$, define the set

$$(2.24) \quad \dots \{ \{ \{ a \} \} \} \dots \in SET$$

with a pair of brackets $\{ \}$ for each $n < \infty$. For that purpose, let us use the following notation

$$\{a\}_0^0 = a, \quad \{a\}_1^1 = \{a\}, \quad \{a\}_2^2 = \{\{a\}\}, \dots$$

Thus the problem is :

How to define in SET

$$(2.25) \quad \{a\}_\omega^\omega \in SET$$

where ω denotes the first infinite ordinal number.

Of course, one would want to define (2.25) as a certain kind of "limit" of the sequence of sets $\{a\}_n^n \in SET, n \geq 0$.

One way to do that for an arbitrary set $a \in SET$ is as follows. Let us denote

$$\begin{aligned} \{a\}_0^0 &= a \\ \{a\}_1^1 &= \{a\}_0^0 \cup \{a\}_0^0 = a \cup \{a\} \\ \{a\}_2^2 &= \{a\}_1^1 \cup \{a, \{a\}_1^1\} = a \cup \{a\} \cup \{a, \{a\}\} \\ \{a\}_3^3 &= \{a\}_2^2 \cup \{a, \{a, \{a\}_2^2\}\} = a \cup \{a\} \cup \{a, \{a\}\} \cup \{a, \{a, \{a\}\}\} \\ \{a\}_4^4 &= \{a\}_3^3 \cup \{a, \{a, \{a, \{a\}_3^3\}\}\} = \\ &= a \cup \{a\} \cup \{a, \{a\}\} \cup \{a, \{a, \{a\}\}\} \cup \{a, \{a, \{a, \{a\}\}\}\} \\ &\vdots \end{aligned}$$

It follows that

$$(2.26) \quad \{a\}_1^1 \subseteq \{a\}_2^2 \subseteq \{a\}_3^3 \subseteq \{a\}_4^4 \dots$$

Thus one can define

$$(2.27) \quad \{a\}_\omega^\omega = \bigcup_{n < \omega} \{a\}_n^n \in SET$$

Clearly, that procedure can be extended to all ordinal numbers α .

Thus the above problem (2.25) got solved in general, although not along its initial formulation.

On the other hand, in the particular case when $a = \Omega \in SET$, then in view of the fact that

$$\Omega = \{a\} \in SET, \quad n < \omega$$

one may come up with a definition of (2.25) considered in its initial formulation, and which hence is simpler than the one given in (2.27), namely

$$(2.28) \quad \{a\} = \Omega \in SET$$

And again, one may extend that definition to all ordinal numbers α , by

$$(2.29) \quad \{a\} = \Omega \in SET$$

3) The obvious difference between (2.20) and (2.21) is that in the second, there is an *outer* pair of brackets $\{ \}$, while in the first there is none. And such an outer pair of brackets does indeed define a set in SET , or for that matter, even in SET_0 , provided that what is within that outer pair of brackets makes sense in the respective version of Set Theory. And clearly, for (2.21) such is the case within SET , as seen in 6) in Examples 2.1. above.

One can also note that the generalization of (2.21) in (2.23) to arbitrary ordinals α always has an outer pair of brackets $\{ \}$. On the other hand, in the generalization (2.29) of (2.20), there is an outer pair of brackets $\{ \}$, only if α is not a limit ordinal.

4) The flat system of equations in 7) in Examples 2.1., can obviously be generalized to arbitrary ordinal numbers α , in a way similar to the generalization in 1) above of 6) in Examples 2.1.

3. Three Basic Operations

In order to pursue the theory, the following three operations, seldom if at all encountered in usual Set Theory, although quite elementary as such, will be needed.

We start with the definition of a fundamental concept.

Definition 3.1.

A set $a \in SET$ is called *transitive*, if and only if

$$b \in a \implies b \subseteq a$$

or equivalently

$$c \in b \in a \implies c \in a$$

Clearly, usual sets in mathematics are *not* transitive. For instance, given a set \mathcal{X} of open subsets in a topological space, then the transitivity of \mathcal{X} would imply that for every open subset $E \in \mathcal{X}$, we must also have $E \subseteq \mathcal{X}$. In other words, \mathcal{X} must also contain as elements all the points $x \in E$, for every $E \in \mathcal{X}$.

And now, the first basic operation.

Definition 3.2.

Given a set $a \in SET$, its *transitive closure* is by definition the smallest transitive set which contains it, and which is denoted by $TC(a)$.

Lemma 3.1.

The transitive closure $TC(a)$ exists for every set $a \in SET$, and it is given by

$$(3.1) \quad TC(a) = \bigcup \{a, \bigcup a, \bigcup \bigcup a, \dots\} \in SET$$

Further, for $a \in SET$, we have

$$(3.2) \quad TC(a) = \{b \mid b \in a\} \cup \{c \in b \in a\} \cup \{d \in c \in b \in a\} \cup \\ \cup \{e \in d \in c \in b \in a\} \cup \dots$$

Here we used the simplifying notation

$$\{c \in b \in a\} = \{c \mid \exists b \in a : c \in b\}$$

$$\{d \in c \in b \in a\} = \{d \mid \exists b \in a : \exists c \in b : d \in c\}$$

⋮

Note.

The meaning of $TC(a)$, for a given set $a \in SET$, is clear from (3.2) which, obviously, can be written in the equivalent form

$$(3.2^*) \quad TC(a) = \{x \mid x \in a_n \in \dots \in a_2 \in a_1 \in a_0 = a, \quad n \geq 0\}$$

Proof.

We note that

$$\bigcup a = \bigcup_{b \in a} b = \{c \in b \in a\}$$

thus

$$\bigcup \bigcup a = \bigcup_{b \in \bigcup a} b = \{c \in b \in \bigcup a\} = \{d \in c \in b \in a\}$$

and so on ...

Therefore

$$\begin{aligned} TC(a) &= \{x \in y \in \{a, \bigcup a, \bigcup \bigcup a, \dots\}\} = \\ &= \{x \in y = a\} \cup \{x \in y = \bigcup a\} \cup \\ &\cup \{x \in y = \bigcup \bigcup a\} \cup \dots = \\ &= a \cup (\bigcup a) \cup (\bigcup \bigcup a) \cup \dots = \\ &= a \cup (\bigcup_{b \in a} b) \cup (\bigcup_{c \in \bigcup a} c) \cup (\bigcup_{d \in \bigcup \bigcup a} d) \dots = \\ &= \{b \in a\} \cup \{c \in b \in a\} \cup \{d \in c \in b \in a\} \dots \end{aligned}$$

Examples 3.1.

1) Given $a \in SET$, then $TC(\{a\})$ is the smallest transitive set which has $a \in SET$ as an element, since

$$\begin{aligned} TC(\{a\}) &= \{a\} \cup \{c \in b \in \{a\}\} \cup \{d \in c \in b \in \{a\}\} \dots = \\ &= \{a\} \cup \{c \in b = a\} \cup \{d \in c \in b = a\} \dots = \\ &= \{a\} \cup \{b \in a\} \cup \{c \in b \in a\} \dots = \{a\} \cup TC(a) \end{aligned}$$

$$2) TC(\phi) = \phi$$

$$3) \text{ If } a \in \mathcal{U}, \text{ then } TC(\{a\}) = \{a\}$$

Note : if $a \in \mathcal{U}$, then $TC(a)$ is *not* defined, since $a \notin SET$

$$4) \text{ If } a \subseteq \mathcal{U}, a \in SET, \text{ then } TC(a) = a$$

5) If $a \in SET$, then

$$TC(a) = \phi \implies \{b \in a\} = \phi \implies a = \phi$$

thus

$$TC(a) = \phi \iff a = \phi$$

6) If $a \subseteq \mathcal{U}$, $a \in SET$, then

$$TC(a) = \phi \implies a = \phi \in SET$$

$$TC(a) = \phi \iff a = \phi$$

7) If $A, B \in SET$, $A, B \neq \phi$, $a \in A$, $b \in B$, then

$$\langle a, b \rangle = \{\{a\}, \{a, b\}\} \in A \times B \in SET$$

and

$$\begin{aligned} TC(\langle a, b \rangle) &= \langle a, b \rangle \cup \{d \in c \in \langle a, b \rangle\} \cup \\ &\cup \{e \in d \in c \in \langle a, b \rangle\} \cup \dots = \\ &= \{\{a\}, \{a, b\}\} \cup \{d \in c = \{a\} \vee d \in c = \{a, b\}\} \cup \\ &\cup \{e \in d \in c = \{a\} \vee e \in d \in c = \{a, b\}\} \cup \dots = \\ &= \{\{a\}, \{a, b\}\} \cup \{d \in \{a, b\}\} \cup \{e \in d \in \{a, b\}\} \cup \dots = \\ &= \{\{a\}, \{a, b\}\} \cup \{a, b\} \cup \{e \in \{a, b\}\} \cup \dots = \\ &= \{\{a\}, \{a, b\}\} \cup TC(\{a, b\}) = \langle a, b \rangle \cup TC(\{a, b\}) \end{aligned}$$

8) If $A, B \in SET$, $A, B \neq \phi$, $R \subseteq A \times B$, then

$$\begin{aligned} TC(R) &= R \cup \{c \in \langle a, b \rangle \in R\} \cup \{d \in c \in \langle a, b \rangle \in R\} \cup \\ &\cup \{e \in d \in c \in \langle a, b \rangle \in R\} \cup \dots = \\ &= R \cup \{c = \{a\} | \langle a, b \rangle \in R\} \cup \{c = \{a, b\} | \langle a, b \rangle \in R\} \cup \\ &\cup \{d \in c = \{a\} | \langle a, b \rangle \in R\} \cup \{d \in c = \{a, b\} | \langle a, b \rangle \in R\} \cup \\ &\cup \{e \in d \in c = \{a\} | \langle a, b \rangle \in R\} \cup \\ &\cup \{e \in d \in c = \{a, b\} | \langle a, b \rangle \in R\} \cup \dots = \\ &= R \cup \{\{a\} | \langle a, b \rangle \in R\} \cup \{\{a, b\} | \langle a, b \rangle \in R\} \cup \end{aligned}$$

$$\cup \{a \mid \langle a, b \rangle \in R\} \cup \{b \mid \langle a, b \rangle \in R\} \cup \\ \cup \{e \in a \mid \langle a, b \rangle \in R\} \cup \{e \in b \mid \langle a, b \rangle \in R\} \cup \dots$$

9) The above goes in particular when R is a function $f : A \rightarrow B$, or when $R = A \times B$, and in the last case we obtain

$$TC(A \times B) = (A \times B) \cup \{\{a\} \mid a \in A\} \cup \{\{a, b\} \mid a \in A, b \in B\} \cup \\ \cup \{a \mid a \in A\} \cup \{b \mid b \in B\} \cup \\ \cup \{e \in a \mid a \in A\} \cup \{e \in b \mid b \in B\} \cup \dots = \\ = (A \times B) \cup \{\{a\} \mid a \in A\} \cup \{\{a, b\} \mid a \in A, b \in B\} \cup TC(A) \cup TC(B)$$

10) If $X \in SET$, then

$$TC(\mathcal{P}(X)) = \mathcal{P}(X) \cup \{c \mid c \in b \in \mathcal{P}(X)\} \cup \\ \cup \{d \mid d \in c \in b \in \mathcal{P}(X)\} \cup \dots = \\ = \mathcal{P}(X) \cup \{c \mid c \in Y \subseteq X\} \cup \{d \mid d \in c \in Y \subseteq X\} \cup \dots = \\ = \mathcal{P}(X) \cup \{c \mid c \in X\} \cup \{d \mid d \in c \in X\} \cup \dots = \\ = \mathcal{P}(X) \cup TC(X)$$

11) If $x, y \in \mathcal{U}$ and $a = \{x, \{y\}\}$, then

$$TC(a) = \{x, \{y\}, y\}$$

since $b \in a \iff b = x \vee b = \{y\}$, thus $c \in b \iff c = y$, hence

$$TC(a) = a \cup \{c \mid c \in b \in a\} \cup \{d \mid d \in c \in b \in a\} \dots = \\ = a \cup \{c \mid c = y\} \cup \{d \mid d \in c = y\} \dots = a \cup \{y\}$$

□

The second important operation is presented in

Definition 3.3.

We define for sets their *support* as follows

$$(3.3) \quad SET \ni a \longmapsto support(a) = TC(a) \cap \mathcal{U}$$

Further, a set $a \in SET$ is called *pure*, if and only if

$$(3.4) \quad support(a) = \phi$$

Note.

The meaning of $support(a)$, for a set $a \in SET$, is easy to see, based on (3.2*), namely

$$(3.3^*) \quad support(a) = \{x \in \mathcal{U} \mid x \in a_n \in \dots \in a_1 \in a_0 = a, \ n \geq 0\}$$

in other words, $support(a)$ is the set of all ur-elements $x \in \mathcal{U}$, if there exist any, with which *finite* descending sequences $x \in a_n \in \dots \in a_1 \in a_0 = a$, with $n \geq 0$, that start with the set a do terminate.

Consequently, pure sets $a \in SET$ do *not* have such finite descending sequences, but only *infinite* ones, namely

$$\dots \in a_n \in a_{n-1} \in \dots \in a_2 \in a_1 \in a_0 = a$$

□

Finally, the third important operation is presented in

Definition 3.4.

$$(3.5) \quad \mathcal{U} \supset A \longmapsto V_{afa}[A] = \{a \in SET \mid support(a) \subseteq A\}$$

and clearly, $V_{afa}[A]$ is always a proper class.

We also denote

$$(3.6) \quad V_{afa}[\phi] = V_{afa} = \{a \in SET \mid a \text{ is a pure set}\}$$

□

Clearly

$$(3.7) \quad V_{afa}[A] \subseteq SET$$

therefore

$$(3.8) \quad V_{afa}[A] \cap \mathcal{U} = A \cap V_{afa}[A] = \phi, \quad A \subseteq \mathcal{U}$$

Also, if $a \in SET$, then we have seen that

$$TC(a) = \phi \iff a = \phi$$

therefore

$$a = \phi \implies \text{support}(a) = \phi$$

Also, if $a \subseteq \mathcal{U}$, then we have seen that

$$TC(a) = a$$

therefore

$$\text{support}(a) = a$$

If $x, y \in \mathcal{U}$ and $a = \{x, \{y\}\}$, then we have seen that

$$TC(a) = \{x, \{y\}, y\}$$

thus

$$\text{support}(a) = TC(a) \cap \mathcal{U} = \{x, \{y\}, y\} \cap \mathcal{U} = \{x, y\}$$

4. Graph Formulation

In this section we follow the presentation in [1], without however the proofs.

We consider *directed graphs* (N, E) , where N is the set of *nodes* and $V \subseteq N \times N$ is the set of *vertices*. A vertex $(n, n') \in V$ can be denoted by $n \rightarrow n'$. Thus

$$n_0 \rightarrow n_1 \rightarrow n_2 \rightarrow \dots$$

is a finite or infinite *path*

The graph (N, E) is called a *well-founded graph*, if and only if it has no infinite path

We also denote

$$N \ni n \longmapsto [n \succ = \{n' \in N \mid (n, n') \in E\}$$

$$N \ni n \longmapsto < n] = \{n' \in N \mid (n', n) \in E\}$$

Given a directed graph (N, E) and $n_0 \in N$, we call $(n_0 \in N, E)$ a *pointed graph*.

A pointed graph $(n_0 \in N, E)$ is called an *accessible graph*, if and only if

$$\forall n \in N : \exists \text{ path } n_0 \rightarrow \dots \rightarrow n$$

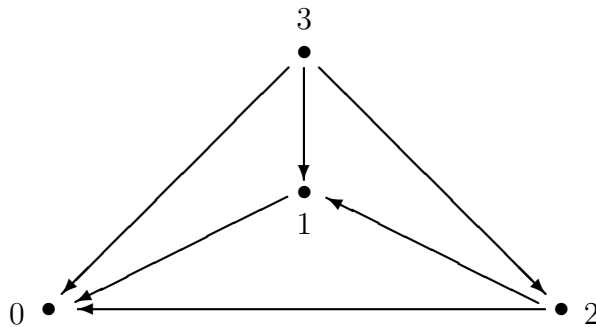
An accessible graph $(n_0 \in N, E)$ is called a *tree with root* n_0 , if and only if

$$\forall n \in N : \exists! \text{ path } n_0 \rightarrow \dots \rightarrow n$$

A *decoration* of a directed graph (N, E) is any mapping $S : N \rightarrow \text{Set}$, such that

$$\forall n \in N : S(n) = \{ S(n') \mid n' \in [n > \}$$

Example 4.1.



let S be any decoration

then $[0 > = \phi$, thus $S(0) = \phi = 0$

and $[1 > = 0$, thus $S(1) = \{S(0)\} = \{\phi\} = 1$

while $[2 > = \{0, 1\}$, thus $S(2) = \{S(0), S(1)\} = \{0, 1\} = 2$

finally $[3 > = \{0, 1, 2\}$, thus $S(3) = \{S(0)S(1), S(2)\} = \{0, 1, 2\} = 3$

Mostowski's Collapsing Lemma 4.1.

Every well-founded graph has a unique decoration.

□

A *picture of a set* A is any accessible graph $(n_0 \in N, E)$ which has a decoration S such that $A = S(n_0)$

Corollary 4.1.

Every well-founded accessible graph is a picture of a unique set.

Proposition 4.1.

Every set has a picture.

Proposition 4.2.

The ANTI-FOUNDATION AXIOM (AFA) has the equivalent formulation :

AFA AXIOM

Every directed graph has a unique decoration.

Corollary 4.2

Every accessible graph is the picture of a unique set.

There exist non-well-founded sets.

5. Comments, and Beyond

Self-reference has for quite a while by now happened to have acquired a rather automatic and somewhat thoughtless bad reputation as being but a source of undesirable paradoxes.

One of the more memorable moments in this regard was in ancient Greece, when the man from Theba came to Athens and stated in front of Athenians that : "All Thebans are liars !"

Nearer to our own days, in 1903, Russell's Paradox reformulated that ancient story within Set Theory which was then emerging as the basis of modern mathematics, and thus further aggravated the age old negative reflexes regarding self-referentiality.

On the other hand, as anthropologists tell us, three fundamental themes in human thought deeply rooted in prehistoric and pre-literate times have been self-referentiality, infinity and change.

Regarding the first, which is of main interest here, countless images of a snake biting its own tail are a testimony. Also, ancient Vedic wisdom saw it as the foundational aspect of reality. As for the ancient Hebrews, in Exodus 3:14 of the Old Testament, they considered it to be nothing less than the very name of God.

In this way, with the self-referential snake - rendered harmless as long as it is busy biting its own tail - as much as with the ancient Hindus or Hebrews, self-referentiality was not at all a horror to be avoided by all means. On the contrary, it was a rather sacred foundational aspect of the whole of reality ...

But then, later, came the man from Theba ...

And in our days, as a reinforcement in the very foundations of mathematics upon Set Theory, we have been facing Russell's Paradox ...

As it happens, however, a turn was taken in [5] back to ancient, pre-Athenian wisdom. And self-referentiality was in fact found to be of a positive practical interest, an interest which could not be addressed in other ways, [1-3].

But to return to what may be seen as more of an everyday mathematics, and actually, physics as well. Recently it was noted that such an elementary and basic concept like *orthogonality* can in fact be defined in arbitrary vector spaces without any scalar product, provided a self-referential definition is employed, [13]. The relevance of that in modern physics is obvious. Indeed, in Quantum Mechanics, for instance, the standard model is based on Hilbert spaces where orthogonality is essential and has considerable physical meaning and interpretation.

However, self-referentiality may turn out to have far larger and deeper impact in mathematics. In this regard, let us mention a few areas where, given long ongoing deeper underlying difficulties so far not treatable without self-referentiality, one may at last find a more appropriate approach by suitable self-referential definitions of basic concepts.

For instance, the usual concept of topology, introduced by Hausdorff in 1914, suffers among others from the fact that the respective category is not Cartesian closed. In other words, given three topological spaces X, Y and Z , we typically do not have the equality $\mathcal{C}(X \times Y, Z) = \mathcal{C}(X, \mathcal{C}(Y, Z))$ between the respective spaces of continuous functions.

That, as well as other deficiencies of the usual concept of topological space have during the last decades been addressed by various more general concepts of so called pseudo-topologies, [15]. The fact, however, remains that the variety of pseudo-topological concepts evolved so far, and defined of course without self-reference, give the impression of a series of ad-hoc disparate steps which do not seem to manage to touch in a more unifying manner the deeper meaning of topology.

Probability theory is another area of mathematics where the standard Kolmogorov model has manifest deficiencies. One that turns up from the very beginning is that each point $x \in X = [0, 1]$ has the probability zero with the usual measure, thus probabilistically it is redundant. Yet the set of all such points cannot be eliminated, since then one would remain with the empty set. And this is in sharp contradistinction with what happens in the case of a finite or countable probability space X , where each point of zero probability can be eliminated, and one remains with a simplified model X_0 which is isomorphic from the point of view of probability. Further well known difficulties with the standard Kolmogorov model are found in the study of continuous time stochastic processes.

It is therefore an open question of some effective interest whether a self-referential definition of probability space may help in overcoming such difficulties. In this regard it is worth noting that, while the Loeb nonstandard approach to probability brought with it a number of advantages, it has nevertheless not been able to address satisfactorily the mentioned, as well as other difficulties.

The concept of computability has been of major interest during the last decades. As for the nature of its definition, the present relevance of the related Church-Turing Thesis can be seen as showing a certain lack of sufficient insight, and thus it can appear as an inadequacy. In this regard, one may consider the possibility of a self-

referential definition of the concept of computability.

Complexity, among other realms in computation, is another fundamental modern concept in mathematics. And then, as its own name may possibly suggest, perhaps, the present day simple non-self-referential definitions for it may actually be rather inappropriate ...

So much for avoiding the alleged horrors of self-referentiality ...

And as a sign of the power of persistence of age old negative connotation attached to self-referentiality, one can note that major recent contributions to the subject still use a negative terminology, such as "non-well-founded sets" or "vicious circles" ...

And now, let us consider a possibly yet older, more universal, and so far incontrovertible horror, namely, that of contradiction.

Indeed, in this regard, there seems not to be found any controversy of any significance whatsoever throughout known human history, with all the evidence pointing to the universal commandment of: "One must avoid contradiction !"

And yet, so strangely, ever since we so essentially use our modern electronic digital computers, we have been basing so much, and in such an essential manner, on a very simple, clear and sharp contradiction.

Indeed, rather not consciously known to most of us, such computers - even when seen as operating only on non-negative integers - function according to the following:

CONTRADICTIONARY SYSTEM OF AXIOMS :

- the well known Peano Axioms

plus the axiom :

- there exists $M \gg 1$, such that $M + 1 = M$

where the respective M , called *machine infinity*, may typically be larger than 10^{100} .

So much for avoiding the alleged horrors of contradictions ...

Here however, apparently not having any known ancient wisdom to return to, a genuinely novel opening was taken with recent studies of so called *inconsistent mathematics*, [11,12].

What may, beyond all that, be indeed a major new opening is the development of mathematics which *brings together* both self-referentiality and contradiction. And the unprecedented vastness of the respective realms that may become available in such a way is only to be guessed at present time ...

6. Axioms of Set Theory

For convenience, we recall here the ZFC Axioms of Set Theory, [8, p. 1].

AXIOM OF EXTENSIONALITY

If two sets X and Y have the same elements, then $X = Y$.

AXIOM OF PAIRING

For any two sets a and b there exists a set $\{a, b\}$ that contains exactly a and b .

AXIOM SCHEMA OF SEPARATION

If P is a property with parameter p , then for any sets X and p there exists a set $Y = \{u \in X \mid P(u, p)\}$ that contains all those elements $u \in X$ which have property P .

AXIOM OF UNION

For any set X there exists a set $Y = \bigcup X$, the union of all elements of X .

AXIOM OF POWER SET

For any set X there exists a set $Y = \mathcal{P}(X)$, the set of all subsets of X .

AXIOM OF INFINITY

There exists an infinite set.

AXIOM SCHEMA OF REPLACEMENT

If a class F is a function, then for any set X there exists a set $Y = F(X) = \{F(x) \mid x \in X\}$.

AXIOM OF REGULARITY OR FOUNDATION

Every nonempty set has an \in -minimal element.

AXIOM OF CHOICE

Every family of nonempty sets has a choice function.

Here, we also recall the additional axioms used above.

ANTI-FOUNDATION AXIOM (AFA)

$\forall \mathcal{E}$ flat system of equations : $\exists!$ s solution

An equivalent formulation of the above Axiom of Regularity or Foundation is given in

AXIOM OF FOUNDATION (FA)

$\forall a \in SET$: $\langle a, \in \rangle$ well-founded

STRONG AXIOM OF PLENITUDE

There is an operation $new(a, b)$, such that

- 1) $\forall a \in SET, b \subset \mathcal{U} : new(a, b) \in \mathcal{U} \setminus a$
- 2) $\forall a, a' \in SET, a \neq a', b \subset \mathcal{U} : new(a, b) \neq new(a', b)$

References

- [1] Aczel P : Non-Wellfounded Sets. CSLI Lecture Notes Number 14, Stanford Univ., 1988
- [2] Barwise J, Moss L : Vicious Circles, on the Mathematics of Non-Wellfounded Phenomena. CSLI Lecture Notes Number 60, Stanford Univ., 1996
- [3] Barwise J, Etchemendy J : The Liar, an Essay on Truth and Circularity. Oxford Univ. Press, 1987
- [4] Bourbaki N : Elements of Mathematics, Set Theory. Springer, New York, 2004
- [5] Forti M, Honsell F : Set theory with free construction principles. Annali Scuola Normale Superiore di Pisa, Classe di Scienze, 1983, 10, 493-522
- [6] Fraenkel A A : Abstract Set Theory, Third, Revised Edition. North-Holland, Amsterdam, 1966
- [7] Henle J M : An Outline of Set Theory. Springer, New York, 1986
- [8] Jech T : Set Theory, The Third Millennium Edition, Revised and Expanded. Springer, New York, 2003
- [9] Lawvere F W, Rosenburgh R : Sets for Mathematics. Cambridge Univ. Press, 2003
- [10] Levy A : Basic Set Theory. Dover, New York, 1979

- [11] Mortensen C : Inconsistent Mathematics. Kluwer, Dordrecht, 1996
- [12] Rosinger E E : On the Safe Use of Inconsistent Mathematics. arXiv:0811.2405
- [13] Rosinger E E, Van Zyl A : Self-Referential Definition of Orthogonality. arXiv:0904.0082
- [14] Sierpinski W : Cardinal and Ordinal Numbers. Hafner, New York, 1958
- [15] Van der Walt J-H : Private Communication
- [16] Zuckerman M M : Sets and Transfinite Numbers. Macmillan, New York, 1974