Article

Identities of Chen-Choi Involving the Euler-Mascheroni's Constant

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Abstract

We give an elementary deduction of the Chen-Choi's identities involving the Euler-Mascheroni's constant.

Keywords: Gamma function, Euler-Mascheroni's constant, Newman-Weierstrass identity.

1. Introduction

Chen-Choi [1, 2] obtained the identities:

$$\prod_{j=1}^{\infty} e^{-\frac{p}{j}}\left(1+\frac{p}{j}+\frac{q}{j^2}\right)=\frac{e^{-p\gamma}}{\Gamma\left(1+\frac{p+\Delta}{2}\right)\Gamma\left(1+\frac{p-\Delta}{2}\right)}\ , \tag{1}$$

$$\Delta=\sqrt{p^2-4q}\ ,$$

$$\prod_{j=1}^{\infty} e^{-\frac{p}{2j-1}}\left(1+\frac{p}{2j-1}+\frac{q}{(2j-1)^2}\right)=\frac{2^{-p}\,\pi\,e^{-p\gamma/2}}{\Gamma\left(\frac{1}{2}+\frac{p+\Delta}{4}\right)\Gamma\left(\frac{1}{2}+\frac{p-\Delta}{4}\right)}\ , \tag{2}$$

where γ =0.5772 1566 4901 5328 6060 … is the Euler-Mascheroni's constant [3-5].

In Sec. 2 we employ the following relation [6] involving an infinite product and the gamma function [5, 7-10]:

$$\prod_{k=m}^{\infty}\frac{(k+z)^2-b}{(k+z)^2-a}=\frac{\Gamma\left(z+m-\sqrt{a}\right)\Gamma\left(z+m+\sqrt{a}\right)}{\Gamma\left(z+m-\sqrt{b}\right)\Gamma\left(z+m+\sqrt{b}\right)}, \tag{3}$$

to give an elementary deduction of (1) and (2).

2. Chen-Choi's formulas

In (3) we can use the values $a=\frac{p^2}{4}$, $b=\frac{\Delta^2}{4}$, $m=1$ and $z=\frac{p}{2}$ to obtain the expression:

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$$\prod_{k=1}^{\infty} \frac{(2k+p)^2-\Delta^2}{(2k+p)^2-p^2} \equiv \prod_{k=1}^{\infty} \left[\frac{1+\frac{p}{k}+\frac{q}{k^2}}{1+\frac{p}{k}}\right] = \frac{\prod_{j=1}^{\infty} e^{-\frac{p}{j}}\left(1+\frac{p}{j}+\frac{q}{j^2}\right)}{\prod_{r=1}^{\infty} e^{-\frac{p}{r}}\left(1+\frac{p}{r}\right)} = \frac{p!}{\Gamma\left(1+\frac{p+\Delta}{2}\right)\Gamma\left(1+\frac{p-\Delta}{2}\right)}, \tag{4}$$

but the Newman (1848)-Weierstrass (1856) relation [8]:

$$z\, e^{\gamma z} \prod_{r=1}^{\infty} e^{-\frac{z}{r}}\left(1+\frac{z}{r}\right) = \frac{1}{\Gamma(z)}, \tag{5}$$

with $z = p$ gives the property:

$$\prod_{r=1}^{\infty} e^{-\frac{p}{r}}\left(1+\frac{p}{r}\right) = \frac{e^{-p\gamma}}{p!}, \tag{6}$$

whose application in (4) implies the Chen-Choi's identity (1) [1, 2], q.e.d.

Now in (3) we apply the values $a = \frac{p^2}{16}$, $b = \frac{\Delta^2}{16}$, $m = 1$, and $z = \frac{p}{4} - \frac{1}{2}$ to deduce the expression:

$$\prod_{k=1}^{\infty} \frac{(4k-2+p)^2-\Delta^2}{(4k-2+p)^2-\Delta^2} \equiv \prod_{k=1}^{\infty} \frac{1+\frac{p}{2k-1}+\frac{q}{(2k-1)^2}}{1+\frac{p}{2k-1}} = \frac{\prod_{j=1}^{\infty} e^{-\frac{p}{2j-1}}\left(1+\frac{p}{2j-1}+\frac{q}{(2j-1)^2}\right)}{\prod_{r=1}^{\infty} e^{-\frac{p}{2r-1}}\left(1+\frac{p}{2r-1}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{p+\Delta}{4}\right)\Gamma\left(\frac{1}{2}+\frac{p-\Delta}{4}\right)}, \tag{7}$$

but we know the relations:

$$\prod_{r=1}^{\infty} e^{-\frac{p}{2r-1}}\left(1+\frac{p}{2r-1}\right) = \frac{\left(\frac{p}{2}\right)!}{p!}\, e^{-\frac{p\gamma}{2}}, \qquad \frac{\left(\frac{p}{2}\right)!}{p!}\,\Gamma\left(\frac{p+1}{2}\right) = \frac{\sqrt{\pi}}{2^p}, \tag{8}$$

then (7) and (8) imply (2), q.e.d.

The identities of Wilf [5, 11, 12] and Choi-Lee-Srivastava [13, 14] are particular cases of (1) and (2).

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