Article

On the Codes over the Ring $Z_4 + uZ_4 + vZ_4$ Cyclic, Constacyclic, Quasi-Cyclic Codes, Their Skew Codes, Cyclic DNA & Skew Cyclic DNA Codes

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Abstract

In the paper, the linear codes over the ring $D = Z_4 + uZ_4 + vZ_4, u^2 = u, v^2 = v, uv = vu = 0$ are studied. A Gray map Φ from D^n to Z_4^{3n} are defined. The Gray images of the cyclic, constacyclic and quasi-cyclic codes over D are determined. Especially, cyclic DNA codes over D are introduced. A nontrivial automorphism is given. The skew cyclic, constacyclic and quasi-cyclic are introduced. The Gray images of them are determined. Furthermore, the skew cyclic DNA codes over D are introduced.

Keywords: Cyclic, constacyclic, quasi-cyclic, code, skew code, cyclic DNA.

1. Introduction

Although a lot of research on error correcting codes are concentrated on codes over finite fields, after Hammons et al's paper [20], a great deal of attention has been given to codes over the finite rings. The certain type of codes over many finite rings are studied such as cyclic, constacyclic, quasi-cyclic codes [7,12,13,15,23-27,29,33]. Their algebraic structure, Gray image, dual, rank, self duality properties were investigated. They were characterized. The generators of them were found. Many of good codes were obtained from them.

Some authors generalized the notions of cyclic, constacyclic and quasi-cyclic code. They introduced skew cyclic,skew constacyclic and skew quasi-cyclic codes over many finite rings [3,4,6,9,10,11,14,17,21,32]. The class of these type codes is more bigger than the others. Many good codes were also obtained from them.

DNA computing were started by Leonhard Adleman in 1994 [5]. Some special error correcting codes over some finite fields and finite rings with 4^n elements where $n \in N$ were used for DNA computing applications. The construction of DNA codes have been discussed by several authors in [1,2,16,18,19,22,28,30,31].

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In this paper is organized as follows. In section 2, some knowledges about the finite ring $D = Z_4 + uZ_4 + vZ_4$, where $u^2 = u, v^2 = v, uv = vu = 0$ are given. A new Gray map from D to Z_4^3 is defined. The Gray images of cyclic, constacyclic and quasi-cyclic codes over D are found. A linear code C over D is represented by means of three codes over Z_4. In section 3, the constacyclic codes are investigated. The cyclic codes of odd length over D satisfy reverse and reverse complement properties are studied in section 4. The binary images of cyclic DNA codes over D are determined in section 5. In section 6, a non trivial automorphism is found on D. The skew cyclic,constacyclic and quasi-cyclic codes over D are introduced. The Gray images of them are determined in section 7. The skew cyclic DNA codes over D are introduced in section 8.

2. Preliminares

Let $D = Z_4 + uZ_4 + vZ_4$, where $u^2 = u, v^2 = v, uv = vu = 0$. The ring D can be also viewed as the quotient ring $Z_4[u,v]/\langle u^2 - u, v^2 - v, uv = vu \rangle$.

Let d be any element of D, which can be expressed uniquely as $d = a + ub + vc$, where $a, b, c \in Z_4$. The ring D has the following properties:

* The finite ring D is with 64 elements.
* The units of the ring D are $1, 3, 1+2u, 1+2v, 2u+3, 2v+3, 1+2u+2v, 3+2u+2v$

Let $(D,*)$ be a D -unit group, $(D,*) \cong (Z_4,*) \otimes (Z_4,*) \otimes (Z_4,*)$, where $(Z_4,*)$ is Z_4 -unit group.

*The ring D has 27 ideals. The trivial ideals are

$$\langle 0 \rangle = \{0\}$$
$$\langle 1 \rangle = \langle 3 \rangle = \langle 1+2v \rangle = \ldots = D$$

The ideals with two elements are

$$\langle 2u \rangle, \langle 2v \rangle, \langle 2+2u+2v \rangle$$

The ideals with four elements are

$$\langle u \rangle = \langle 3u \rangle, \langle v \rangle = \langle 3v \rangle, \langle 2+2u \rangle, \langle 2+2v \rangle, \langle 2u+2v \rangle, \langle 1+3u+3v \rangle$$

The ideals with eight elements are

$$\langle 2 \rangle, \langle 2u+v \rangle = \langle 2u+3v \rangle, \langle 2+2u+v \rangle = \langle 2+2u+3v \rangle, \langle 1+3u+v \rangle = \langle 3+u+3v \rangle,$$
$$\langle 3+3u+v \rangle = \langle 1+u+3v \rangle, \langle 2+3u+2v \rangle = \langle 2+u+2v \rangle, \langle u+2v \rangle = \langle 3u+2v \rangle$$

The ideals with sixteen elements are

$$\langle 2+u\rangle=\langle 2+3u\rangle,\langle 3+u\rangle=\langle 1+3u\rangle=\langle 3+u+2v\rangle=\langle 1+3u+2v\rangle,$$
$$\langle 3u+3v\rangle=\langle u+v\rangle=\langle 3u+v\rangle=\langle u+3v\rangle,\langle 2+3v\rangle=\langle 2+v\rangle$$
$$\langle 3+2u+v\rangle=\langle 1+3v\rangle=\langle 3+v\rangle=\langle 1+2u+3v\rangle,\langle 1+u+v\rangle=\langle 3+3u+3v\rangle$$

The ideals with thirty two elements are

$$\langle 3+3u\rangle=\langle 1+u\rangle=\langle 1+u+2v\rangle=\langle 3+3u+2v\rangle$$
$$\langle 1+2u+v\rangle=\langle 1+v\rangle=\langle 3+3v\rangle=\langle 3+2u+3v\rangle$$
$$\langle 2+3u+v\rangle=\langle 2+u+3v\rangle=\langle 2+3u+3v\rangle=\langle 2+u+v\rangle$$

* D is a principal ideal ring
* D is not a finite chain ring.

A code of length n over D is a subset of D^n. C is a linear iff C is an D submodule of D^n. The elements of the code (linear code) is called codewords.

Let $\sigma,\sigma_\lambda,\zeta$ be maps from D^n to D^n given by

$$\sigma(\alpha_1,...,\alpha_n)=(\alpha_n,\alpha_0,...,\alpha_{n-1})$$
$$\sigma_\lambda(\alpha_1,...,\alpha_n)=(\lambda\alpha_n,\alpha_0,...,\alpha_{n-1})$$
$$\zeta(\alpha_1,...,\alpha_n)=(-\alpha_{n-1},\alpha_0...,\alpha_{n-2})$$

where λ is a unit in D. Let C be a linear code of length n over D. Then C is said to be cyclic if $\sigma(C)=C$, λ-constacyclic if $\sigma_\lambda(C)=C$, negacyclic, if $\zeta(C)=C$.

Let $a\in Z_4^{3n}$ with $a=(a_0,a_1,...,a_{3n-1})=(a^{(0)}\mid a^{(1)}\mid a^{(2)})$, $a^{(i)}\in Z_4^n$ for $i=0,1,2$. Let e be a map from Z_4^{3n} to Z_4^{3n} given by $\varphi(a)=(\sigma(a^{(0)})|\sigma(a^{(1)})|\sigma(a^{(2)}))$, where σ is a cyclic shift from Z_4^n to Z_4^n given by $\sigma(a^{(i)})=((a^{(i,n-1)}),(a^{(i,0)}),(a^{(i,1)}),...,(a^{(i,n-2)}))$ for every $a^{(i)}=(a^{(i,0)},...,a^{(i,n-1)})$, where $a^{(i,j)}\in Z_4$, $j=0,1,...,n-1$. A code of length $3n$ over Z_4 is said to be a quasi-cyclic code of index 3 if $\varphi(C)=C$.

The Lee weights of $0,1,2,3\in Z_4$ are defined by $w_L(0)=0,w_L(1)=w_L(3)=1,w_L(2)=2$.

Let $d=a+ub+vc$ be an element of D, then Lee weight of d is defined as $w_L(d)=w_L(a,a+b,a+c)$, where $a,b,c\in Z_4$. The Lee weight of a vector $c=(c_0,...,c_1)\in D^n$ to be the sum of Lee weights its components. For any elements $c_1,c_2\in D^n$, the Lee distance between c_1 and c_2 is given by $d_L(c_1,c_2)=w_L(c_1-c_2)$. The minimum Lee distance of C is defined as $d_L(C)=\min d_L(c,c)$, where for any $c\in C,c\neq c$.

For any $x=(x_0,...,x_{n-1}),y=(y_0,...,y_{n-1})$ the inner product is defined as

$$xy=\sum_{i=0}^{n-1}x_i y_i$$

If $xy=0$, then x and y are said to be orthogonal. Let C be a linear code of length n over D, the dual of C

$$C^{\perp}=\{x\ :\forall y\in C, xy=0\}$$

which is also a linear code over D of length n. A code C is self orthogonal, if $C\subset C^{\perp}$ and self dual, if $C=C^{\perp}$.

We define the Gray map as follows

$$\Phi : D\rightarrow Z_4^3$$
$$a+ub+vc\mapsto (a,a+b,a+c)$$

This map is extended componentwise to

$$\Phi : D^n \rightarrow Z_4^{3n}$$
$$(\alpha_1,...,\alpha_n)=(a_1,...,a_n,a_1+b_1,...,a_n+b_n,a_1+c_1,...,a_n+c_n)$$

where $\alpha_i=a_i+ub_i+vc_i$ with $i=1,2,...,n.$ Φ is a Z_4-module isomorphism.

Theorem 1 *The Gray map Φ is distance preserving map from (D^n, Lee distance) to (Z_4^{3n}, Lee distance).*

Proof Let $z_1=(z_{1,0},...,z_{1,n-1}),z_2=(z_{2,0},...,z_{2,n-1})$ be the elements of D^n, where

$$z_{1,i}=a_{1,i}^0+ua_{1,i}^1+va_{1,i}^2 \text{ and } z_{2,i}=a_{2,i}^0+ua_{2,i}^1+va_{2,i}^2,\ i=0,1,...,n-1.$$

Then

$$z_1-z_2=(z_{1,0}-z_{2,0},...,z_{1,n-1}-z_{2,n-1}) \text{ and } \Phi(z_1-z_2)=\Phi(z_1)-\Phi(z_2)$$

So, $d_L(z_1,z_2)=w_L(z_1-z_2)=w_L(\Phi(z_1-z_2))=w_L(\Phi(z_1)-\Phi(z_2))=d_L(\Phi(z_1),\Phi(z_2)).$

Theorem 2 *If C is self orthogonal, so is $\Phi(C)$.*

Proof Let $x_1=a_1+ub_1+vc_1, x_2=a_2+ub_2+vc_2$, where $a_1,b_1,c_1,a_2,b_2,c_2\in Z_4$.
From $x_1x_2=a_1a_2+u(a_1b_2+b_1a_2+b_1b_2)+v(a_1c_2+a_2c_1+c_1c_2)$. Since C is self orthogonal, so we have $a_1a_2=0, a_1b_2+b_1a_2+b_1b_2=0, a_1c_2+a_2c_1+c_1c_2=0.$
From $\Phi(x_1)\Phi(x_2)=(a_1,a_1+b_1,a_1+c_1)(a_2,a_2+b_2,a_2+c_2)=0$. Therefore $\Phi(C)$ is self orthogonal.

Proposition 3 *Let Φ be Gray map from D^n to Z_4^{3n}, let σ be the cyclic shift and let φ be a map as above. Then $\Phi\sigma=\varphi\Phi.$*

Proof Let $a=(a_0,...,a_{n-1})\in D^n$. Let $a_i=a_i^0+ua_i^1+va_i^2$ where $a_i^0,a_i^1,a_i^2\in Z_4$, for $i=0,1,...,n-1$. From definition Φ, we have

$$\Phi(a)=(a_0^0,a_1^0,...,a_{n-1}^0,a_0^0+a_0^1,...,a_{n-1}^0+a_{n-1}^1,a_0^0+a_0^2,...,a_{n-1}^0+a_{n-1}^2) .$$

By applying φ, we have

$$\varphi(\Phi(a))=(a_{n-1}^0,a_0^0,...,a_{n-2}^0,a_{n-1}^0+a_{n-1}^1,...,a_{n-2}^0+a_{n-2}^1,a_{n-1}^0+a_{n-1}^2,...,a_{n-2}^0+a_{n-2}^2).$$

On the other hand, $\sigma(a)=(a_{n-1},a_0,...,a_{n-2})$. If we apply Φ, we have

$$\Phi(\sigma(a))=(a_{n-1}^0,a_0^0,...,a_{n-2}^0,a_{n-1}^0+a_{n-1}^1,...,a_{n-2}^0+a_{n-2}^1,a_{n-1}^0+a_{n-1}^2,...,a_{n-2}^0+a_{n-2}^2)$$

Theorem 4 *Let σ and φ be in section 2. A code C of length n over D is a cyclic code iff $\Phi(C)$ is a quasi-cyclic code of index 3 over Z_4 with length $3n$.*

Proof Let C be a cyclic code. Then $\sigma(C)=C$. If we apply Φ, we have $\Phi(\sigma(C))=\Phi(C)$. By using Proposition 3, $\Phi(\sigma(C))=\varphi(\Phi(C))=\Phi(C)$. Hence, $\Phi(C)$ is a quasi- cyclic code of index 3.

For the other part, if $\Phi(C)$ is a quasi-cyclic code of index 3 , then we have $\varphi(\Phi(C))=\Phi(C)$. By using Proposition 3, we have $\varphi(\Phi(C))=\Phi(\sigma(C))=\Phi(C)$. Since Φ is injective, we have $\sigma(C)=C$.

Let A_1,A_2,A_3 be linear codes.

$$A_1\otimes A_2\otimes A_3=\{(a_1,a_2,a_3):\ a_i\in A_i, i=1,2,3\}$$

and

$$A_1\oplus A_2\oplus A_3=\{a_1+a_2+a_3\ :\ a_i\in A_i, i=1,2,3\}$$

Definition 5 *Let C be a linear code of length n over D . Define*

$$C_1=\{a\ :\exists\, b,c\in Z_4^n, a+ub+vc\in C\}$$

$$C_2=\{a+b\ :\exists\, c\in Z_4^n, a+ub+vc\in C\}$$

$$C_3=\{a+c\ :\exists\, b\in Z_4^n, a+ub+vc\in C\}$$

where C_1,C_2 and C_3 are linear codes over Z_4 of length n .

Theorem 6 *Let C be a linear code of length n over D. Then $\Phi(C)=C_1\otimes C_2\otimes C_3$ and $|C|=|C_1||C_2||C_3|$.*

Corollary 7 *If $\Phi(C)=C_1\otimes C_2\otimes C_3$, then $C=(1-u-v)C_1\oplus uC_2\oplus vC_3$.*

Theorem 8 *Let* $C=(1-u-v)C_1 \oplus uC_2 \oplus vC_3$ *be a linear code of length* n *over* D. *Then* C *is a cyclic code over* D *if and only if* C_1, C_2 *and* C_3 *are all cyclic codes over* Z_4.

Proof It is proved that as in Proof of Theorem in [13].

Lemma 9 [8,15] *Let* n *be an odd positive integer and* $x^n-1=\prod_{i=1}^{r} f_i(x)$ *be the unique factorization of* x^n-1, *where* $f_1(x),...,f_r(x)$ *are basic irreducible polynomials over* Z_4.
Let C *be a cyclic code of odd length* n *over* Z_4, *then*

$$C=\left(f_0(x),2f_1(x)\right)=\left(f_0(x)+2f_1(x)\right)$$

where $f_0(x)$ *and* $f_1(x)$ *are monic factors of* x^n-1 *and* $f_1(x) \mid f_0(x)$.

If C *is a linear code of any length* n *over* Z_4, *then there exist monic polynomials* $f(x), g(x), p(x) \in Z_4$ *such that*

$$C=\left(f(x)+2p(x), 2g(x)\right)$$

where $g(x) \mid f(x) \mid x^n-1$, $g(x) \mid p(x)[x^n-1/f(x)]$ *and* $|C|=2^{2n-\deg f(x)-\deg g(x)}$.

Theorem 11 *Let* $C=(1-u-v)C_1 \oplus uC_2 \oplus vC_3$ *be a cyclic code of any length* n *over* D *there exist* $f_i(x), g_i(x), p_i(x) \in Z_4[x]$ *for* $i=1,2,3$ *such that* $C_i=\left(f_i(x)+2p_i(x), 2g_i(x)\right)$, *then*

$$C=((1-u-v)f_1(x)+uf_2(x)+vf_3(x)+2[(1-u-v)p_1(x)+up_2(x)+vp_3(x)],$$
$$2[(1-u-v)g_1(x)+ug_2(x)+vg_3(x)]).$$

If n *is odd, then* $C=((1-u-v)(f_1(x)+2g_1(x))+u(f_2(x)+2g_2(x))+v(f_3(x)+2g_3(x)))$.

Proof It is proved that as in Proof of Theorem in [26].

Definition 12 *A subset* C *of* D^n *is called a quasi-cyclic code of length* $n=sl$ *with index* l *if* C *is satisfies the following conditions*
i) C *is a submodule of* D^n
ii) if $e=(e_{0,0},...,e_{0,l-1},e_{1,0},...,e_{1,l-1},...,e_{s-1,0},...,e_{s-1,l-1}) \in C$, *then*
$T_{s,l}(e)=(e_{s-1,0,...,}e_{s-1,l-1},e_{0,0},...,e_{0,l-1},...,e_{s-2,0},...,e_{s-2,l-1}) \in C$.

Definition 13 *Let* $a \in Z_4^{3n}$ *with* $a=(a_0,a_1,...,a_{3n-1})=\left(a^{(0)}\middle|a^{(1)}\middle|a^{(2)}\right)$, $a^{(i)} \in Z_4^n$, *for* $i=0,1,2$. *Let* Γ *be a map from* Z_4^{3n} *to* Z_4^{3n} *given by*

$$\Gamma(a)=\left(\mu\left(a^{(0)}\right)\middle|\mu\left(a^{(1)}\right)\middle|\mu\left(a^{(2)}\right)\right)$$

where μ *is the map from* Z_4^n *to* Z_4^n *given by*

$$\mu\left(a^{(i)}\right)=((a^{(i,s-1)}),(a^{(i,0)}),...,(a^{(i,s-2)}))$$

for every $a^{(i)}=\left(a^{(i,0)},...,a^{(i,s-1)}\right)$ *where* $a^{(i,j)} \in Z_4^l$, $j=0,1,...,s-1$ *and* $n=sl$. *A code of length*

$3n$ *over* Z_4 *is said to be* l*-quasi cyclic code of index* 3 *if* $\Gamma(C) = C$.

Proposition 14 *Let* $T_{s,l}$ *be the quasi-cyclic shift on* D. *Then* $\Phi T_{s,l} = \Gamma\Phi$ *where* Γ *is as above.*

Proof Let $e = (e_{0,0},\ldots,e_{0,l-1},e_{1,0},\ldots,e_{1,l-1},\ldots,e_{s-1,0},\ldots,e_{s-1,l-1})$ with $e_{i,j} = a_{i,j} + ub_{i,j} + vc_{i,j}$, where $i = 0,1,\ldots,s-1$ and $j = 0,1,\ldots,l-1$. We have

$$T_{s,l}(e) = (e_{s-1,0,\ldots,}e_{s-1,l-1},e_{0,0},\ldots,e_{0,l-1},\ldots,e_{s-2,0},\ldots,e_{s-2,l-1}).$$

If we apply Φ, we have

$$\Phi(T_{s,l}(e)) = (a_{s-1,0},\ldots,a_{s-1,l-1},\ldots,a_{s-2,0},\ldots,a_{s-2,l-1},a_{s-1,0} + b_{s-1,0},\ldots,a_{s-2,l-1} + b_{s-2,l-1},a_{s-1,0} + c_{s-1,0},\ldots,a_{s-2,l-1} + c_{s-2,l-1}).$$

On the other hand,

$$\Gamma(\Phi(e)) = (a_{s-1,0},\ldots,a_{s-1,l-1},\ldots,a_{s-2,0},\ldots,a_{s-2,l-1},a_{s-1,0} + b_{s-1,0},\ldots,a_{s-2,l-1} + b_{s-2,l-1}, a_{s-1,0} + c_{s-1,0},\ldots,a_{s-2,l-1} + c_{s-2,l-1}).$$

So, we have $\Phi T_{s,l} = \Gamma\Phi.$

Theorem 15 *The Gray image of a quasi-cyclic code over* D *of length n with index* l *is a* l*-quasi-cyclic code of index* 3 *over* Z_4 *with length* $3n$.

Proof Let C be a quasi-cyclic code over D of length n with index l. So $T_{s,l}(C) = C$. By applying Φ, we have $\Phi(T_{s,l}(C)) = \Phi(C)$. By using Proposition 14, $\Phi(T_{s,l}(C)) = \Gamma(\Phi(C)) = \Phi(C)$. Hence, $\Phi(C)$ is a l-quasi- cyclic code of index 3 over Z_4 with length $3n$.

3. Constacyclic Codes over *D*

We investigate λ -cyclic codes over D, where λ is unit. The ring D has got eight units. They are $1,3,1+2u,1+2v,3+2u,3+2v,1+2u+2v,3+2u+2v$.

Theorem 16 *Let* $C = (1-u-v)C_1 \oplus uC_2 \oplus vC_3$ *be a linear code of length* n *over* D. *Then* C *is* $(1+2v)$*-cyclic code over* D *if and only if* C_1, C_2 *are cyclic codes and* C_3 *is a negacyclic code of length* n *over* Z_4.

Proof Let $\alpha = (\alpha_0,\ldots,\alpha_{n-1}) \in C \subset D^n$, where $\alpha_i = (1-u-v)a_i + ub_i + vc_i, a_i, b_i, c_i \in Z_4, 0 \le i \le n-1$. Then $s_1 = (a_0,\ldots,a_{n-1}), t_1 = (b_0,\ldots,b_{n-1}), r_1 = (c_0,\ldots,c_{n-1}) \in Z_4^n$. By the definition of the $(1+2v)$-cyclic shift ϑ, we have $\vartheta(\alpha) = ((1-u-v)a_{n-1} + ub_{n-1} - vc_{n-1},\alpha_0,\ldots,\alpha_{n-2})$.
Then $\sigma(s_1) = (a_{n-1},a_0,\ldots,a_{n-2}), \sigma(t_1) = (b_{n-1},b_0,\ldots,b_{n-2})$ and $\zeta(r_1) = (-c_{n-1},c_0,\ldots,c_{n-2})$. That means of C is $(1+2v)$-cyclic code of length n over D, then C_1, C_2 are cyclic codes and C_3 is a negacyclic code of length n over Z_4. Other part is seen easily.

Similarly, it can be obtain the following

C	C_1	C_2	C_3
$(1+2u)$-cyclic	cyclic	negacyclic	cyclic
$(3+2u)$-cyclic	negacyclic	cyclic	negacyclic
$(3+2v)$-cyclic	negacyclic	negacyclic	cyclic
$(1+2u+2v)$-cyclic	cyclic	negacyclic	negacyclic
$(3+2u+2v)$-cyclic	negacyclic	cyclic	cyclic

4. The reverse and reverse complement codes over D

In this section, we study cyclic codes of odd length over D satisfy reverse and reverse complement properties.

The elements $0,1,2,3$ of Z_4 are in one to one correspondence with the nucleotide DNA bases A,T,C,G such that $0 \to A, 1 \to T, 2 \to C$ and $3 \to G$. The Watson Crick Complement is given by $\overline{A} = T, \overline{T} = A, \overline{G} = C, \overline{C} = G$.

Since the ring D is cardinality 4^3, then we give a one to one correspondence between the elements of D and the 64 codons over the alphabet $\{A,T,G,C\}^3$ by using the Gray map. For example

Elements	Gray image	Codons
0	$(0,0,0)$	AAA
1	$(1,1,1)$	TTT
2	$(2,2,2)$	CCC
3	$(3,3,3)$	GGG
u	$(0,1,0)$	ATA
$1+u$	$(1,2,1)$	TCT
$\vdots$	$\vdots$	$\vdots$

The codons satisfy the Watson Crick Complement.

Definition 17 *For $x = (x_0, x_1, ..., x_{n-1}) \in D^n$, the vector $(x_{n-1}, x_{n-2}, ..., x_1, x_0)$ is called the reverse of x and is denoted by x^r. A linear code C of length n over D is said to be reversible if $x^r \in C$ for every $x \in C$.*

For $x=(x_0,x_1,\ldots,x_{n-1})\in D^n$, *the vector* $(\overline{x_0},\overline{x_1},\ldots,\overline{x_{n-1}})$ *is called the complement of* x *and is denoted by* x^c. *A linear code* C *of length* n *over* D *is said to be complement if* $x^c\in C$ *for every* $x\in C$.
For $x=(x_0,x_1,\ldots,x_{n-1})\in D^n$, *the vector* $(\overline{x_{n-1}},\overline{x_{n-2}},\ldots,\overline{x_1},\overline{x_0})$ *is called the reversible complement of* x *and is denoted by* x^{rc}. *A linear code* C *of length* n *over* D *is said to be reversible complement if* $x^{rc}\in C$ *for every* $x\in C$.

Definition 18 *Let* $f(x)=a_0+a_1x+\ldots+a_rx^r$ *with* $a_r\neq 0$ *be polynomial. The reciprocal of* $f(x)$ *is defined as* $f^*(x)=x^r f(\frac{1}{x})$. *It is easy to see that* $\deg f^*(x)\leq \deg f(x)$ *and if* $a_0\neq 0$, *then* $\deg f^*(x)=\deg f(x)$. $f(x)$ *is called a self reciprocal polynomial if there is a constant* m *such that* $f^*(x)=mf(x)$.

Lemma 19 *Let* $f(x),g(x)$ *be polynomials in* $R[x]$. *Suppose* $\deg f(x)-\deg g(x)=m$ *then,*
i) $(f(x)g(x))^*=f^*(x)g^*(x)$
ii) $(f(x)+g(x))^*=f^*(x)+x^m g^*(x)$

Theorem 20 *Let* $C=(1-u-v)C_1\oplus uC_2\oplus vC_3$ *be a cyclic code of odd length over* D. *Then* C *is reversible code if and only if* C_1,C_2 *and* C_3 *are reversible codes over* Z_4.

Proof Let C_1,C_2 and C_3 be reversible codes. For any $b\in C, b=(1-u-v)b_1+ub_2+vb_3$, where $b_i\in C_i$, for $1\leq i\leq 3$. Since C_1,C_2 and C_3 are reversible, $b_1^r\in C_1, b_2^r\in C_2$ and $b_3^r\in C_3$. So, $b^r=(1-u-v)b_1^r+ub_2^r+vb_3^r\in C$. Hence C is reversible code.

On the other hand, let C be a reversible code over D. So for any $b=(1-u-v)b_1+ub_2+vb_3$, where $b_1\in C_1, b_2\in C_2$ and $b_3\in C_3$, we get $b^r=(1-u-v)b_1^r+ub_2^r+vb_3^r\in C$.
Let $b^r=(1-u-v)b_1^r+ub_2^r+vb_3^r=(1-u-v)s_1+us_2+vs_3$, where $s_1\in C_1, s_2\in C_2$ and $s_3\in C_3$. Therefore C_1,C_2 and C_3 are reversible codes over Z_4.

Lemma 21 *For any* $c\in D$, *we have* $c+\overline{c}=3+3u+3v$.

Lemma 22 *For any* $a\in D$, *we have* $\overline{a}+3\overline{0}=3a$.

Theorem 23 *Let* $C=(1-u-v)C_1\oplus uC_2\oplus vC_3$ *be a cyclic code of odd length over* D. *Then* C *is reversible complement over* D *iff* C *is reversible over* D *and* $(\overline{0},\overline{0},\ldots,\overline{0})\in C$.

Proof Since C is reversible complement, for any $c=(c_0,c_1,\ldots,c_{n-1})\in C, c^{rc}=(\overline{c_{n-1}},\overline{c_{n-2}},\ldots,\overline{c_0})\in C$

Since C is a linear code, so $(0,0,...,0) \in C$. Since C is reversible complement, so $(\bar{0},\bar{0},...,\bar{0}) \in C$. By using Lemma 21, we get

$$3c^r = 3(c_{n-1}, c_{n-2},...,c_0) = (\bar{c}_{n-1}, \bar{c}_{n-2},...,\bar{c}_0) + 3(\bar{0},\bar{0},...,\bar{0}) \in C$$

Hence for any $c \in C$, we have $c^r \in C$.

On the other hand, let C be reversible code over D. So, for any $c = (c_0, c_1,...,c_{n-1}) \in C$, then $c^r = (c_{n-1}, c_{n-2},...,c_0) \in C$. For any $c \in C$,

$$c^{rc} = (\bar{c}_{n-1}, \bar{c}_{n-2},...,\bar{c}_0) = 3(c_{n-1}, c_{n-2},...,c_0) + (\bar{0},\bar{0},...,\bar{0}) \in C$$

So, C is reversible complement code over D.

Theorem 24 *Let S_1 and S_2 be two reversible complement cyclic codes of length n over D. Then $S_1 + S_2$ and $S_1 \cap S_2$ are reversible complement cyclic codes.*

Proof Let $d_1 = (c_0, c_1,...,c_{n-1}) \in S_1, d_2 = (c_0^1, c_1^1,...,c_{n-1}^1) \in S_2$. Then,

$$(d_1 + d_2)^{rc} = \left(\overline{(c_{n-1} + c_{n-1}^1)},...,\overline{(c_1 + c_1^1)}, \overline{(c_0 + c_0^1)}\right)$$

By using $\overline{a+b} = \bar{a} + \bar{b} - 3(1+u)(1+v)$ we have

$$\begin{aligned} &= \left(\overline{c_{n-1}} + \overline{c_{n-1}^1} - 3(1+u)(1+v),...,\overline{c_0} + \overline{c_0^1} - 3(1+u)(1+v)\right) \\ &= \left(\overline{c_{n-1}} - 3(1+u)(1+v),...,\overline{c_0} - 3(1+u)(1+v)\right) + \left(\overline{c_{n-1}^1},...,\overline{c_0^1}\right) \\ &= \left(d_1^{rc} - 3(1+u)(1+v)\right) + d_2^{rc} \in S_1 + S_2 \end{aligned}$$

This shows that $S_1 + S_2$ is reversible complement cyclic code. It is clear that $S_1 \cap S_2$ is reversible complement cyclic code.

5. Binary images of cyclic DNA codes over *D*

The 2-adic expansion of $c \in Z_4$ is $c = \alpha(c) + 2\beta(c)$ such that $\alpha(c) + \beta(c) + \gamma(c) = 0$ for all $c \in Z_4$

c	$\alpha(c)$	$\beta(c)$	$\gamma(c)$
0	0	0	0
1	1	0	1
2	0	1	1
3	1	1	0

The Gray map is given by

$$\Psi : Z_4 \to Z_2^2$$
$$c \mapsto \Psi(c) = (\beta(c), \gamma(c))$$

for all $c \in Z_4$ in [28]. We define

$$\breve{O} : D \to Z_2^6$$
$$\begin{aligned} a+ub+vc \mapsto \breve{O}(a+ub+vc) &= \Psi(\Phi(a+ub+vc)) \\ &= \Psi(a, a+b, a+c) \\ &= (\beta(a), \gamma(a), \beta(a+b), \gamma(a+b), \beta(a+c), \gamma(a+c)) \end{aligned}$$

where Φ is a Gray map from D to Z_4^3.

Let $a+ub+vc$ be any element of the ring D. The Lee weight w_L of element of the ring D is defined as follows

$$w_L(a+ub+vc) = w_L(a, a+b, a+c)$$

where $w_L(a, a+b, a+c)$ described the usual Lee weight on Z_4^3. For any $c_1, c_2 \in D$ the Lee distance d_L is given by $d_L(c_1, c_2) = w_L(c_1 - c_2)$.

The Hamming distance $d(c_1, c_2)$ between two codewords c_1 and c_2 is the Hamming weight of the codewords $c_1 - c_2$. Binary images of the codons;

$$\begin{array}{ccc} AAA & \to & 000000 \\ TTT & \to & 010101 \\ GGG & \to & 101010 \\ CCC & \to & 111111 \\ \vdots & \vdots & \vdots \end{array}$$

Lemma 25 *The Gray map $\breve{O}$ is a distance preserving map from (D^n, Lee distance) to (Z_2^{6n}, Hamming distance). It is also Z_2-linear.*

Proof For $c_1, c_2 \in D^n$, we have $\breve{O}(c_1 - c_2) = \breve{O}(c_1) - \breve{O}(c_2)$. So,

$$d_L(c_1, c_2) = w_L(c_1 - c_2) = w_H(\breve{O}(c_1 - c_2)) = w_H(\breve{O}(c_1) - \breve{O}(c_2)) = d_H(\breve{O}(c_1), \breve{O}(c_2)).$$

Hence, the Gray map $\breve{O}$ is distance preserving map. For Z_2-linear, it is easily seen that $\breve{O}(k_1c_1 + k_2c_2) = k_1\breve{O}(c_1) + k_2\breve{O}(c_2)$, where $c_1, c_2 \in D^n, k_1, k_2 \in Z_2$.

Proposition 26 *Let σ be the cyclic shift of D^n and η be the 6-quasi-cyclic shift of Z_2^{6n}. Let $\breve{O}$ be the Gray map from D^n to Z_2^{6n}. Then $\breve{O}\sigma = \eta\breve{O}$.*

Proof Let $c = (c_0, c_1, ..., c_{n-1}) \in D^n$, we have $c_i = a_{1i} + ua_{2i} + va_{3i}$ with $a_{1i}, a_{2i}, a_{3i} \in Z_4, 0 \le i \le n-1$. By applying the Gray map, we have

$$\breve{O}(c)=\begin{pmatrix}\beta(a_{10}),\gamma(a_{10}),\beta(a_{10}+a_{20}),\gamma(a_{10}+a_{20}),\beta(a_{10}+a_{30}),\gamma(a_{10}+a_{30}),...,\\ \beta(a_{1n-1}+a_{3n-1}),\gamma(a_{1n-1}+a_{3n-1})\end{pmatrix}.$$

Hence

$$\eta(\breve{O}(c))=\begin{pmatrix}\beta(a_{1n-1}),\gamma(a_{1n-1}),\beta(a_{1n-1}+a_{2n-1}),\gamma(a_{1n-1}+a_{2n-1}),...,\beta(a_{10}),\gamma(a_{10})\\ ,...,\beta(a_{1n-2}),\gamma(a_{1n-2}),...,\gamma(a_{1n-2}+a_{3n-1})\end{pmatrix}.$$

On the other hand, $\sigma(c)=(c_{n-1},c_0,c_1,...,c_{n-2})$. We have

$$\breve{O}(\sigma(c))=\begin{pmatrix}\beta(a_{1n-1}),\gamma(a_{1n-1}),\beta(a_{1n-1}+a_{2n-1}),\gamma(a_{1n-1}+a_{2n-1}),...,\beta(a_{10}),\gamma(a_{10})\\ ,...,\beta(a_{1n-2}),\gamma(a_{1n-2}),...,\gamma(a_{1n-2}+a_{3n-1})\end{pmatrix}.$$

Therefore, $\breve{O}\sigma=\eta\breve{O}.$

Theorem 27 *If C is a cyclic DNA code of length n over D then $\breve{O}(C)$ is a binary quasi-cyclic DNA code of length $6n$ with index 6.*

Proof Let C be a cyclic DNA code of length n over D. So, $\sigma(C)=C$. By using the Proposition 25, we have $\breve{O}(\sigma(C))=\eta(\breve{O}(C))=\breve{O}(C)$. Hence $\breve{O}(C)$ is a set of length $6n$ over the alphabet Z_2 which is a quasi-cyclic code of index 6.

6. Skew codes over *D*

We are interested in studying skew codes over D, in this section. Firstly, we define a nontrivial automorphism θ on the ring D as follows,

$$\theta: D\to D$$
$$a+ub+vc\mapsto a+uc+bv$$

where $a,b,c\in Z_4$.

The ring $D[x,\theta]=\{a_0+a_1x+...+a_{n-1}x^{n-1} \ : \ a_i\in D, n\in N\}$ is called skew polynomial ring. The ring is a non-commutative ring. The addition in the ring $D[x,\theta]$ is the usual polynomial additional and multiplication is defined using the rule, $(ax^i)(bx^j)=a\theta^i(b)x^{i+j}$. The order of the automorphism θ is 2.

Definition 28 *A subset C of D^n is called a skew cyclic code of length n if C satisfies the following conditions,*

i) C is a submodule of D^n,

ii) If $c=(c_0,c_1,...,c_{n-1})\in C$, then $\sigma_\theta(c)=(\theta(c_{n-1}),\theta(c_0),...,\theta(c_{n-2}))\in C$

Let $f(x)+\langle x^n-1\rangle$ be an element in the set $S_n=D[x,\theta]/\langle x^n-1\rangle$ and let $r(x)\in D[x,\theta]$.

Define multiplication from left as follows,

$$r(x)(f(x)+\langle x^n-1\rangle)=r(x)f(x)+\langle x^n-1\rangle$$

for any $r(x)\in D[x,\theta]$.

Theorem 29 *S_n is a left $D[x,\theta]$-module where multiplication defined as in above.*

Theorem 30 *A code C in S_n of length n is a skew cyclic code if and only if C is a left $D[x,\theta]$-submodule of the left $D[x,\theta]$-module S_n.*

Theorem 31 *Let C be a skew cyclic code over D of length n and let $f(x)$ be a polynomial in C of minimal degree. If $f(x)$ is monic polynomial, then $C=\langle f(x)\rangle$, where $f(x)$ is a right divisor of x^n-1.*

Definition 32 *A subset C of D^n is called a skew quasi-cyclic code of length n if C satisfies the following conditions,*

i) C is a submodule of D^n,

ii) If $e=(e_{0,0},...,e_{0,l-1},e_{1,0},...,e_{1,l-1},...,e_{s-1,0},..,e_{s-1,l-1})\in C$, then

$\tau_{\theta,s,l}(e)=(\theta(e_{s-1,0}),...,\theta(e_{s-1,l-1}),\theta(e_{0,0}),...,\theta(e_{0,l-1}),\theta(e_{s-2,0}),...,\theta(e_{s-2,l-1}))\in C$.

We note that x^s-1 is a two sided ideal in $D[x,\theta]$ if $m\mid s$ where m is the order of θ. So $D[x,\theta]/(x^s-1)$ is well defined.

The ring $R_s^l=(D[x,\theta]/(x^s-1))^l$ is a left $R_s=D[x,\theta]/(x^s-1)$ module by the following multiplication on the left $f(x)(g_1(x),...,g_l(x))=(f(x)g_1(x),...f(x)g_l(x))$. If the map Λ is defined by

$$\Lambda\ :\ D^n\to R_s^l$$

$(e_{0,0},...,e_{0,l-1},e_{1,0},...,e_{1,l-1},...,e_{s-1,0},...,e_{s-1,l-1})\mapsto\ (c_0(x),...,c_{l-1}(x))$ such that $c_j(x)=\sum_{i=0}^{s-1}e_{i,j}x^i\in R_s$

where $j=0,1,...,l-1$ then the map Λ gives a one to one correspondence D^n and the ring R_s^l.

Theorem 33 *A subset C of D^n is a skew quasi-cyclic code of length $n=sl$ and index l if and only if $\Lambda(C)$ is a left R_s-submodule of R_s^l.*

Definition 34 *Let θ be an automorphism of D, λ be a unit in D, C be a linear code D. A linear code C is said to be a skew constacyclic code if C is closed under the $\theta-\lambda$-cyclic shift $\tau_{\theta,\lambda}\ :\ D^n\to D^n$ defined by*

$$\tau_{\theta,\lambda}(c_0,...,c_{n-1})=(\theta(\lambda c_{n-1}),\theta(c_0),...,\theta(c_{n-2}))$$

7. The Gray images of skew cyclic, quasi-cyclic and constacyclic codes over *D*

Proposition 35 *Let* σ_θ *be the skew cyclic shift on* D^n, *let* Φ *be the Gray map from* D^n *to* Z_4^{3n} *and* φ *be as in the preliminaries. Then* $\Phi\sigma_\theta = \upsilon\varphi\Phi$, *where* υ *is map such that* $\upsilon(x,y,z) = (x,z,y)$ *for* $x \in Z_4^n$.

Proof It is proved that as in the proof the Proposition 25.

Theorem 36 *The Gray image of a skew cyclic code over* D *of length* n *is permutation equivalent to a quasi-cyclic code of index* 3 *with length* $3n$.

Proof It is proved that as in the proof the Theorem 26.

Proposition 37 *Let* $\tau_{\theta,s,l}$ *be the skew quasi-cyclic shift,* Γ *be as in the preliminaries,* Φ *be the Gray map from* D^n *to* Z_4^{3n}. *Then* $\Phi\tau_{\theta,s,l} = \upsilon\Gamma\Phi$, *where* υ *is map such that* $\upsilon(x,y,z) = (x,z,y)$ *for* $x \in Z_4^n$.

Proof It is proved that as in the proof the Proposition 25.

Theorem 38 *The Gray image of a skew quasi-cyclic code over* D *of length* n *is permutation equivalent to a* l*-quasi-cyclic code of index* 3 *with length* $3n$.

Proof It is proved that as in the proof the Theorem 26.

Proposition 39 *Let* $\tau_{\theta,\lambda}$ *be the* θ-λ*-cyclic shift, let* Φ *be the Gray map from* D^n *to* Z_4^{3n} *and* σ_λ *be constacyclic shift. Then* $\Phi\tau_{\theta,\lambda} = \upsilon\Phi\sigma_\lambda$ *where* υ *is map such that* $\upsilon(x,y,z) = (x,z,y)$ *for* $x \in Z_4^n$.

Proof It is proved that as in the proof the Proposition 25.

Theorem 40 *The Gray image of a skew constacyclic code over* D *of length* n *is permutation to the Gray image of a constacyclic code over* D *of length* n.

Proof It is proved that as in the proof the Theorem 26.

8. Skew cyclic DNA codes over *D*

In this section, we introduce a family of DNA skew cyclic codes over D. We study its property of being reverse complement.

For all $x \in D$, we have

$$\theta(x) + \theta(\overline{x}) = 3 + 3u + 3v$$

Theorem 41 *Let $C = \langle f(x) \rangle$ be a skew cyclic code over D, where $f(x)$ is a monic polynomial in C of minimal degree. If C is reversible complement, the polynomial $f(x)$ is self reciprocal and $(3+3u+3v, 3+3u+3v, \ldots, 3+3u+3v) \in C$.*

Proof Let $C = \langle f(x) \rangle$ be a skew cyclic code over D, where $f(x)$ is a monic polynomial in C. Since $(0,0,\ldots,0) \in C$ and C is reversible complement, we have

$$\left(\overline{0},\overline{0},\ldots,\overline{0}\right) = (3+3u+3v, 3+3u+3v, \ldots, 3+3u+3v) \in C.$$

Let $f(x) = 1 + a_1 x + \ldots + a_{t-1}x^{t-1} + x^t$. Since C is reversible complement, we have $f^{rc}(x) \in C$. That is

$$f^{rc}(x) = (3+3u+3v) + (3+3u+3v)x + \ldots + (3+3u+3v)x^{n-t-2} + \\ (2+3u+3v)x^{n-t-1} + \overline{a}_{t-1}x^{n-t} + \ldots + \overline{a}_1 x^{n-2} + (2+3u+3v)x^{n-1}$$

Since C is a linear code, we have $f^{rc}(x) - (3+3u+3v)\frac{x^n-1}{x-1} \in C$. This implies that

$$-x^{n-t-1} + (\overline{a}_{t-1} - (3+3u+3v))x^{n-t} + \ldots + (\overline{a}_1 - (3+3u+3v))x^{n-2} - x^{n-1} \in C$$

By multiplying on the right by x^{t+1-n}, we have

$$-1 + (\overline{a}_{t-1} - (3+3u+3v))\theta(1)x + \ldots + (\overline{a}_1 - (3+3u+3v))\theta^{t-1}(1)x^{t-1} - \theta^t(1)x^t \in C$$

By using $a + \overline{a} = 3 + 3u + 3v$, for $a \in D$, we have

$$-1 - a_{t-1}x - a_{t-2}x^2 - \ldots - a_1 x^{t-1} - x^t = 3f^*(x) \in C$$

Since $C = \langle f(x) \rangle$, there exist $q(x) \in D[x,\theta]$ such that $3f^*(x) = q(x)f(x)$. Since $\deg f(x) = \deg f^*(x)$, we have $q(x) = 1$. Since $3f^*(x) = f(x)$, we have $f^*(x) = 3f(x)$. So, $f(x)$ is self reciprocal.

Theorem 42 *Let $C = \langle f(x) \rangle$ be a skew cyclic code over D, where $f(x)$ is a monic polynomial in C of minimal degree. If $(3+3u+3v, 3+3u+3v, \ldots, 3+3u+3v) \in C$ and $f(x)$ is self reciprocal, then C is reversible complement.*

Proof Let $f(x) = 1 + a_1 x + \ldots + a_{t-1}x^{t-1} + x^t$ be a monic polynomial of the minimal degree. Let $c(x) \in C$. So, $c(x) = q(x)f(x)$, where $q(x) \in D[x,\theta]$. By using Lemma 18, we have $c^*(x) = (q(x)f(x))^* = q^*(x)f^*(x)$. Since $f(x)$ is self reciprocal, so $c^*(x) = q^*(x)ef(x)$, where $e \in Z_4 \setminus \{0\}$. Therefore $c^*(x) \in C = \langle f(x) \rangle$. Let $c(x) = c_0 + c_1 x + \ldots + c_t x^t \in C$. Since C is a cyclic code, we get

$$c(x)x^{n-t-1} = c_0 x^{n-t-1} + c_1 x^{n-t} + ... + c_t x^{n-1} \in C$$

The vector correspond to this polynomial is

$$(0, 0, ..., 0, c_0, c_1, ..., c_t) \in C$$

Since $(3+3u+3v, 3+3u+3v, ..., 3+3u+3v) \in C$ and C linear, we have

$$(3+3u+3v, 3+3u+3v, ..., 3+3u+3v) - (0, 0, ..., 0, c_0, c_1, ..., c_t) = (3+3u+3v,$$
$$..., 3+3u+3v, (3+3u+3v) - c_0, ..., (3+3u+3v) - c_t) \in C$$

By using $a + \bar{a} = 3 + 3u + 3v$, for $a \in D$, we get

$$(3+3u+3v, 3+3u+3v, ..., 3+3u+3v, \bar{c}_0, ..., \bar{c}_t) \in C$$

which is equal to $\left(c(x)^*\right)^{rc}$. This shows that $\left(\left(c(x)^*\right)^{rc}\right)^* = c(x)^{rc} \in C.$

9. Conclusion

Firstly, the finite ring $D = Z_4 + uZ_4 + vZ_4, u^2 = u, v^2 = v, uv = vu = 0$ is introduced. The cyclic, constacyclic, quasi-cyclic codes and their skew codes over D are studied. After a Gray map Φ is defined, the Gray image of them are determined. Moreover, the cyclic DNA and skew cyclic DNA codes over D are introduced.

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