Article

p-Adic Square Root Function and p-Adic Light-cone

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Abstract

The argument of the article demonstrates that the extension allowing square roots of ordinary p-adic numbers is 4-dimensional for $p < 2$ and 8-dimensional for $p = 2$. The region of convergence of the p-adic square root function can be regarded as the counterpart of light-cone.

Keywords: p-Adic analysis, 2-adic numbers, square root function, light-cone.

Contents

1 Introduction

This little article appears as an appendix to one of the chapters [2] of an online book trying to formulate the ideas related to the question what p-adic physics describes, what it means, and how one could fuse real and p-adic physics to a larger coherent whole [1]. I take the article as a clumsy and innocent mathematical exercise of a theoretical physicist trying to gain an ntuitive understanding about what p-adic numbers are but for some reason it has generated more interest than the work that I take myself much more seriously. Perhaps the reason is that it contains formulas instead of philosophizing with a lot of 'maybe's and 'on the other hand's.

The basic argument of the article is that the extension allowing square roots of ordinary p-adic numbers is 4-dimensional for $p < 2$ and 8-dimensional for $p = 2$. The region of convergence of the p-adic square root function can be regarded as the counterpart of light-cone. The obvious question was whether the dimensions of space-time and the 8-D imbedding space in which space-time surfaces reside in TGD Universe could relate to these algebraic dimensions. It turned out that in TGD Universe classical number fields (reals,complex numbers, quaternions, and octonions) provide the manner to understand these dimensions and that algebraic dimensions assignable to the extensions of p-adic numbers are not identifiable as real physical dimensions. Apart from these after-thoughts the article is in the original form.

2 p-Adic square root function and square root allowing extension of p-adic numbers

The following arguments demonstrate that the extension allowing square roots of ordinary p-adic numbers is 4-dimensional for $p < 2$ and 8-dimensional for $p = 2$.

2.1 $p > 2$ resp. $p = 2$ corresponds to $D = 4$ resp. $D = 8$ dimensional extension

What is important is that only the square root of ordinary p-adic numbers is needed: the square root need not exist outside the real axis. It is indeed impossible to find a finite-dimensional extension allowing square root for all ordinary p-adic numbers numbers. For $p > 2$ the minimal dimension for algebraic extension allowing square roots near real axis is $D = 4$. For $p = 2$ the dimension of the extension is $D = 8$.

For $p > 2$ the form of the extension can be derived by the following arguments.

1. For $p > 2$ a p-adic number y in the range $(0, p-1)$ allows square root only provided there exists a p-adic number $x \in \{0, p-1\}$ satisfying the condition $y = x^2 \ mod \ p$. Let x_0 be the smallest integer, which does not possess a p-adic square root and add the square root θ of x_0 to the number field. The numbers in the extension are of the form $x + \theta y$. The extension allows square root for every $x \in \{0, p-1\}$ as is easy to see. p-adic numbers $mod \ p$ form a finite field $G(p, 1)$ [3] so that any p-adic number y, which does not possess square root can be written in the form $y = x_0 u$, where u possesses square root. Since θ is by definition the square root of x_0 then also y possesses square root. The extension does not depend on the choice of x_0.

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The square root of -1 does not exist for $p\ mod\ 4 = 3$ [4] and $p = 2$ but the addition of θ gurantees its existence automatically. The existence of $\sqrt{-1}$ follows from the existence of $\sqrt{p-1}$ implied by the extension by θ. $\sqrt{(-1+p)-p}$ can be developed in power in powers of p and series converges since the p-adic norm of coefficients in Taylor series is not larger than 1. If $p-1$ does not possess a square root, one can take θ to be equal to $\sqrt{-1}$.

2. The next step is to add the square root of p so that the extension becomes 4-dimensional and an arbitrary number in the extension can be written as

$$Z = (x+\theta y) + \sqrt{p}(u+\theta v) \ . \tag{2.1}$$

In $p = 2$ case 8-dimensional extension is needed to define square roots. The addition of $\sqrt{2}$ implies that one can restrict the consideration to the square roots of odd 2-adic numbers. One must be careful in defining square roots by the Taylor expansion of square root $\sqrt{x_0 + x_1}$ since n:th Taylor coefficient is proportional to 2^{-n} and possesses 2-adic norm 2^n. If x_0 possesses norm 1 then x_1 must possess norm smaller than 1/8 for the series to converge. By adding square roots $\theta_1 = \sqrt{-1}, \theta_2 = \sqrt{2}$ and $\theta_3 = \sqrt{3}$ and their products one obtains 8-dimensional extension.

The emergence of the dimensions $D = 4$ and $D = 8$ for the algebraic extensions allowing the square root of an ordinary p-adic number stimulates an obvious question: could one regard space-time as this kind of an algebraic extension for $p > 2$ and the imbedding space $H = M^4_+ \times CP_2$ as a similar 8-dimensional extension of the 2-adic numbers? Contrary to the first expectations, it seems that algebraic dimension cannot be regarded as a physical dimension, and that quaternions and octonions provide the correct framework for understanding space-time and imbedding space dimensions. One could perhaps say that algebraic dimensions are additional dimensions of the world of cognitive physics rather than those of the real physics and there presence could perhaps explain why we can imagine all possible dimensions mathematically.

By construction, any ordinary p-adic number in the extension allows square root. The square root for an arbitrary number sufficiently near to p-adic axis can be defined through Taylor series expansion of the square root function $\sqrt{Z}$ at a point of p-adic axis. The subsequent considerations show that the p-adic square root function does not allow analytic continuation to R^4 and the points of the extension allowing a square root consist of disjoint converge cubes forming a structure resembling future light cone in certain respects.

2.2 p-Adic square root function for $p > 2$

The study of the properties of the series representation of a square root function shows that the definition of the square root function is possible in certain region around the real p-adic axis. What is nice that this region can be regarded as the p-adic analog (not the only one) of the future light cone defined by the condition

$$N_p(Im(Z)) < N_p(t = Re(Z)) = p^k \ , \tag{2.2}$$

where the real p-adic coordinate $t = Re(Z)$ is identified as a time coordinate and the imaginary part of the p-adic coordinate is identified as a spatial coordinate. The p-adic norm for the four-dimensional extension is analogous to ordinary Euclidian distance. p-Adic light cone consists of cylinders parallel to time axis having radius $N_p(t) = p^k$ and length $p^{k-1}(p-1)$. As a real space (recall the canonical correspondence) the cross section of the cylinder corresponds to a parallelpiped rather than ball.

The result can be understood heuristically as follows.

1. For the four-dimensional extension allowing square root ($p > 2$) one can construct square root at each point $x(k,s) = sp^k$ represented by ordinary p-adic number, $s = 1, ..., p-1$, $k \in Z$. The task is to show that by using Taylor expansion one can define square root also in some neighbourhood of each of these points and find the form of this neighbourhood.

2. Using the general series expansion of the square root function one finds that the convergence region is p-adic ball defined by the condition

$$N_p(Z - sp^k) \leq R(k) \ , \tag{2.3}$$

and having radius $R(k) = p^d, d \in Z$ around the expansion point.

3. A purely p-adic feature is that the convergence spheres associated with two points are either disjoint or identical! In particular, the convergence sphere $B(y)$ associated with any point inside convergence sphere $B(x)$ is identical with $B(x)$: $B(y) = B(x)$. The result follows directly from the ultra-metricity of the p-adic norm. The result means that stepwise analytic continuation is not possible and one can construct square root function only in the union of p-adic convergence spheres associated with the points $x(k,s) = sp^k$ which correspond to ordinary p-adic numbers.

4. By the scaling properties of the square root function the convergence radius $R(x(k,s)) \equiv R(k)$ is related to $R(x(0,s)) \equiv R(0)$ by the scaling factor p^{-k}:

$$R(k) = p^{-k}R(0) , \tag{2.4}$$

so that the convergence sphere expands as a function of the p-adic time coordinate. The study of the convergence reduces to the study of the series at points $x = s = 1, ..., k-1$ with a unit p-adic norm.

5. Two neighboring points $x = s$ and $x = s+1$ cannot belong to the same convergence sphere: this would lead to a contradiction with the basic results of about square root function at integer points. Therefore the convergence radius satisfies the condition

$$R(0) < 1 . \tag{2.5}$$

The requirement that the convergence is achieved at all points of the real axis implies

$$\begin{aligned} R(0) &= \frac{1}{p} , \\ R(p^k s) &= \frac{1}{p^{k+1}} . \end{aligned} \tag{2.6}$$

If the convergence radius is indeed this, then the region, where the square root is defined, corresponds to a connected light cone like region defined by the condition $N_p(Im(Z)) = N_p(Re(Z))$ and $p > 2$-adic space time is the p-adic analog of the M^4 lightcone. If the convergence radius is smaller, the convergence region reduces to a union of disjoint p-adic spheres with increasing radii.

How the p-adic light cone differs from the ordinary light cone can be seen by studying the explicit form of the p-adic norm for $p > 2$ square root allowing extension $Z = x + iy + \sqrt{p}(u + iv)$

$$\begin{aligned} N_p(Z) &= (N_p(det(Z)))^{\frac{1}{4}} , \\ &= (N_p((x^2+y^2)^2 + 2p^2((xv-yu)^2 + (xu-yv)^2) + p^4(u^2+v^2)^2))^{\frac{1}{4}} , \end{aligned} \tag{2.7}$$

where $det(Z)$ is the determinant of the linear map defined by a multiplication with Z. The definition of the convergence sphere for $x = s$ reduces to

$$N_p(det(Z_3)) = N_p(y^4 + 2p^2y^2(u^2+v^2) + p^4(u^2+v^2)^2)) < 1 . \tag{2.8}$$

For physically interesting case $p \ mod \ 4 = 3$ the points (y, u, v) satisfying the conditions

$$\begin{aligned} N_p(y) &\leq \frac{1}{p} , \\ N_p(u) &\leq 1 , \\ N_p(v) &\leq 1 , \end{aligned} \tag{2.9}$$

belong to the sphere of convergence: it is essential that for all u and v satisfying the conditions one has also $N_p(u^2 + v^2) \leq 1$. By the canonical correspondence between p-adic and real numbers, the real counterpart of the sphere $r = t$ is now the parallelpiped $0 \leq y < 1, 0 \leq u < p, 0 \leq v < p$, which expands with an average velocity of light in discrete steps at times $t = p^k$.

2.3 Convergence radius for square root function

In the following it will be shown that the convergence radius of $\sqrt{t+Z}$ is indeed non-vanishing for $p > 2$. The expression for the Taylor series of $\sqrt{t+Z}$ reads as

$$\begin{aligned} \sqrt{t+Z} &= = \sqrt{x}\sum_n a_n , \\ a_n &= (-1)^n \frac{(2n-3)!!}{2^n n!} x^n , \\ x &= \frac{Z}{t} . \end{aligned} \tag{2.10}$$

The necessary criterion for the convergence is that the terms of the power series approach to zero at the limit $n \to \infty$. The p-adic norm of the n:th term is for $p > 2$ given by

$$N_p(a_n) = N_p(\frac{(2n-3)!!}{n!})N_p(x^n) < N_p(x^n)N_p(\frac{1}{n!}) . \tag{2.11}$$

The dangerous term is clearly the $n!$ in the denominator. In the following it will be shown that the condition

$$U \equiv \frac{N_p(x^n)}{N_p(n!)} < 1 \ for \ \ N_p(x) < 1 \ , \tag{2.12}$$

holds true. The strategy is as follows:
a) The norm of x^n can be calculated trivially: $N_p(x^n) = p^{-Kn}, K \geq 1$.
b) $N_p(n!)$ is calculated and an upper bound for U is derived at the limit of large n.

2.3.1 p-Adic norm of $n!$ for $p > 2$

Lemma 1: Let $n = \sum_{i=0}^{k} n(i)p^i$, $0 \leq n(i) < p$ be the p-adic expansion of n. Then $N_p(n!)$ can be expressed in the form

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} N(i)^{n(i)} \ , \\ N(1) &= \frac{1}{p} \ , \\ N(i+1) &= N(i)^{p-1}p^{-i} \ . \end{aligned} \tag{2.13}$$

An explicit expression for $N(i)$ reads as

$$N(i) = p^{-\sum_{m=0}^{i} m(p-1)^{i-m}} \ . \tag{2.14}$$

Proof: $n!$ can be written as a product

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} X(i, n(i)) \ , \\ X(k, n(k)) &= N_p((n(k)p^k)!) \ , \\ X(k-1, n(k-1)) &= N_p(\prod_{i=1}^{n(k-1)p^{k-1}} (n(k)p^k + i)) = N_p((n(k-1)p^{k-1})!) \ , \\ X(k-2, n(k-2)) &= N_p(\prod_{i=1}^{n(k-2)p^{k-2}} (n(k)p^k + n(k-1)p^{k-1} + i) \ ,) \\ &= N_p((n(k-2)p^{k-2})!) \ , \\ X(k-i, n(k-i)) &= N_p((n(k-i)p^{k-i})!) \ . \end{aligned} \tag{2.15}$$

The factors $X(k, n(k))$ reduce in turn to the form

$$\begin{aligned} X(k, n(k)) &= \prod_{i=1}^{n(k)} Y(i, k) \ , \\ Y(i, k) &= \prod_{m=1}^{p^k} N_p(ip^k + m) \ . \end{aligned} \tag{2.16}$$

The factors $Y(i, k)$ in turn are indentical and one has

$$\begin{aligned} X(k, n(k)) &= X(k)^{n(k)} \ , \\ X(k) &= N_p(p^k!) \ . \end{aligned} \tag{2.17}$$

The recursion formula for the factors $X(k)$ can be derived by writing explicitely the expression of $N_p(p^k!)$ for a few lowest values of k:
1) $X(1) = N_p(p!) = p^{-1}$.
2) $X(2) = N_p(p^2!) = X(1)^{p-1}p^{-2}$ ($p^2!$ decomposes to $p-1$ products having same norm as $p!$ plus the last term equal to p^2.
i) $X(i) = X(i-1)^{p-1}p^{-i}$

Using the recursion formula repeatedly the explicit form of $X(i)$ can be derived easily. Combining the results one obtains for $N_p(n!)$ the expression

$$\begin{aligned} N_p(n!) &= p^{-\sum_{i=0}^{k} n(i)A(i)} \ , \\ A(i) &= \sum_{m=1}^{i} m(p-1)^{i-m} \ . \end{aligned} \tag{2.18}$$

The sum $A(i)$ appearing in the exponent as the coefficient of $n(i)$ can be calculated by using geometric series

$$\begin{aligned} A(i) &= (\frac{p-1}{p-2})^2(p-1)^{i-1}(1+\frac{i}{(p-1)^{i+1}}-\frac{(i+1)}{(p-1)^i}) \ , \\ &\leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \ . \end{aligned} \tag{2.19}$$

2.3.2 Upper bound for $N_p(\frac{x^n}{n!})$ for $p > 2$

By using the expressions $n = \sum_i n(i)p^i$, $N_p(x^n) = p^{-Kn}$ and the expression of $N_pn!$ as well as the upper bound

$$A(i) \leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \ . \tag{2.20}$$

For $A(i)$ one obtains the upper bound

$$N_p(\frac{x^n}{n!}) \leq p^{-\sum_{i=0}^k n(i)p^i(K-(\frac{(p-1)}{(p-2)})^2(\frac{(p-1)}{p})^{i-1})} \ . \tag{2.21}$$

It is clear that for $N_p(x) < 1$ that is $K \geq 1$ the upper bound goes to zero. For $p > 3$ exponents are negative for all values of i: for $p = 3$ some lowest exponents have wrong sign but this does not spoil the convergence. The convergence of the series is also obvious since the real valued series $\frac{1}{1-\sqrt{N_p(x)}}$ serves as a majorant.

2.4 $p = 2$ case

In $p = 2$ case the norm of a general term in the series of the square root function can be calculated easily using the previous result for the norm of $n!$:

$$N_p(a_n) = N_p(\frac{(2n-3)!!}{2^n n!})N_p(x^n) = 2^{-(K-1)n+\sum_{i=1}^k n(i)\frac{i(i+1)}{2^{i+1}}} \ . \tag{2.22}$$

At the limit $n \to \infty$ the sum term appearing in the exponent approaches zero and convergence condition gives $K > 1$, so that one has

$$N_p(Z) \equiv (N_p(det(Z)))^{\frac{1}{8}} \leq \frac{1}{4} \ . \tag{2.23}$$

The result does not imply disconnected set of convergence for square root function since the square root for half odd integers exists:

$$\sqrt{s+\frac{1}{2}} = \frac{\sqrt{2s+1}}{\sqrt{2}} \ , \tag{2.24}$$

so that one can develop square as a series in all half odd integer points of the p-adic axis (points which are ordinary p-adic numbers). As a consequence, the structure for the set of convergence is just the 8-dimensional counterpart of the p-adic light cone. Space-time has natural binary structure in the sense that each $N_p(t) = 2^k$ cylinder consists of two identical p-adic 8-balls (parallelpipeds as real spaces).

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