Review Article

Gravitation, Entropy & the Mandelbrot Set

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Abstract

The fundamental forces are often seen, in Physics, to arise in the context of symmetries and symmetry groups. Perhaps the reason we lack a full unification of gravitation with the other forces is that gravity results from a fundamental asymmetry instead. Some assert that gravitation is different from the strong and weak nuclear forces and electromagnetism, and should not be considered a fundamental force, but should rather be seen as a residual of the other forces. Jacobson wrote that gravity appears to be a consequence of thermodynamic entropy. Verlinde and Padmanabhan further developed entropic theories of gravity. This work shows that both Newton's gravitation and Einstein's equations can arise from entropic considerations. But is the basis for dissipative processes found in pure Mathematics? The Mandelbrot Set reproduces the Verhulst dynamic and further displays the monotonic progress of entropy. Furthermore; it is maximally asymmetrical along the real axis – exhibiting fundamental asymmetry. This paper asserts that $\mathcal{M}$ is the missing link for gravity theorists, because it is asymmetrical on the whole but contains many exact symmetries. $\mathcal{M}$ therefore reveals a natural balance between preserved and broken symmetry, which illustrates how symmetry-breaking is essential to the Physics of gravitation.

Keywords: Thermodynamic gravity, entropic gravity, symmetry groups, symmetry breaking, fundamental asymmetry, Mandelbrot Set.

Introduction

Entropic theories of gravity are an accommodation to the existence of a deep asymmetry in nature, which has been only partly addressed in theoretical Physics to date. While the bulk of the Physics community has studied an extensive range of symmetry-preserving mathematical objects, in their search for answers; the author devoted many hours over more than 30 years to studying a maximally asymmetric object – the Mandelbrot Set – and probing its connections to Physics [1]. It only recently became apparent, however, that the connection $\mathcal{M}$ has with physical reality and theoretical Physics is largely *because* of its fundamental asymmetry [2].

It can be shown that the Mandelbrot Set is a complement to the symmetry groups which are often used as the root of theoretical Physics – and form the basis of the Standard Model – because it is the representation of a universal organizing principle. The Mandelbrot Mapping Conjecture posits that $\mathcal{M}$ mimics processes at the extreme of high temperature on the cusp at $(0.25, 0i)$, and

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the cessation of vibratory phenomena at absolute zero, in the main antenna tip at $(-2,0i)$. Ergo it models dissipative processes especially well, when they are treated as a globally broken symmetry. Recent interest among gravity theorists in thermodynamic [3] and entropic [4], theories of gravitation may mean that the Physics of Gravity is where this property of $\mathcal{M}$ has the greatest applicability. While thermodynamic gravity has come under fire, with claims of refutation by Carroll and Remmen [5]; the proposed construction may survive intact, because it has a unique theoretical basis utilizing the spreading metaphor for entropy [6], which answers their objection.

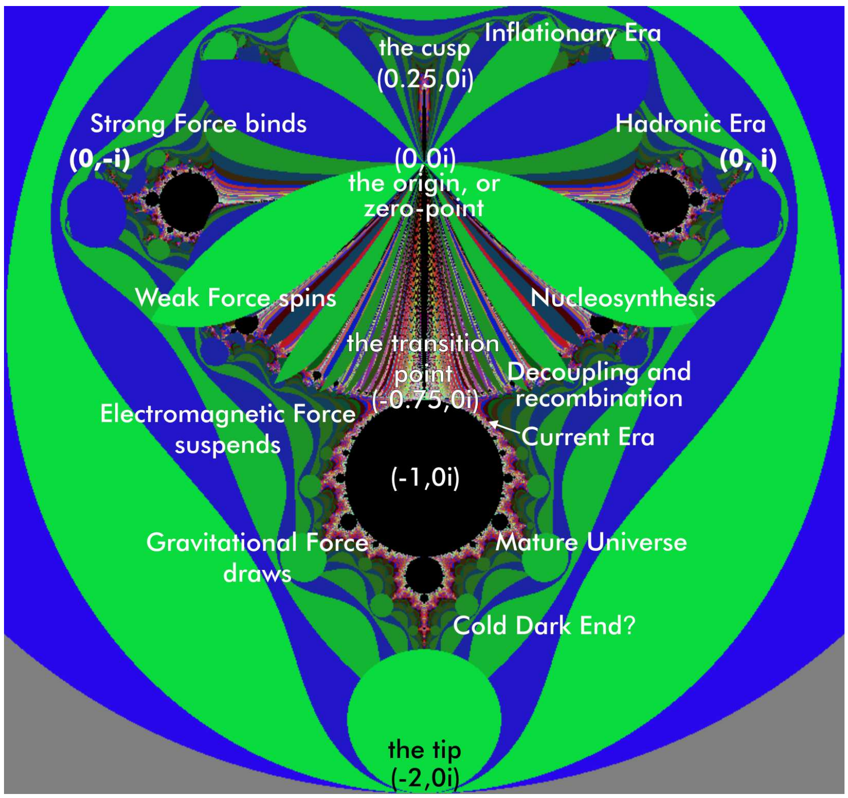

Fig. 1. Mandelbrot Butterfly annotated as a cosmic thermometer

Entropic theories of gravity are formulated by studying the thermodynamics with respect to a local horizon, which is typically any local Rindler horizon, but it is clearer and more general with a true holographic bound – dimensional reduction at a boundary between spaces of higher and lower dimension. While the most familiar example of this is compaction of 3-d space to a 2-d surface at the event horizon of a black hole, where Physics in 4-d spacetime is holographically projected; there can be a holographic boundary with the next higher dimension too. This is a feature in the model suggested by Pourhasan, Afshordi, and Mann [7], where the source of our present day cosmos is a 5-d black hole $\rightarrow$ white hole in 4-d spacetime, with a holographic boundary between the prior universe and ours. Their idea was based on the braneworld model in DGP gravity [8], which was proposed to explain long-range behavior of gravity and the accelerating expansion of the cosmos. But the idea of a 5-d black hole → white hole in 4-d also arises in the work of Poplawski [9], which is based on the Einstein-Cartan model. Since these theories were proposed; extensions and variations have appeared, including a theory called cascading DGP [10], where the dimensionality cascades from 6-d at extreme distances, to 5-d at great distance, then to 4-d spacetime in the local universe. Since what we observe at the greatest distances corresponds to cosmic events that happened ages ago, this means a precursor to the current universe was higher-dimensional, if any of these theories hold true.

Mandelbrot Gravity theory (or MGT) likewise suggests today's 4-d spacetime is the product of a 5-d volume, which was the prior universe. In analogies of the Mandelbrot Set with Cosmology – using some conventional assumptions about the early universe – folding of space occurs at decoupling; suggesting the surface of last scattering is a holographic screen where the pre-decoupling universe was 5-dimensional. This is modeled in $\mathcal{M}$ at (-0.75,0*i*), where the cardioid meets a circular disc centered at (-1,0*i*). We propose that the universe was turned inside-out when the mass-energy contained within a volume by the opacity of the primordial plasma ball was forcibly ejected – and injected into the available space – once it was allowed to do so. This theory treats the 5-d → 4-d shift as an actual eversion of space, however, such that formerly outward-facing directions became inward-facing toward the centers of independent massive objects.

To explain gravity, it is assumed the cosmological action of fermionic mass was initially to press outward the boundaries of spacetime, since 3-d particles tend to be concentrated on the skin or hypersurface of a higher-dimensional sphere, along the fabric or brane. Thus the presently weak gravity is a relic of its formerly outward-pressing energetic action creating spacetime in the early universe – as an inward-facing force possessed by independent masses. By thus treating gravity as a residual force of energy in the early cosmos; the cosmology based on $\mathcal{M}$ I have developed suggests a picture similar to entropic gravity theories – by Jacobson [11], Verlinde [12], Padmanabhan [13], and others – but it offers a unique route to a similar result, which avoids some pitfalls.

The Misiurewicz point near (-1.54,0*i*) depicts and thus predicts a connection between the event horizon of a Schwarzschild black hole and a Bose-Einstein condensate or BEC. Recent work by Dvali and Gomez [14] treats black holes as a condensate of gravitons, where the event horizon is the quantum critical point. They state that black holes are the most powerful quantum computers, but communication limits prevent that power from being harnessed for useful calculations [15], because what escapes is no more correlated than Hawking radiation. We are not prevented from studying the horizon in $\mathcal{M}$ however. $\mathcal{M}$ is a purely calculational system, so it lets us peer behind boundaries that nature makes opaque. Fortuitously, the Misiurewicz point for the quantum critical point is tractable analytically, allowing us to obtain exact solutions and high-precision numerical values [16]. This spot at about (-1.543689,0i) , is unassuming and could easily be missed. When magnified; we see rows of telephone poles that shrink to nothing as they approach it, and then grow in size (but in opposite phase) on the other side. Scale reduction of repeating similar forms happens at all Misiurewicz points; but forms do not grow again on the other side at terminal points.

We have shown previously that when areas where the magnitude of the iterand monotonically diminishes are colored in, there are disks populating the periphery at the branching points of the Mandelbrot Butterfly figure – which are basins of attraction between a branching point and a terminus. At our location of interest, in the tail of $\mathcal{M}$, the band-merging point of the bifurcation diagram and the leading edge of the largest disk coincide – so we will examine the dynamics at this Misiurewicz point in some detail.

The Theoretical Physics Setting for MGT

Some ideas from Einstein – that there is a mass-energy equivalence, and that gravity results from deformation of the spacetime fabric – are essential to this theory. We also assume portions of the Big Bang-Inflationary Universe model are true. But we note that Paul Steinhardt compellingly argues [17] conventional inflation has unforeseen problems and unseen artifacts – including that *all* inflation is eternal and chaotic – which suggests we should consider a different basis for inflation-like cosmologies. What best fits how gravity arises in cosmologies based on the Mandelbrot Set [18], is a flavor of octonionic inflation where the dimensionality of space evolves.

We see several theories where spacetime goes from being 2-d initially – then evolves with a running D to 4-d spacetime – including CDT [19], Quantum Einstein Gravity [20], Loop Quantum Gravity [21], Hořava-Lifshitz gravity [22], and Rainbow Gravity [23]. D varies continuously from the Planck scale, in these theories, so spacetime assumes a fractional dimension at various phases on the way from 2-d to 4-d, and it therefore has a microscopic fractal structure – while evolving from a lower dimension to the current 4-d spacetime. However; in theories like DGP gravity, a higher-d origin for the cosmos is assumed instead, where in cascading DGP D cascades from 6-d $\rightarrow$ 5-d $\rightarrow$ 4-d, so it is nearly the opposite of the phenomenology above. Thus we have theories for both rising and falling dimensionality, from the early universe to the present.

In MGT; it is assumed that the dimensionality of spacetime evolves in an octonionic embedding space. Since geometry is non-associative under the octonion algebra; creating spacetime in 8-d is more complicated than constructing forms in 3-d space. It is necessary to speak of both an upper and a lower bound for D, when charting its evolution in the early universe – instead of saying spacetime has a specific dimensionality at any given time, in the earliest epoch. In spaces with a non-associative geometry; D is sensitively relative to the order and sequence observations are made, as well as one's viewpoints while observing, because the properties of interiority/exteriority and size/distance are relational and emergent, as creation of spacetime and form progresses in the early universe. [24] Things can appear higher-dimensional in the bulk but are only 2-d at the microscale.

Thus, one can have a process where the dimensionality of space and dimensions of spacetime evolve from a lower limit upward and from an upper limit downward, simultaneously. But the sequentially evolutive property of the octonions provides a directed evolution, not a purely random or chaotic one. In one scenario; we see spacetime initially arising as a collection of differently-oriented planar crystal segments in a jewel-like pattern evolving into a fractal pattern – where uncorrelated segments are linked together into surfaces that assume volumetric arrangements of increasing order. But once structures that can serve as containers emerge; this expanding volume rapidly takes on the character of a sphere of some dimension.

We should expect the speed of light to be greater, when the mass of the universe is less or when light is constrained to 2-d planar segments of spacetime. This constraint effect is the difference in propagation speed observed for surface plasmons and photons in 3-d space, except that the quantum-mechanical surface is a facet of spacetime. But there is also a rationale for light's speed limit to be exceeded, or to go away entirely, in a universe that is devoid of all mass. We can speak of mass-energy and acknowledge that radiation is self-gravitating, but matter and energy behave differently. A massive body radiating strongly enough reaches the Eddington limit, so its surface radiation pressure repels any masses that its gravity would otherwise attract.

Thus pure energy can attract *or* repel, while its equivalent mass is purely attractive, at least at a low enough temperature. Since mass is involved in the mass-energy equation, we posit the presence of mass in the universe sets the speed of light. If we rearrange terms in $E = mc^2$ to yield $c^2 = E/m$, and then let $m \to 0$, we see that $c^2 \to \infty$. [25] Thus; in the matter-free regime, before the appearance of massive particles, the speed of light is essentially infinite. If light's speed is unconstrained before mass appears, at the Planck time, and geometrogenesis must occur for space to become volumetric (admitting particles); this clearly suggests a mechanism to replace conventional inflation. While the space created in MGT is slightly different, where the early universe grows to a 5-d hypervolume before everting into a 4-d spacetime, the phenomenology is roughly the same as in recent work by Afshordi and Magueijo [26].

This dynamic arises partly from basic Physics considerations, and partly relates to subtleties of Math which inexorably shape the way the universe unfolds. How can one deal with the massless realm or matter-free regime, of the Cosmos' first instant? We assume that prior to conditions where massive particles could form; the mass-energy of the universe was almost exclusively energetic – a concentration of pure energy – where the emergence of sub-atomic particles converts energy into mass that is flung outward from the region(s) of highest energy. Ergo; the initial action of mass and gravity was almost exclusively outward – to create the volumetric expanse of spacetime through the relativistic action of outward-pressing mass bending spacetime – which then becomes an arena for the emerging particles to inhabit.

So the Physics is about how concentrated energy pushes out space into a volume, during that brief time (as with the onset of inflation) – which then allows 3-d particles to exist – and how those particles become the outflowing mass stretching the fabric of space further, allowing its contents to cool. However; it must be noted that the associated Mathematics is also emergent, which means the familiar rules apply only when certain conditions are met. Specifically; we cannot assume the metric spaces of the seminal universe are commutative and associative, because there are logical reasons to assume the opposite is true – that the geometry of the earliest phase of the universe is both non-commutative and non-associative. After discussing the spheres in the context of inflation; I will summarize how this mathematical subtlety relates to the early universe, and to cosmic evolution.

If we consider the properties of higher-dimensional spheres, we find that their surfaces have volume. This follows from the simple pattern that begins with a circle or 1-sphere. The circle is called a 1-sphere because its topological boundary, separating the interior from the exterior, is a 1-d line. The degenerate case called a 0-sphere consists of a pair of points on a line, which bracket the space between them. But the edge or rim of a circle is more clearly like a container, which can hold items within its interior separate from those items that are outside the rim. This pattern continues as we go up in dimensions, because a common sphere is a 2-sphere, where the boundary is a 2-dimensional surface, and a ball so constructed can be filled to contain a volume of air (for example).

When we go one step further; we find that the surface of a sphere in 4-d is a 3-d volume, and a 5-d sphere's surface is a 4-d hypervolume. So we find that 3-d particles that can travel through the bulk in a 3-d volume of space are excluded from the bulk of a hypersphere's enclosed volume, and are constrained to travel on the skin of the ball – along or across the brane – instead. Ergo; the fermionic mass is concentrated in the outward-pressing hypersurface in the early cosmos, contributing to the stretching and curving of space. However; the same mass that pushed space out in the early universe now pulls other objects in toward its centers.

We are used to the properties of spaces and objects being stable, or static, but those who see space as emergent favor a different view. Pioneer in Non-commutative geometry, Alain Connes famously declared "Noncommutative measure spaces evolve with time" [27], and said this is a template for other important statements one can make about NCG. In this field we see how size and distance measurements shift over time, or depend on the level of scale and the order measurements are taken. Thus; many familiar concepts take on the flavor of quantum mechanical variables. Examining how non-associative spaces behave, we see this evolutive dynamism is extended, and the comments of Paul C. Kainen [28] about the Octonions are helpful.

"Of course, multiplication in the octaval arithmetic fails to be either commutative or associative, but that could be a blessing in disguise. If multiplication depends on the order of the elements being multiplied together and even on how they are grouped, then at one fell swoop, geometry enters the calculation in an organic way. The Principle of Indeterminacy could then arise in a natural fashion from relativistic considerations, making quantum theory a consequence of an underlying 8-dimensional hidden-variable process, very much in the flavor of the theories of de Broglie and Bohm. Uncertainty of measurement would be a corollary of our inability to absolutely order events or to absolutely control the way in which they are grouped." This implies that the non-associative property of the Octonions confers a sequential evolution to objects and spaces fostered therein. And this property forces things to evolve beyond that non-associative geometric state.

The dynamism of the Octonions is likely enough to set geometrogenesis in motion with only a modicum of energy, allowing emergent properties to arise, but it also makes forms created in the early universe transitory or ephemeral. There is a simile between the algebraic action of placing terms within parentheses and the geometric action of placing items within containers – so for non-associative geometric figures, the interiority and exteriority implied by the boundaries of objects is defined relationally. Ergo; we find that geometrogenesis is a necessary precursor to the emergence of sub-atomic particles, which must inhabit a well-defined volume. So what drives geometrogenesis and defines particles? If we consider the classification of the Lie groups by Killing and Cartan, there are three infinite families – $SO(n)$, $SU(n)$, and $Sp(n)$ – which come from the Real, Complex, and Quaternion numbers, and the five exceptional groups – G_2, F_4, E_6, E_7, and E_8 – which all spring from the Octonions!

The Lie groups can explain particle physics well. But to explain inflation and/or cosmological expansion, we must examine higher-d spheres, and chart where their surface area and volume are maximized – the dimensionality where the hypersurface and hypervolume are maximal. We see in Fig. 2 and Fig. 3 below [29], that the curves rise to a peak then decline. While hypersurface area is maximal for the 8-d 7-sphere, the sphere's filled interior or ball is maximal in 5-d. It was discussed in work with departed colleague Ray B. Munroe [30], that this unequal relation is a driver of octonionic inflation, which arises naturally in Mandelbrot Set Cosmology – by assuming the 2-d Set is a projection of a higher-dimensional figure that resides in the quaternions and octonions.

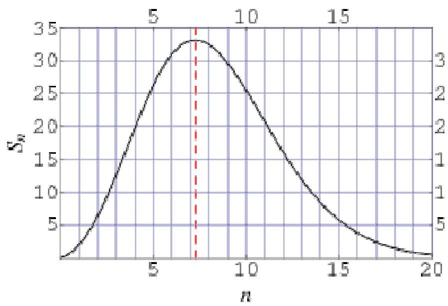

Fig. 2. hypersurface area of the n-sphere

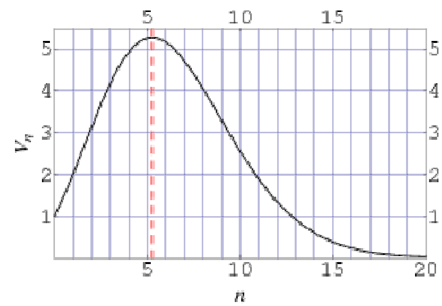

Fig. 3. hypervolume of the n-ball

When discussing Entropic Gravity, a clear notion of how entropy is defined is essential. But this question has a deep relevance to Cosmology, as well, and thus to the Physics behind theories of Cosmology or Gravitation based on the Mandelbrot Set and its fundamental asymmetry. As I have emphasized in past writing; the oft-quoted notion that entropy is about things becoming disordered is misleading, and this overused generalization should be avoided where better descriptive terminology is available. I find great utility in the metaphor of "spatial and temporal spreading" advocated by Leff [31] and the notion that entropy is the "dispersal of energy" championed by Lambert [32]. This works especially well to explain how entropy drives spontaneous inflation, in scenarios similar to the work of Carroll and Chen [33], and explains why entropy could fuel cosmic expansion.

The idea that energy tends to spread out, if not constrained, is adequate motivation of itself – to support this claim. And these are precisely the terms in which Lambert defines entropy. But the broader notion suggested by Leff, that spreading is entropy's signature dynamic, allows a visible connection to be made with structures along the periphery of the Mandelbrot Set, most especially in the branching patterns at the Misiurewicz points. As first noted by Tan Lei [34], there is almost precise symmetry near the center, but greater asymmetry at the edges, with strong similarity between Mandelbrot and Julia Set forms – at these points. So in $\mathcal{M}$ we see multiple local symmetries spreading into a backdrop of global asymmetry, and the interplay between these two opposing dynamics around every symmetric structure.

This interplay is a result of the fundamental asymmetry possessed by $\mathcal{M}$ as a whole. It is also an example of the global self-similarity of structures in the Mandelbrot Set with the entire Set. Any portion of $\mathcal{M}$ shows a view of, and properties of, the whole structure – much as a square cut from a hologram will reveal the entire object that was imaged. So what we see are beautiful symmetries framed against fundamental asymmetry, which creates symmetry-breaking structures in every possible variety. By displaying exact symmetries and periodic behaviors against a backdrop of fundamental asymmetry, the Mandelbrot Set gives us a unique opportunity to examine how symmetry is broken in the general case, and to find specific examples relevant to Physics. This can be realized only in combination with a host of other mathematical objects of special importance, such as the exceptional Lie groups, the Fischer-Greiss or Monster group, and things at the opposite end of the spectrum like the simplexes, spheres, and tori.

The Cosmology based on the Mandelbrot Set suggests a Mathematical Universe like Tegmark's [35], or Plato's Ideal Forms [36], but in an even broader sense where nature knows about and employs the totality of Math and uses all of the objects of pure Mathematics [37] – because they are the most efficient means to an end. In Physics phenomenology; natural law is what results when Nature tries to maximize, minimize, or optimize its systems and structures. The universe seeks and realizes optimal structures, as though cosmic evolution is an exercise in the Calculus of Variations.

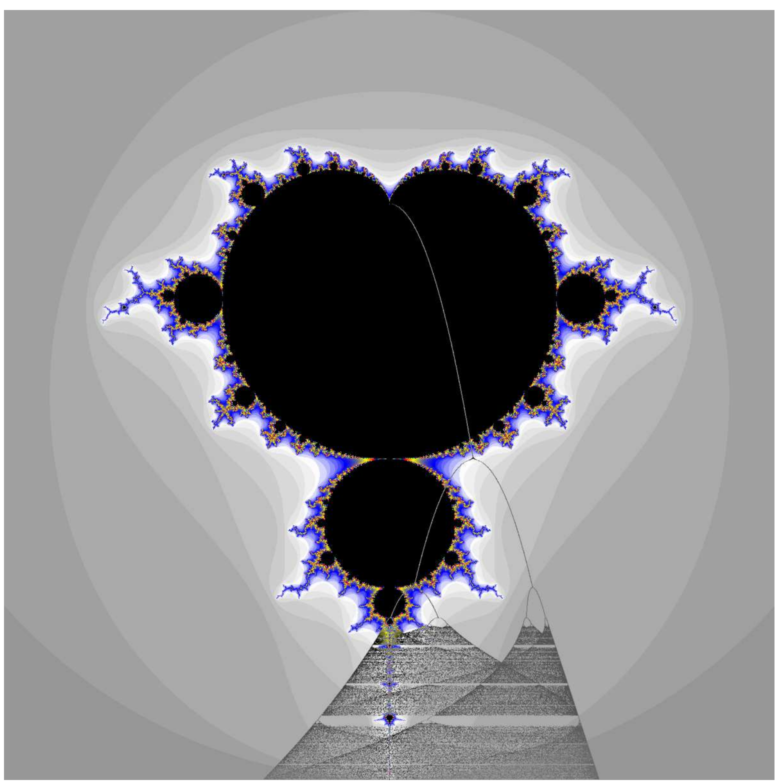

Fig. 4. Mandelbrot Set with its bifurcation diagram superimposed

Where the totality of all Math is integrated into Physics; the symmetry groups have gotten the most attention, but they do not tell the whole story. While the largest exceptional group E_8 is the epitome of symmetric order as the maximally symmetric mathematical object, $\mathcal{M}$ is maximally asymmetrical. The bifurcation diagram for $c \to c^2 + c$, the generating equation for $\mathcal{M}$ over the reals, reproduces the Verhulst process, illustrates the progression to chaos, and so on. We know that since $\mathcal{M}$ displays higher-order periodic branching, it defines higher-order chaos, and multifurcations in every order or period. $\mathcal{M}$ shows us the interplay between symmetry and asymmetry, and between order and chaos, as we have noted.

But to use the insights we obtain from the Mandelbrot Set for Physics, we must extend the arguments or analogies into higher dimensions. One might ask what a 2-d figure like $\mathcal{M}$ has to do with, or can tell us about, higher-dimensional reality. The Mandelbrot Set as it is normally viewed is the cross-section of a higher dimensional figure living in the quaternions and octonions. We see only its shadow or a projection of $\mathcal{M}$ in higher-d space. But the common form of $\mathcal{M}$ encodes or displays important facts about higher-dimensional reality. It shows us how a higher-d precursor can create the conditions we observe now, and it illustrates the progression from high to low dimensionality via broken symmetry. It reveals how the cosmos evolves from non-associative geometry in the early universe to a space that is both associative and commutative today.

This explains the Mandelbrot Set's relevance to Physics, because $\mathcal{M}$ best represents the interplay of symmetry and symmetry-breaking – of all mathematical objects. It is the maximal example of fundamental asymmetry, and based on a minimal formula and algorithm. When we see the 2-d Mandelbrot as a projection of a higher-d figure existing in the Quaternions and Octonions, we see that octonionic geometrization (by exercising repetitive triangulation) bumps into limits imposed by $\mathcal{M}$, on its way to inflating space and expanding spacetime. This imposition of an asymmetry or direction forces ordering to take place. Without a gradient or fundamental asymmetry as a driver, ordering the elements of spacetime in octonionic inflation is almost random, or appears so, because interiority and exteriority are relative among geometric elements. This can be seen as an artifact of non-associativity itself, since while non-commutative geometry imposes a varying definition of size and distance, non-associative geometry asserts that the sense of what is inside, or outside, of a given reference frame also changes.

At the outset of inflation, spacetime obeys E_8 symmetries, and has a fundamental asymmetry or gradient. This is the slope at the cusp of the cardioid region of the Mandelbrot Set. And given its overall shape, $\mathcal{M}$ also contains Cartan's rolling ball in 3:1 ratio analogy for G_2 [38], where the edge folds back on itself at $(-0.75, 0i)$. This location is an endpoint, or final singularity for our precursor, where the maximum (hyper-) volume is reached in 5-d. This deserves further explanation, however.

It is commonly observed that thermodynamic cooling during universal expansion is the dynamic that brings about a threshold which decouples matter from energy. However; we should note that – for a given quantity of mass-energy – there are absolute maxima imposed by geometry as well. Once the (hyper-) surface reaches its maximum extent, there is no further it can go, so this ends octonionic inflation. But by that time, geometrization has resulted in well-defined spatial volumes, so volumetric expansion of spacetime has commenced, and is at its most rapid. If volumetric expansion goes to its maximum, and D is a running variable; the limit is reached in dimension 5. This maximum coincides with $(-0.75, 0i)$ on the Mandelbrot Set. If the distance from the origin to the cusp at $(0.25, 0i)$ is the radius of a circle, then the location $(-0.75, 0i)$ is three times as distant. But this point is surmounted by a disc with radius $.25$, centered at $(-1, 0i)$, exactly duplicating Cartan's rolling ball analogy of G_2 symmetries.

Knowing this analogy works best in five dimensions, we see this is connected to the fact the 5-ball is the maximal volume for the *n*-ball, where *n* can take on any integer value. So if the process of geometrogenesis does proceed by adding dimensions, as well as by adding to the extent of space; it is geometrically exhausted – in terms of the potential for further growth – once spacetime reaches 5-d. And in MGT; this then delivers us into a 4-d spacetime bubble – post-decoupling.

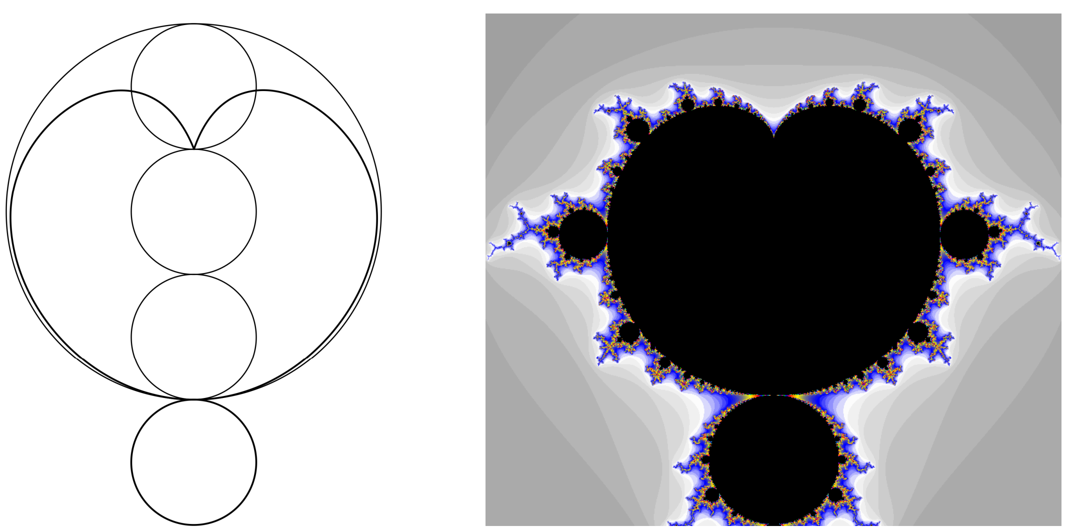

Fig. 5. The diagram at left shows the construction of the cardioid and circular disc in $\mathcal{M}$ by rolling a circle of radius .25 on another, and then placing one at the extremum

What we see represented at $(-0.75, 0i)$ is the folding of space, where $\mathcal{M}$ depicts how the fabric of spacetime folds back on itself, and initiates a 5-d $\rightarrow$ 4-d transition. This feature precisely mimics the place where the rolling ball rests on the larger surface, in Cartan's analogy for G_2 symmetries. This suggests G_2 is involved with, or describes, the process of eversion or transitions where surface normals and tangents turn inside out. One could describe this as a bounce, but the prior universe is higher-dimensional. In MGT; 4 of the octonions 7 rotation axes (*i,j,k,I,J,K,L*) are fixed in the 5-d precursor to our cosmos, while the remaining 3 axes (*i,j,k*) are fixed in the ball riding on its surface – that we inhabit – which is a quaternionic bubble. The throat at $(-0.75, 0i)$ depicts the transition from non-associative geometry in the early universe, to spaces with familiar properties, in MGT.

This transition correspondence has been explored somewhat; papers by Griffin and Joshi [39], Kricker and Joshi [40], and a book chapter by Joshi [41] detailing how the generalized Mandelbrot Set can reveal and depict transition points in the octonionic quadratic domain, to map where algebraic functions are non-associative, and where they become associative. And every location in $\mathcal{M}$ has its own Julia Set. So $\mathcal{M}$ is a kind of master map, catalog, or table of contents, for a large family of objects. [42] But the Julia Sets for Misiurewicz points are special because they embody fundamental symmetries and archetypal symmetry breaking. So it should not be surprising that our discussion will later visit how one of the lowest-order Misiurewicz points in $\mathcal{M}$ reveals something important about gravity.

Theories of Thermodynamic and Entropic Gravity

Schwarzschild black holes are a good model or testing ground for gravity, because they have no charge or spin so their only attributes are mass and size, making them a purely gravitational object – well almost. It was shown by Hawking that black holes don't only absorb everything that enters them, but also radiate back energy at a characteristic spectrum for a black body of a particular temperature. A black body is something that absorbs all light or other radiation when cold, but glows brightly when heated to a high temperature. So even a Schwarzschild black hole, which has no charge or spin, has a specific mass, size, and temperature. A large black hole's temperature is very low, so it does appear nearly black, but it radiates for a characteristic *T*. Later Unruh found that the observed temperature is not an absolute but varies depending on the observer's acceleration toward, or away from, such a massive black-body object. This fact allows us to find a correspondence between any gravitational acceleration and a temperature that we can then use to forge a link between thermodynamic variables and Einstein's equations or Newton's laws. The discovery by Bekenstein and Hawking that a black hole's surface area and entropy are quantized in Planck area units allows us to put this relation into strict mathematical terms. So these powerful insights set the stage for later developments in thermodynamic and entropic gravity theories. Let us briefly review those developments.

$$S_{BH} = \frac{kA}{4L_P^2} = \frac{Akc^3}{4G\hbar}$$

The Bekenstein-Hawking equation shows that
entropy is quantized at four times the Planck area

A 1995 paper by Ted Jacobson derives the Einstein equation from the proportionality of entropy and horizon area [43], using a variation of the first law of black hole thermo-dynamics or Clausius' formula. The entropy-area relation above was formulated for a dimensional reduction at a true holographic boundary. This happens at the radius of gravitation, which for a Schwarzschild object is its event horizon, the Schwarzschild radius:

$$r_G = r_S = \frac{2GM}{c^2}$$

Jacobson proposed a broader definition, and said that we should consider the implied relation of Bekenstein-Hawking proportionality and Unruh's theorem to gravity when we look at entropy for any local Rindler horizon. Rindler devised coordinates to simplify calculations for frames of reference near but not at the event horizon of a black hole. But Jacobson asserted that we could use that framework as a tool to examine gravity as a consequence of thermodynamic relations, in a broader context (i.e. – in the vicinity of a gravitational source). He used the entropy-horizon area proportionality and employed the relation $\partial Q = TdS$ (from Clausius) to represent the impinging energy flux, temperature, and change in entropy to show that Einstein's equation emerges, when we use Unruh's theorem to let T describe the gravitational acceleration of an observer just behind the screen.

Then early in 2010; Erik Verlinde argued along similar lines [44], but asserted that the source of gravitation is not thermodynamics as such, but holography and/or entropy. In the related work of Padmanabhan [45] also early in 2010; we find a detailed treatment of both thermodynamic and entropic models of gravity. In both of these cases, and with Jacobson; gravity is treated as a residual or remainder, rather than a fundamental force. In Jacobson's relation $\partial Q = TdS$ the left hand side represents the energy flux at the horizon, and it is implied that gravity represents a sink for energy, where the universe is the source, as with a black hole. The expansion of the universe, the cosmic background energy, and the quantum vacuum energy, all push inward toward the center of any mass – in such a scenario – and they therefore constitute the source of the gravitational force. This is reminiscent of the 'induced gravity' proposed by Sakharov [46,47] in 1967, where microcausal quantum interactions can replace curvature as the source of gravitation.

But in other theories of emergent gravity, including the later work of Verlinde [48]; the emphasis is less on where the energy comes from and more on where information goes. We find that entropy is not just a thermodynamic reality, because it makes an appearance in information theory and as quantum entanglement. But since the author will treat this in later work; and to explain entropic gravity more simply, we will leave further elaboration aside for now.

The principles of entropy are a remarkable way to make sense of gravity, and this is rather surprising. Using the above insights and following the method of Verlinde, Newton's universal law of gravitation can be straightforwardly derived.

$$N = \frac{A}{l_{Pl}^2} \quad \text{where} \quad l_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \quad \text{yields} \quad N = \frac{Ac^3}{\hbar G} \quad \text{using} \quad E = \frac{1}{2}NkT$$

N information bits at horizon degrees of freedom the equipartition theorem

$$\text{and } E = mc^2 \quad T = \frac{\hbar a}{2\pi ck} \quad A = 4\pi r^2 \text{ and } F = ma \text{ we obtain } F = G\frac{mM}{r^2}$$

we use equivalency and Unruh for T keeping in mind Newton's law of gravitation

This gives some perspective on the power of entropic gravity, and the ability of this metaphor to connect disparate subjects in Physics. But it also tells us something about natural law, that there should be such a connection. Furthermore; it could tell us that the universe as a whole is running down, or becoming depleted of energy over time. This may favor the model of a cold dark end to the cosmos. It may in fact be a consequence of entropic theories of gravity that they predict the heat death of the universe. But that brings us to the possible connection of gravity with condensation. The thermodynamics of an expanding universe predicts continued cooling, where as things cool they condense. When cooled to near absolute zero, some gases will undergo Bose-Einstein condensation, and this process is found to be another good model for gravity. So this is what we will explore next.

Condensation at a Misiurewicz Point

Recent papers by Dvali and Gomez [49] have focused a larger segment of the Physics community on the notion that Bose-Einstein condensation is a faithful model for gravity, because the event horizon of a Schwarzschild black hole is like the quantum critical point of Bose-Einstein condensation. In their model; the Schwarzschild object is a condensate of gravitons, where the graviton occupation number N denotes quantum mechanical degrees of freedom at the horizon. The Classical degrees of freedom diminish as the dimensions are reduced at the horizon, but the Quantum degrees of freedom go to a large N, for the gravitons condensing.

When a Bose-Einstein condensate is formed; atoms or particles that demanded their own space become gregarious, and thus willing to share a space instead. So at the quantum critical point;

there is an inward force toward a bounded space with a smaller volume, having a paradoxically large interior, upon BEC formation. The inward force of condensation thus mimics the action of gravity. This echoes the earlier thoughts of Sakharov, who wrote that condensed matter transitions could mimic gravity's action. But this feature is also seen in the work on 'Emergent Relativity,' of Laughlin [50] and earlier with Chapline and Santiago [51], where details of the analogy with BEC formation can be found. However; the observation by Dvali and Gomez, that it is a condensate of gravitons which comprises a Schwarzschild black hole, opens a new chapter for the theoretical Physics community, by showing us the phenomenology of gravitons.

There is a Misiurewicz point in $\mathcal{M}$ designated $M_{3,1}$ which vividly illustrates the formation of a condensate and the event horizon of a black hole. I have been studying this location for more than 20 years. I thought it quite curious that gravity should be connected with something that takes place only at a very low temperature, so I was reluctant to point out what the Mandelbrot Set told me – that there is a connection between event horizons and BEC formation. But explorations in analog gravity which feature a connection between BEC quantum critical points and black hole event horizons have a rich history [52] including recent experiments by Jeff Steinhauer [53]. The location in $\mathcal{M}$ that mimics this analogy is a location along the real axis, partway down the tail, spike, or main antenna, the Misiurewicz point $M_{3,1}$. This point at about (-1.543689,0i) is significant because it is one of the lowest order Misiurewicz points in $\mathcal{M}$, which makes it tractable analytically, so we can obtain an exact answer for its location and a numerical value to any desired level of precision.

But it has a deeper significance, because it reveals archetypical information about the Misiurewicz points in $\mathcal{M}$. Julia Sets for the Misiurewicz points of multi-armed features are almost perfectly symmetric at the center, and less so on the fringes, as noted by Tan Lei [54]. But the feature at (-1.543689,0i) has one point of entry and one point of exit, and it is anti-symmetric, so it represents purely broken symmetry. It is thus exceptional among the Misiurewicz points, but this makes it a good indicator of what makes all those locations important to Physics.

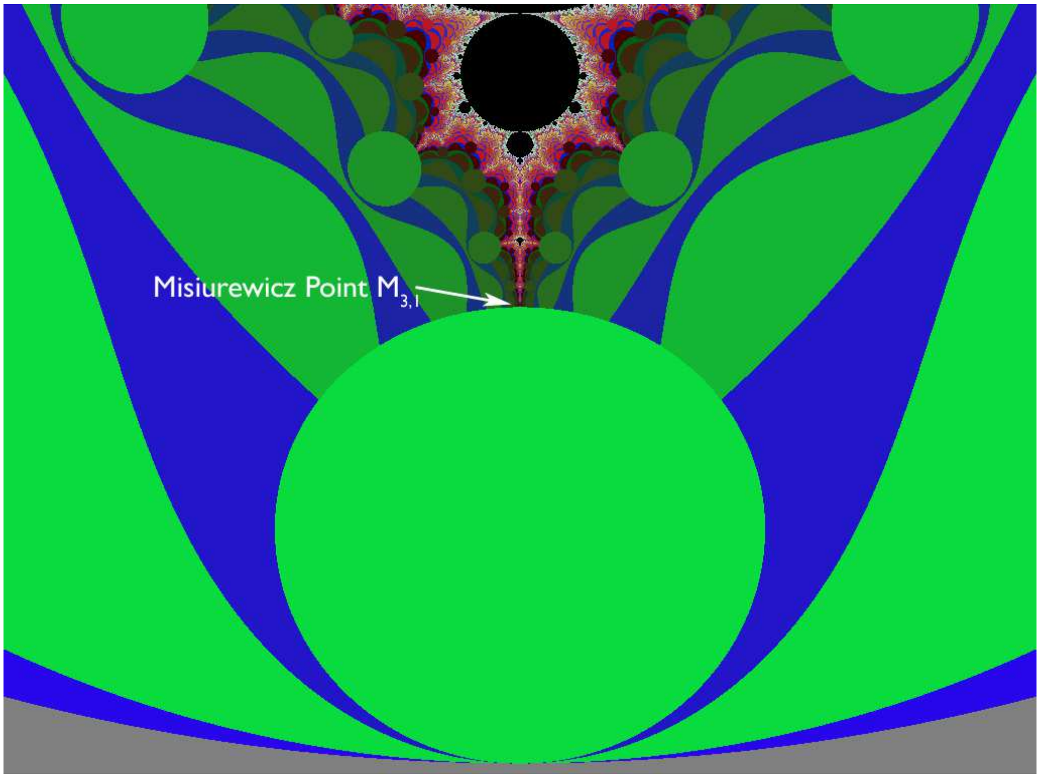

Fig. 6. Base of Mandelbrot Butterfly as seen in Fig. 1
annotated to show the location of Misiurewicz point $M_{3,1}$

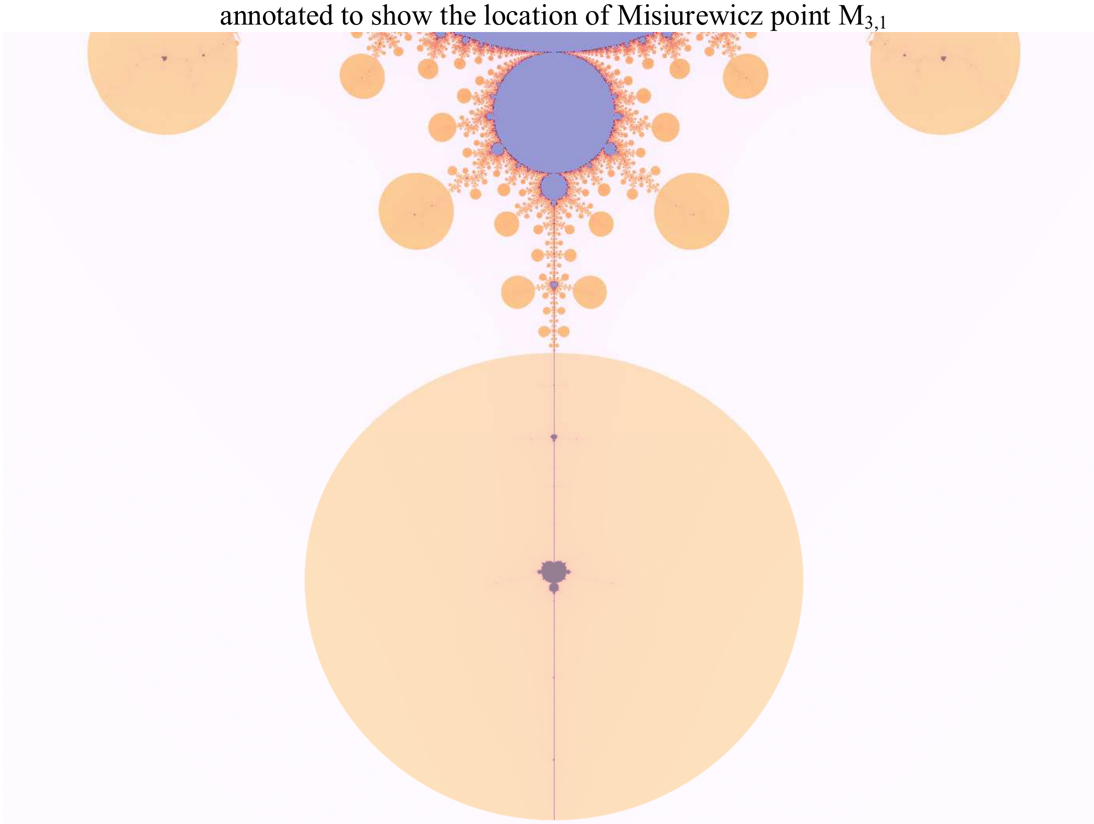

Fig. 7. The same region of the Mandelbrot Butterfly in
an image by Paul Bourke shows mini-$\mathcal{M}$s inside discs

During formation of a condensate, individual particles or atoms are seen to coalesce into a single entity, so that many forms take up the space of one or a few, and share attributes as a unified whole – rather than a collection of parts. The discs that line the entire periphery of the Butterfly figure and lie beyond the primary Misiurewicz points in $\mathcal{M}$ are thought by this author to represent basins of attraction having a specific period, associated with the mini-$\mathcal{M}$ at the center of each disc. The Mandelbrot Butterfly is obtained simply by coloring in areas where the iterand magnitude diminishes over 3 iterations. But this algorithm can easily be changed so the lower-order solutions are suppressed, allowing us to see beneath those structures and behind the boundaries.

This is like being able to peer behind an event horizon, or look at what happens while separated atoms coalesce into one blob in a condensate. One can zoom in to any desired degree to examine any point in the process of crossing, or animate the process to see how it unfolds over time. Not only can one animate a zoom, into areas in $\mathcal{M}$ of interest, or pan across regions with interesting dynamics. One can also examine the Julia Sets using the algorithmic adaptation for the Butterfly figure, and animate the coalescence process itself – which I have done. In Fig. 8 below; I show how suppressing the lowest-order solution allows us to strip away the layer with the largest disc at the base of the Mandelbrot Butterfly figure. The image on the left has all its layers, while on the right we see what appears in unseen formations when the largest disc is removed.

Fig. 8. The simulation of horizons/condensation in $\mathcal{M}$ allows us to zoom in, and to peer behind layers of form. We examine the Mandelbrot Butterfly near $M_{3,1}$, magnified in the images below, and with one layer removed in images on the right.

Another compelling feature of this location is that it is the band merging point of the bifurcation diagram for $c \to c^2 + c$, which is the Mandelbrot formula, iterated over the reals. As I noted earlier; the bifurcation diagram splits everywhere the boundary of $\mathcal{M}$ folds back on itself – along the real axis. But at $M_{3,1}$ we see a remarkable reversal, where trajectories from every branch converge in a single location. This feature is illustrated in Fig. 9 below, which is an adjusted representation of Fig. 4, annotated to highlight where the spot in $\mathcal{M}$ and the band merging point in the bifurcation diagram line up. This is the spot that coincides with the Schwarzschild event horizon and the quantum critical point, in the above diagrams and analogies.

I learned of the band merging feature from Michel Planat, when I was seeking to find out how to calculate the exact location of $M_{3,1}$ [55], and he pointed me to Peitgen and Richter [56], who cite Grossman and Thomae [57]. But it is a very special spot indeed. It satisfies the equation $((c^2+c)^2+c)^2+c = (c^2+c)^2+c$, which is three iterations of the Mandelbrot formula on the left hand side and two on the right. Our location of interest is where the split trajectories all come together, as pointed out below. In my mind; it only makes sense that trajectories would come together at an event horizon or critical surface. This appears to automatically predict maximal complexity at the horizon, as seen in the recent work by Susskind [58]. This correspondence deepens the analogy with physical phenomena at $M_{3,1}$ and increases the number of ways that $\mathcal{M}$ can help us understand gravity.

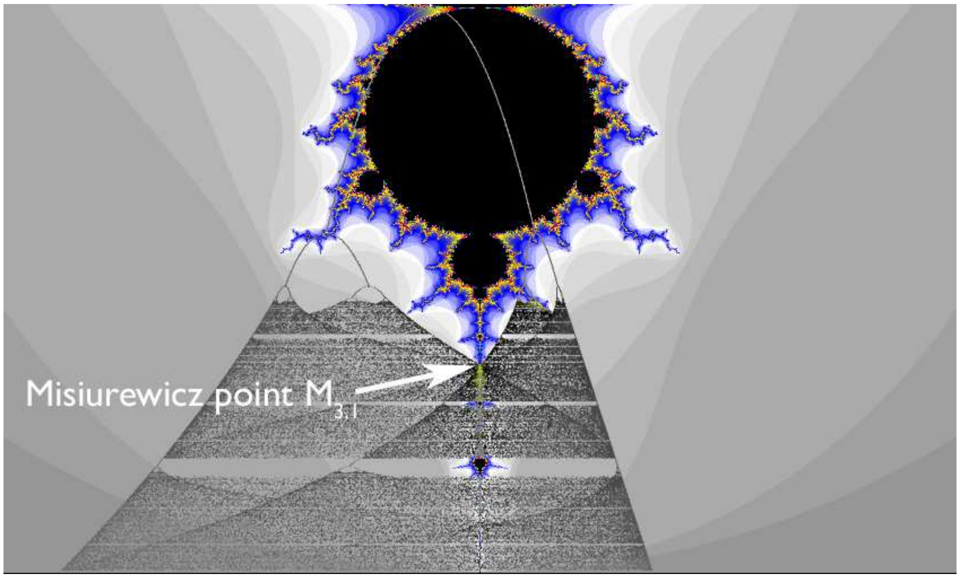

Fig. 9. The band merging point of the bifurcation diagram is at the Misiurewicz point $M_{3,1}$ and mini-$\mathcal{M}$s are in the band gaps.

Concluding Remarks

When I set out to prepare for GR21; I knew fairly well what my own research was telling me, and I was pleased to be able to present so much of it on my poster. But I was not nearly as prepared to learn how much my work overlaps with that of others. A lot of ground has been covered in modified and quantum gravity theories, by my fellow researchers, and I am pleased about the non-trivial area of overlap. From the 5-d $\rightarrow$ 4-d transition seen in DGP gravity and elsewhere to BH-BEC analogies seen in a variety of theories and experiments; I found there are explorers covering almost every part of the theoretical and phenomenological territory $\mathcal{M}$ had showed me.

So I learned about the significance of what I am working on by comparing it to what others are doing. One might think that a theory of gravity spawned from the Mandelbrot Set by someone with a fertile imagination would have little resemblance to Modern Physics. But in theoretical Physics, there are a lot of people with fertile imaginations. Lo and behold; the mathematical or geometric factors that shape $\mathcal{M}$ creep into Physics in other ways, as well. Even if the Mandelbrot Set is not a causative agent for the laws of nature, a common basis might make both what they are, so the resemblance is still natural. But I suspect that a deeper connection exists, and that the research of others is running into some of the same natural limits in the form of geometrical and procedural constraints I have already begun to explore studying $\mathcal{M}$.

It turns out the Mandelbrot Set can tell us quite a lot about gravity, and the extent of commonality I see of my research based on $\mathcal{M}$ with the work of more mainstream researchers helps put what I've learned in perspective. In the search for unifying theories of Physics; it is perfectly natural that symmetry and the symmetry groups get a lot of attention. But if the Lie groups arise in accord with the number types (normed division algebras), there is common ground with the Mandelbrot Set – at least if we include generalizations of $\mathcal{M}$ into the quaternions and octonions. And $\mathcal{M}$ gives us a missing piece, by showing us the interplay between preserved and broken symmetries.

While objects like E_8 are purely symmetrical, the Mandelbrot Set is anti-symmetric, along the real axis. So when well-informed authors like Sabine Hossenfelder decry [59] the preoccupation of people in Physics for beautiful Maths; I point out that this is why we must look at the ugly Maths too. In fact; I think the problem is not one where physicists are too immured in Mathematics, but rather one where they deliberately ignore some of the general cases mathematicians tend to focus on, in favor of learning a few specific examples that are useful in basic Physics. Mikhail Kovalyov pointed out at FFP10 [60] that physicists tend to linearize overmuch. They make limiting assumptions in order to obtain a simple linear equation they can plug numbers into, and use that to formulate hypotheses, despite the fact that actual non-linearities arise in nature – which don't get incorporated into these simplified expressions.

In that context, it is not surprising the Mandelbrot Set would escape notice, or might not be taken seriously as a source of Physics insights. I was not eager to share the insights I had years ago, before making serious efforts to learn various topics in Physics and Mathematics relevant to pursuing or demonstrating any such connection. But more recently; I have felt a need to keep on top of developments in Physics – especially in modified and/or quantum gravity – or risk being 'scooped' by other researchers, for a story I knew well and never told. Indeed; learning about DGP gravity and its successors left me feeling like I now have quite a lot of catching up to do, even though my insights from 30 years ago put me ahead of the pack in some areas. So in trying to follow up on what I presented at and what I learned from GR21, and to prepare for GR22 which I hope to attend; I find myself in good company, with insights that are echoed in the work of top researchers. This provides an incentive to see this line of research through, because the impact of any insights I can offer will be greater, if similar ideas have already cropped up in mainstream research. So it is likely additional follow-ups will appear.

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