Article

An Approach to Riemannian Geometry within Conformable Fractional Derivative

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Abstract

In this paper, we introduce the conformable fractional Christoffel index symbols of a first and second kind and also we study some of its basic properties. The transformations of conformable fractional Christoffel symbols are obtained. We modify the covariant and absolute derivatives of tensor by involving conformable fractional derivative and obtained conformable fractional covariant, absolute derivatives of addition of two tensors. An illustrative example is presented.

Keywords: Conformable fractional, Christoffel index symbols, covariant derivative, absolute derivative, fractional divergence.

1. Introduction

In 1965 L' Hospital wrote a letter to Leibnitz and asked about what happen if number of times a function was differentiated is not an integer. From this early beginning the subject of fractional derivative was born. Since then many researchers have been trying to generalize the concept of usual derivative to fractional derivative. Now a day's many definitions of fractional derivatives are available such as Riemann- Liouville, Caputo, Hadmard, Grunewald- Letnikov, Riez [1-4].Out of these definitions of fractional derivatives Riemann- Liouville and Caputo are popular, but basic properties of usual derivatives like product rule and chain rule are lost in these derivatives. The fractional calculus has many applications in Relativity and Cosmology.

In recent times El-Nabulsi [5] developed a new approach to cosmology and obtained fractional Friedman equations. E. Raberi et.al. [6] formulate Hamiltonian- Jacobi with Fractional derivative. EL-Nabulsi Ahmed Rami [7-9] studied Friedman Robertson Walker universe, cosmology with fractional action principle, differential geometry and modern cosmology respectively. V.K. shchigolev [10] studied cosmological models with fractional derivative. F.Riewe [11] were investigated Non– Conservative Lagrangian and Hamiltonian mechanics with in fractional calculus. M. Klimek [12] was studied Lagrangian and Hamiltonian fractional sequential mechanics models with symmetric fractional derivative. M. Naber [13] studied Schrodinger equation with in Caputo fractional derivative. Recently Alireza K. Golmakhaneh

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et.al. [14] generalized the tensor calculus by using the local fractional calculus on fractals and obtained Einstein field equations within local fractional calculus.

Recently Khalil et.al. [15] Introduced new definition of fractional derivative depending on the basic limit of the derivative called as conformable fractional derivative. The new definition is simpler, more efficient and satisfies product rule of derivatives, quotient law of two functions and has simple chain rule [16]. T. Abdeljawad [17] discussed Conformable fractional version of Grownwall's inequality, Taylor power series expansion, and Laplace Transforms. A. Tallafha et.al. [18] Were studied Total and Directional fractional derivatives.Matheus J. Lazo et.al. [19] Formulate an action principle for the particles under fractional forces. W.S. Chung [20] introduced conformable fractional version of the calculus of variation and constructed the fractional Euler-Lagrange equation.

This paper will be divided into several parts as follows, the definition of conformable fractional derivative, notation and properties, formulae of some standard functions are presented. In section 2. In section 3 the conformable fractional christoffel index symbols are obtained and discussed some of its properties, also we obtained conformable fractional transformation of christoffel symbols. The conformable fractional derivatives of tensors and conformable fractional divergence are studied in section 4.and section 5 is devoted to our conclusions.

2. A Review of Conformable Fractional Derivative.

In this section we review the conformable fractional derivative and its properties. Also we give the conformable fractional derivatives of some standard functions.

2.1. Conformable Fractional Derivative

Suppose $f:[0,\infty)\to R$, then the conformable fractional derivative of f of order α is defined as

$$D_\alpha f(t)=\frac{d^\alpha f(t)}{dt^\alpha}=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}\quad for\,all\,t>0,\alpha\in(0,1] \tag{1}$$

If the conformable fractional derivative of f of order α exists then we say that f is α differentiable.

2.2. Properties of Conformable Fractional Derivative

Let $\alpha\in(0,1]$ and f,g be α differentiable functions at point $t>0$, then

1. $D_\alpha(af+bg)=a\,D_\alpha(f)+b\,D_\alpha(g)$
2. $D_\alpha(\lambda)=0$ for all constant function $f(t)=\lambda$
3. $D_\alpha(fg)=f\,D_\alpha(g)+g\,D_\alpha(f)$

4. $D_\alpha\left(\frac{f}{g}\right)=\frac{g\,D_\alpha(f)-f\,D_\alpha(g)}{g^2}$

5. $D_\alpha\left(t^p\right)=p\,t^{p-\alpha}$

6. If f be α differentiable then $D_\alpha\left(f(t)\right)=t^{1-\alpha}\frac{df}{dt}$

2.3. Conformable Fractional Derivative of Certain Functions

The conformable fractional derivative of some standard functions [16] are

1. $D_\alpha\left(e^{ax}\right)=a\,x^{1-\alpha}\,e^{ax},\forall$ real a

2. $D_\alpha(\sin ax)=a\,x^{1-\alpha}\cos ax,\forall$ real a

3. $D_\alpha(\cos ax)=-a\,x^{1-\alpha}\sin ax,\forall$ real a

4. $D_\alpha(\tan ax)=a\,x^{1-\alpha}\sec^2 ax,\forall$ real a

5. $D_\alpha(\cot ax)=-a\,x^{1-\alpha}\,\mathrm{cosec}^2\,ax,\forall$ real a

6. $D_\alpha(\sec ax)=a\,x^{1-\alpha}\sec ax.\tan ax,\forall$ real a

7. $D_\alpha(\cos ecax)=-a\,x^{1-\alpha}\,\mathrm{c}oescax.\cot ax,\forall$ real a

8. $D_\alpha\left(e^{\frac{1}{\alpha}x^\alpha}\right)=e^{\frac{1}{\alpha}x^\alpha}$

9. $D_\alpha\left(\sin\frac{1}{\alpha}x^\alpha\right)=\cos\frac{1}{\alpha}x^\alpha$

10. $D_\alpha\left(\cos\frac{1}{\alpha}x^\alpha\right)=-\sin\frac{1}{\alpha}x^\alpha$

3. Fundamental Metric Tensor & Conformable Fractional Christoffel Index Symbols

In this section we defined fundamental metric tensor in fractional Riemann Space. also we modify christoffel index symbols of first and second kind by involving conformable fractional derivative and studied some of its basic properties and obtained transformation of conformable fractional christoffel index symbols. Each christoffel symbol is essentially a triplet of three indices i, j and k. Associated with different combination of values of the three indices are different functions.

3.1. Fundamental Metric Tensor

The squared fractional distance between two points x^r and $x^r + dx^r$ in conformable fractional Riemann space is given by

$$\left(d^{\alpha} s\right)^2 = g_{mn}\, d^{\alpha} x^m\, d^{\alpha} x^n \tag{2}$$

Eq. (2) gives fractal line element

where

$$g_{mn} = \frac{\partial y^p}{\partial x^m} \frac{\partial y^P}{\partial x^n}\ , m, n = 1, 2, ..., N \tag{3}$$

g_{mn} is called as metric tensor, which is symmetric in m and n. The inverse metric tensor is given by $g^{mn} = \dfrac{1}{g_{mn}}$ and it is symmetric in m and n. Also $g_{pq}\, g^{mp} = \delta_q^m$

3.2. Conformable Fractional Christoffel index symbol of first kind

Consider a fractional Riemannian manifold $\left(M^{\alpha}, g^{\alpha}\right)$ and a chart; we can define the conformable fractional christoffel index symbol of first kind as

$$\left[mn, r\right]^{\alpha} = \frac{1}{2}\left\{\frac{\partial^{\alpha}\left(g_{mr}\right)}{\partial x^n} + \frac{\partial^{\alpha}\left(g_{nr}\right)}{\partial x^m} - \frac{\partial^{\alpha}\left(g_{mn}\right)}{\partial x^r}\right\} \tag{4}$$

$m, n, r = 0, 1, ..., N, 0 < \alpha \leq 1$

Where $\left[mn, r\right]^{\alpha}$, g_{mn} and α are called fractional index symbol of first kind, fundamental metric tensor and fractal dimension respectively.

3.3. Conformable Fractional Christoffel index symbol of second kind

The christoffel index symbol of the second kind is defined on fractional Riemannian manifold $\left(M^{\alpha}, g^{\alpha}\right)$ and a given chart as

$${}^{\alpha}\Gamma_{mn}^{r} = g^{rs}\left[mn, s\right]^{\alpha} = \frac{1}{2}\, g^{rs}\left\{\frac{\partial^{\alpha}\left(g_{ms}\right)}{\partial x^{n\alpha}} + \frac{\partial^{\alpha}\left(g_{ns}\right)}{\partial x^{m\alpha}} - \frac{\partial^{\alpha}\left(g_{mn}\right)}{\partial x^{s\alpha}}\right\} \tag{5}$$

$m,n,r=0,1,............,N,0<\alpha\le 1$

Example1. Let V_4^α be a fractal space with fractal line element $\left(d^\alpha s\right)^2 = d^\alpha t^2 - R_\alpha^2\, d^\alpha r^2 - S_\alpha^2\left(d^\alpha \theta^2 + \sin^2\theta\, d^\alpha \phi^2\right)$, where $R_\alpha = R_\alpha(t)$ and $S_\alpha = S_\alpha(t)$ are α differential functions of time t. Then using eq. (2) the elements of metric tensors are

$$g_{11} = -R_\alpha^2,\ g_{22} = -S_\alpha^2,\ g_{33} = -S_\alpha^2\sin^2\theta,\ g_{44} = 1 \tag{6}$$

Here consider $x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = t$

The inverse of metric tensor is obtained by using the formula

$$g^{mn} = \frac{1}{g_{mn}}$$

$$\therefore\ g^{11} = \frac{-1}{R_\alpha^2},\ g_{22} = \frac{-1}{S_\alpha^2},\ g_{33} = \frac{-1}{S_\alpha^2\sin^2\theta},\ g_{44} = 1 \tag{7}$$

Using Eq. (4) the conformable fractional christoffel symbols of first kind are

$[11,4]^\alpha = t^{1-\alpha} R_\alpha \dot{R}_\alpha$, where $\dot{R}_\alpha = \frac{dR_\alpha}{dt}$

$[14,1]^\alpha = -t^{1-\alpha} R_\alpha \dot{R}_\alpha = [41,1]^\alpha$, $\quad [22,4]^\alpha = t^{1-\alpha} S_\alpha \dot{S}_\alpha$

$[24,2]^\alpha = -t^{1-\alpha} S_\alpha \dot{S}_\alpha = [42,2]^\alpha$, $[33,4]^\alpha = t^{1-\alpha} S_\alpha \dot{S}_\alpha \sin^2\theta$

$[43,3]^\alpha = -t^{1-\alpha} S_\alpha \dot{S}_\alpha \sin^2\theta = [34,3]^\alpha$, $[33,2]^\alpha = \theta^{1-\alpha} S_\alpha^2 \sin\theta\cos\theta$

$[23,3]^\alpha = -\theta^{1-\alpha} S_\alpha^2 \sin\theta\cos\theta = [32,3]^\alpha$

The non vanishing conformable fractional christoffel index symbols of second kind are obtained using Eq. (5)

${}^\alpha\Gamma_{11}^4 = t^{1-\alpha} R_\alpha \dot{R}_\alpha$, where $\dot{R}_\alpha = \frac{dR_\alpha}{dt}$

${}^\alpha\Gamma_{14}^1 = {}^\alpha\Gamma_{41}^1 = -t^{1-\alpha}\frac{\dot{R}_\alpha}{R_\alpha}$, ${}^\alpha\Gamma_{22}^4 = -t^{1-\alpha} S_\alpha \dot{S}_\alpha$, ${}^\alpha\Gamma_{24}^2 = {}^\alpha\Gamma_{42}^2 = -t^{1-\alpha}\frac{\dot{S}_\alpha}{S_\alpha}$

$$^{\alpha}\Gamma^{4}_{33} = -t^{1-\alpha} S_{\alpha} \dot{S}_{\alpha} \sin^{2}\theta \ , \ ^{\alpha}\Gamma^{3}_{43} = {}^{\alpha}\Gamma^{3}_{34} = t^{1-\alpha} \frac{\dot{S}_{\alpha}}{S_{\alpha}} \ , \ ^{\alpha}\Gamma^{2}_{33} = \theta^{1-\alpha} \sin\theta\cos\theta$$

$$^{\alpha}\Gamma^{3}_{23} = {}^{\alpha}\Gamma^{3}_{32} = \theta^{1-\alpha} \cot\theta$$

All the result in this example will leads to standard results by choosing $\alpha = 1$

3.4. Properties of Conformable Fractional Christoffel index symbols

1. The conformable fractional christoffel index symbols are symmetric in m and n

$$i.e. [mn, r]^{\alpha} = [nm, r]^{\alpha} \text{ and } {}^{\alpha}\Gamma^{r}_{mn} = {}^{\alpha}\Gamma^{r}_{nm}$$

Proof:

$$[mn, r]^{\alpha} = \frac{1}{2}\left\{ \frac{\partial^{\alpha}(g_{mr})}{\partial x^{n\alpha}} + \frac{\partial^{\alpha}(g_{nr})}{\partial x^{m\alpha}} - \frac{\partial^{\alpha}(g_{mn})}{\partial x^{r\alpha}} \right\}$$

$$= \frac{1}{2}\left\{ \frac{\partial^{\alpha}(g_{nr})}{\partial x^{m\alpha}} + \frac{\partial^{\alpha}(g_{mr})}{\partial x^{n\alpha}} - \frac{\partial^{\alpha}(g_{nm})}{\partial x^{r\alpha}} \right\} = [nm, r]^{\alpha}$$

$$^{\alpha}\Gamma^{r}_{mn} = \frac{1}{2} g^{rs} \left\{ \frac{\partial^{\alpha}(g_{ms})}{\partial x^{n\alpha}} + \frac{\partial^{\alpha}(g_{ns})}{\partial x^{m\alpha}} - \frac{\partial^{\alpha}(g_{mn})}{\partial x^{s\alpha}} \right\}$$

$$= \frac{1}{2} g^{rs} \left\{ \frac{\partial^{\alpha}(g_{ns})}{\partial x^{m\alpha}} + \frac{\partial^{\alpha}(g_{ms})}{\partial x^{n\alpha}} - \frac{\partial^{\alpha}(g_{mn})}{\partial x^{s\alpha}} \right\} = {}^{\alpha}\Gamma^{r}_{nm}$$

2. $[mn, r]^{\alpha} = g_{rs} \ {}^{\alpha}\Gamma^{s}_{mn}$

Proof:

$$^{\alpha}\Gamma^{s}_{mn} = g^{sp}[mn, p]^{\alpha}$$

$$g_{rs} \ {}^{\alpha}\Gamma^{s}_{mn} = g_{rs} \, g^{sp}[mn, p]^{\alpha} = \delta^{p}_{r}[mn, p]^{\alpha} = [mn, r]^{\alpha}$$

3. The conformable fractional derivative of the metric tensor can be expressed as

Sum of conformable fractional christoffel index symbol of first kind

$$\text{i.e. } \frac{\partial^{\alpha}(g_{mn})}{\partial x^{r\alpha}} = [mr, n]^{\alpha} + [nr, m]^{\alpha}$$

Proof:

$$[mr,n]^{\alpha}+[nr,m]^{\alpha}=\frac{1}{2}\left\{\frac{\partial^{\alpha}(g_{mn})}{\partial x^{r\alpha}}+\frac{\partial^{\alpha}(g_{rn})}{\partial x^{m\alpha}}-\frac{\partial^{\alpha}(g_{mr})}{\partial x^{n\alpha}}\right\}+\frac{1}{2}\left\{\frac{\partial^{\alpha}(g_{nm})}{\partial x^{r\alpha}}+\frac{\partial^{\alpha}(g_{rm})}{\partial x^{n\alpha}}-\frac{\partial^{\alpha}(g_{nr})}{\partial x^{m\alpha}}\right\}$$

$$=\frac{\partial^{\alpha}(g_{mn})}{\partial x^{r\alpha}}$$

4. $\dfrac{\partial^{\alpha}\left(g^{mn}\right)}{\partial x^{r\alpha}}=-g^{ms}\,{}^{\alpha}\Gamma^{n}_{sr}-g^{ns}\,{}^{\alpha}\Gamma^{m}_{sr}$

Proof: we know that

$$g^{mp}\,g_{pq}=\delta^{m}_{q}$$

Taking conformable fractional derivative of above equation, we get

$$g^{mp}\frac{\partial^{\alpha}\left(g_{pq}\right)}{\partial x^{r\alpha}}+g_{pq}\frac{\partial^{\alpha}\left(g^{mp}\right)}{\partial x^{r\alpha}}=0$$

$$g^{qn}\,g^{mp}\frac{\partial^{\alpha}\left(g_{pq}\right)}{\partial x^{r\alpha}}+g^{qn}\,g_{pq}\frac{\partial^{\alpha}\left(g^{mp}\right)}{\partial x^{r\alpha}}=0$$

$$g^{qn}\,g^{mp}\frac{\partial^{\alpha}\left(g_{pq}\right)}{\partial x^{r\alpha}}+\delta^{n}_{p}\frac{\partial^{\alpha}\left(g^{mp}\right)}{\partial x^{r\alpha}}=0$$

$$\frac{\partial^{\alpha}\left(g^{mn}\right)}{\partial x^{r\alpha}}=-g^{qn}\,g^{mp}\frac{\partial^{\alpha}\left(g_{pq}\right)}{\partial x^{r\alpha}}=-g^{mp}\left\{g^{qn}\left[pr,q\right]^{\alpha}\right\}-g^{qn}\left\{g^{mp}\left[qr,p\right]^{\alpha}\right\}$$

$$=-g^{mp}\,{}^{\alpha}\Gamma^{n}_{pr}-g^{qn}\,{}^{\alpha}\Gamma^{m}_{qr}=-g^{ms}\,{}^{\alpha}\Gamma^{n}_{sr}-g^{sn}\,{}^{\alpha}\Gamma^{m}_{sr}$$

5. Contraction of the conformable fractional christoffel index symbol of the second kind

$${}^{\alpha}\Gamma^{m}_{mn}=\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left(\ln\sqrt{g}\right)$$

Proof:

$$g^{mn}=\frac{\text{cofactor of } g_{mn}}{g}=\frac{\Delta^{mn}}{g}$$

Now $\dfrac{\partial^{\alpha}g}{\partial x^{n\alpha}}=\Delta^{mp}\dfrac{\partial^{\alpha}\left({}^{\alpha}g^{mp}\right)}{\partial x^{n\alpha}}=g^{mp}.g\dfrac{\partial^{\alpha}\left(g^{mp}\right)}{\partial x^{n\alpha}}$

$$\frac{1}{g}\frac{\partial^\alpha g}{\partial x^{n\alpha}} = {}^\alpha\Gamma^m_{mn} + {}^\alpha\Gamma^p_{pn} = 2\,{}^\alpha\Gamma^m_{mn}$$

$$\therefore\ {}^\alpha\Gamma^m_{mn} = \frac{1}{2g}\frac{\partial^\alpha g}{\partial x^n} = \frac{\partial^\alpha}{\partial x^n}\left(\ln\sqrt{g}\right)$$

3.5. Transformation of conformable Fractional Christoffel Symbols

We know that g^{mn} is a contarvariant second order tensor

$$\therefore g'_{mn} = \frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}} g_{ab} \tag{8}$$

Taking the conformable fractional derivative of eq. (8) w. r. t. x^r we get

$$\frac{\partial^\alpha g'_{mn}}{\partial x'^{r\alpha}} = \frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\frac{\partial^\alpha g_{ab}}{\partial x'^{r\alpha}} + \frac{\partial^\alpha}{\partial x'^{r\alpha}}\left(\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\right)\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}} g_{ab} + \frac{\partial^\alpha x^a}{\partial x'^{r\alpha}}\frac{\partial^\alpha}{\partial x'^{r\alpha}}\left(\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\right)\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}} g_{ab}$$

$$\frac{\partial^\alpha g'_{mn}}{\partial x'^{r\alpha}} = \frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\frac{\partial^\alpha x^c}{\partial x'^{r\alpha}}\frac{\partial g_{ab}}{\partial x^c} + \left\{\left(x'^r\right)^{1-\alpha}\left(x'^m\right)^{1-\alpha}\frac{\partial^2 x^a}{\partial x'^r\partial x'^m} + (1-\alpha)\left(x'^r\right)^{-\alpha}\frac{\partial^\alpha x^a}{\partial x'^{r\alpha}}\right\}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}} g_{ab} +$$

$$\left\{\left(x'^r\right)^{1-\alpha}\left(x'^n\right)^{1-\alpha}\frac{\partial^2 x^b}{\partial x'^r\partial x'^n} + (1-\alpha)\left(x'^r\right)^{-\alpha}\frac{\partial^\alpha x^b}{\partial x'^{r\alpha}}\right\}\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}} g_{ab} \tag{9}$$

Rotating m, n, and r cyclically (and also a, b, c in the first term on RHS of eq. (9) we get

$$\frac{\partial^\alpha g'_{nr}}{\partial x'^{m\alpha}} = \frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\frac{\partial^\alpha x^c}{\partial x'^{r\alpha}}\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial g_{bc}}{\partial x^a} + \left\{\left(x'^m\right)^{1-\alpha}\left(x'^n\right)^{1-\alpha}\frac{\partial^2 x^a}{\partial x'^m\partial x'^n} + (1-\alpha)\left(x'^m\right)^{-\alpha}\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\right\}\frac{\partial^\alpha x^b}{\partial x'^{r\alpha}} g_{ab} +$$

$$\left\{\left(x'^m\right)^{1-\alpha}\left(x'^r\right)^{1-\alpha}\frac{\partial^2 x^b}{\partial x'^m\partial x'^r} + (1-\alpha)\left(x'^m\right)^{-\alpha}\frac{\partial^\alpha x^b}{\partial x'^{m\alpha}}\right\}\frac{\partial^\alpha x^a}{\partial x'^{n\alpha}} g_{ab} \tag{10}$$

$$\frac{\partial^\alpha g'_{rm}}{\partial x'^{n\alpha}} = \frac{\partial^\alpha x^c}{\partial x'^{r\alpha}}\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\frac{\partial g_{ca}}{\partial x^c} + \left\{\left(x'^n\right)^{1-\alpha}\left(x'^r\right)^{1-\alpha}\frac{\partial^2 x^a}{\partial x'^n\partial x'^r} + (1-\alpha)\left(x'^n\right)^{-\alpha}\frac{\partial^\alpha x^a}{\partial x'^{n\alpha}}\right\}\frac{\partial^\alpha x^b}{\partial x'^{m\alpha}} g_{ab} +$$

$$\left\{\left(x'^n\right)^{1-\alpha}\left(x'^m\right)^{1-\alpha}\frac{\partial^2 x^b}{\partial x'^n\partial x'^m} + (1-\alpha)\left(x'^m\right)^{-\alpha}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\right\}\frac{\partial^\alpha x^a}{\partial x'^{r\alpha}} g_{ab} \tag{11}$$

Eq. (10) +Eq. (11)-Eq. (9) gives.

$$\frac{\partial^\alpha g'_{nr}}{\partial x'^{m\alpha}} + \frac{\partial^\alpha g'_{rm}}{\partial x'^{n\alpha}} - \frac{\partial^\alpha g'_{mn}}{\partial x'^{r\alpha}} = \frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\frac{\partial^\alpha x^c}{\partial x'^{r\alpha}}\left[\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ca}}{\partial x^c} - \frac{\partial g_{ab}}{\partial x^c}\right] + B_1 + B_2 + B_3$$

$$2\left[mn,r\right]^{'\alpha}=2\left[ab,c\right]\frac{\partial^{\alpha}x^{a}}{\partial x^{'m\alpha}}\frac{\partial^{\alpha}x^{b}}{\partial x^{'n\alpha}}\frac{\partial^{\alpha}x^{c}}{\partial x^{'r\alpha}}+B_1+B_2+B_3 \tag{12}$$

Where $B_1=-\left(x^{'r}\right)^{1-\alpha}\left(x^{'m}\right)^{1-\alpha}\frac{\partial^{2}x^{a}}{\partial x^{'r}\partial x^{'m}}\frac{\partial^{\alpha}x^{b}}{\partial x^{'n\alpha}}g_{ab}+\left(x^{'m}\right)^{1-\alpha}\left(x^{'r}\right)^{1-\alpha}\frac{\partial^{2}x^{b}}{\partial x^{'m}\partial x^{'r}}\frac{\partial^{\alpha}x^{a}}{\partial x^{'n\alpha}}g_{ab}$

$$-\left(1-\alpha\right)\left(x^{'r}\right)^{-\alpha}\frac{\partial^{\alpha}x^{a}}{\partial x^{'r\alpha}}\frac{\partial^{\alpha}x^{b}}{\partial x^{'n\alpha}}g_{ab}+\left(1-\alpha\right)\left(x^{'m}\right)^{-\alpha}\frac{\partial^{\alpha}x^{b}}{\partial x^{'m\alpha}}\frac{\partial^{\alpha}x^{a}}{\partial x^{'n\alpha}}g_{ab} \tag{13}$$

By interchanging a & b in the second and third term and replacing m by r in the last term of Eq. (13), we get

$$B_1=0 \tag{14}$$

$$B_2=-\left(x^{'r}\right)^{1-\alpha}\left(x^{'n}\right)^{1-\alpha}\frac{\partial^{2}x^{b}}{\partial x^{'r}\partial x^{'n}}\frac{\partial^{\alpha}x^{a}}{\partial x^{'m\alpha}}g_{ab}+\left(x^{'n}\right)^{1-\alpha}\left(x^{'r}\right)^{1-\alpha}\frac{\partial^{2}x^{a}}{\partial x^{'r}\partial x^{'n}}\frac{\partial^{\alpha}x^{b}}{\partial x^{'m\alpha}}g_{ab}$$

$$-\left(1-\alpha\right)\left(x^{'r}\right)^{-\alpha}\frac{\partial^{\alpha}x^{a}}{\partial x^{'m\alpha}}\frac{\partial^{\alpha}x^{b}}{\partial x^{'r\alpha}}g_{ab}+\left(1-\alpha\right)\left(x^{'n}\right)^{-\alpha}\frac{\partial^{\alpha}x^{b}}{\partial x^{'m\alpha}}\frac{\partial^{\alpha}x^{a}}{\partial x^{'n\alpha}}g_{ab} \tag{15}$$

By interchanging a & b in the second and third term and replacing n by r in the last term of Eq. (15), we get

$$B_2=0 \tag{16}$$

$$B_3=\left(x^{'m}\right)^{1-\alpha}\left(x^{'n}\right)^{1-\alpha}\frac{\partial^{2}x^{a}}{\partial x^{'m}\partial x^{'n}}\frac{\partial^{\alpha}x^{b}}{\partial x^{'r\alpha}}g_{ab}+\left(x^{'m}\right)^{1-\alpha}\left(x^{'n}\right)^{1-\alpha}\frac{\partial^{2}x^{b}}{\partial x^{'m}\partial x^{'n}}\frac{\partial^{\alpha}x^{a}}{\partial x^{'r\alpha}}g_{ab}$$

$$+\left(1-\alpha\right)\left(x^{'m}\right)^{-\alpha}\frac{\partial^{\alpha}x^{a}}{\partial x^{'m\alpha}}\frac{\partial^{\alpha}x^{b}}{\partial x^{'r\alpha}}g_{ab}+\left(1-\alpha\right)\left(x^{'n}\right)^{-\alpha}\frac{\partial^{\alpha}x^{b}}{\partial x^{'n\alpha}}\frac{\partial^{\alpha}x^{a}}{\partial x^{'r\alpha}}g_{ab} \tag{17}$$

By interchanging a & b in the second and fourth term and replacing n by m in the last term of Eq. (17), we get

$$B_3=2\left\{\left(x^{'m}\right)^{1-\alpha}\left(x^{'n}\right)^{1-\alpha}\frac{\partial^{2}x^{a}}{\partial x^{'m}\partial x^{'n}}+\left(1-\alpha\right)\left(x^{'m}\right)^{-\alpha}\frac{\partial^{\alpha}x^{a}}{\partial x^{'m\alpha}}\right\}\frac{\partial^{\alpha}x^{b}}{\partial x^{'r\alpha}}g_{ab}$$

$$\therefore B_3=2\frac{\partial^{\alpha}x^{b}}{\partial x^{'r\alpha}}\frac{\partial^{\alpha}}{\partial x^{'m\alpha}}\left(\frac{\partial^{\alpha}x^{a}}{\partial x^{'n\alpha}}\right)g_{ab} \tag{18}$$

Substituting $B_1, B_2 \text{ and } B_3$ in eq. (12) we get,

$$\therefore [mn,r]'^{\alpha} = [ab,c]\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\frac{\partial^\alpha x^c}{\partial x'^{r\alpha}} + \frac{\partial^\alpha x^b}{\partial x'^{r\alpha}}\frac{\partial^\alpha}{\partial x'^{m\alpha}}\left(\frac{\partial^\alpha x^a}{\partial x'^{n\alpha}}\right)g_{ab} \tag{19}$$

Eq. (19) gives the transformation of conformable fractional christoffel symbols of first kind.

Also we know that $g^{\prime rp}$ is a covariant tensor of second order

$$\therefore g'^{rp} = \frac{\partial^\alpha x'^r}{\partial x^{s\alpha}}\frac{\partial^\alpha x'^p}{\partial x^{q\alpha}}g^{sq} \tag{20}$$

$$\therefore\ [mn,r]'^{\alpha} g'^{rp} = [ab,c]\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\frac{\partial^\alpha x^c}{\partial x^{s\alpha}}\frac{\partial^\alpha x'^p}{\partial x'^{q\alpha}}g^{sq} + \frac{\partial^\alpha}{\partial x'^{m\alpha}}\left(\frac{\partial^\alpha x^a}{\partial x'^{n\alpha}}\right)\frac{\partial^\alpha x^b}{\partial x'^{r\alpha}}\frac{\partial^\alpha x'^r}{\partial x^{s\alpha}}\frac{\partial^\alpha x'^p}{\partial x^{q\alpha}}g^{sq}g_{ab}$$

$$= [ab,c]\delta^c_s\left(x^s\right)^{1-\alpha} g^{sq}\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\frac{\partial^\alpha x'^p}{\partial x^{q\alpha}} + \frac{\partial^\alpha}{\partial x'^{m\alpha}}\left(\frac{\partial^\alpha x^a}{\partial x'^{n\alpha}}\right)\left(x^s\right)^{1-\alpha}\delta^b_s\frac{\partial^\alpha x'^p}{\partial x^{q\alpha}}g^{sq}g_{ab}$$

$$= [ab,s]g^{sq}\left(x^s\right)^{1-\alpha}\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\frac{\partial^\alpha x'^p}{\partial x^{q\alpha}} + \frac{\partial^\alpha}{\partial x'^{m\alpha}}\left(\frac{\partial^\alpha x^a}{\partial x'^{n\alpha}}\right)\left(x^s\right)^{1-\alpha}\frac{\partial^\alpha x'^p}{\partial x^{q\alpha}}g^{sq}g_{as}$$

$$= \Gamma^q_{ab}\left(x^s\right)^{1-\alpha}\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}}\frac{\partial^\alpha x'^p}{\partial x^{q\alpha}} + \frac{\partial^\alpha}{\partial x'^{m\alpha}}\left(\frac{\partial^\alpha x^q}{\partial x'^{n\alpha}}\right)\left(x^s\right)^{1-\alpha}\frac{\partial^\alpha x'^p}{\partial x^{q\alpha}}$$

$${}^{\alpha}\Gamma'^{p}_{mn} = \left\{\Gamma^s_{ab}\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}} + \frac{\partial^\alpha}{\partial x'^{m\alpha}}\left(\frac{\partial^\alpha x^s}{\partial x'^{n\alpha}}\right)\right\}\left(x^s\right)^{1-\alpha}\frac{\partial^\alpha x'^p}{\partial x^{s\alpha}}$$

$$\therefore {}^{\alpha}\Gamma'^{r}_{mn} = \left\{\Gamma^s_{ab}\frac{\partial^\alpha x^a}{\partial x'^{m\alpha}}\frac{\partial^\alpha x^b}{\partial x'^{n\alpha}} + \frac{\partial^\alpha}{\partial x'^{m\alpha}}\left(\frac{\partial^\alpha x^s}{\partial x'^{n\alpha}}\right)\right\}\left(x^s\right)^{1-\alpha}\frac{\partial^\alpha x'^r}{\partial x^{s\alpha}} \tag{21}$$

Eq. (21) gives the transformation of conformable fractional christoffel symbols of second kind.

4. Conformable Fractional Derivatives of Tensor

In this section we study two types of conformable fractional derivatives of tensors; first one is conformable fractional Absolute derivative and second is conformable fractional covariant derivative. The covariant derivative is generalization of partial derivative, while the absolute derivative is generalization of ordinary derivative.

4.1 Conformable Fractional Absolute Derivative

Let C^α be a fractal space curve which is described by the parametric equation in V_N^α such as $C^\alpha : x^{r\alpha} = x^{r\alpha}(u)$ where u is a parameter. Then for any contravariant vector T^r along the curve C^α we define conformable fractional absolute derivative as

$$\frac{\delta^\alpha T^r}{\delta u^\alpha} = \frac{d^\alpha T^r}{du^\alpha} + {}^\alpha\Gamma^r_{mn} T^m \frac{d^\alpha x^n}{du^\alpha} \tag{22}$$

and for covariant vector T_r, the conformable fractional absolute derivative is given by

$$\frac{\delta^\alpha T_r}{\delta u^\alpha} = \frac{d^\alpha T_r}{du^\alpha} - T_m \, {}^\alpha\Gamma^r_{mn} \frac{d^\alpha x^n}{du^\alpha} \tag{23}$$

Similarly we can define the conformable fractional absolute derivative of higher order tensors

$$\frac{\delta^\alpha T^{rs}}{\delta u^\alpha} = \frac{d^\alpha T^{rs}}{du^\alpha} + T^{ms} \, {}^\alpha\Gamma^r_{mn} \frac{d^\alpha x^n}{du^\alpha} + T^{rm} \, {}^\alpha\Gamma^s_{mn} \frac{d^\alpha x^n}{du^\alpha} \tag{24}$$

$$\frac{\delta^\alpha T_{rs}}{\delta u^\alpha} = \frac{d^\alpha T_{rs}}{du^\alpha} - T_{ms} \, {}^\alpha\Gamma^r_{mn} \frac{d^\alpha x^n}{du^\alpha} - T_{rm} \, {}^\alpha\Gamma^m_{sn} \frac{d^\alpha x^n}{du^\alpha} \tag{25}$$

$$\frac{\delta^\alpha T^r_s}{\delta u^\alpha} = \frac{d^\alpha T^r_s}{du^\alpha} + T^m_s \, {}^\alpha\Gamma^r_{mn} \frac{d^\alpha x^n}{du^\alpha} - T^r_m \, {}^\alpha\Gamma^m_{sn} \frac{d^\alpha x^n}{du^\alpha} \tag{26}$$

Conformable fractional absolute derivative describes the rate of change of vector field along the curve. And it also preserves the order and type of the tensor.

Theorem: Let A and B be any tensors then $\frac{\delta^\alpha}{\delta u^\alpha}(A+B) = \frac{\delta^\alpha A}{\delta u^\alpha} + \frac{\delta^\alpha B}{\delta u^\alpha}$

Proof:

$$\frac{\delta^\alpha}{\delta u^\alpha}(A+B) = \frac{d^\alpha}{du^\alpha}(A+B) + {}^\alpha\Gamma^r_{mn}(A+B)\frac{d^\alpha x^n}{du^\alpha}$$

$$= \left(\frac{d^\alpha A}{du^\alpha} + {}^\alpha\Gamma^r_{mn} A \frac{d^\alpha x^n}{du^\alpha}\right) + \left(\frac{d^\alpha B}{du^\alpha} + {}^\alpha\Gamma^r_{mn} B \frac{d^\alpha x^n}{du^\alpha}\right) = \frac{\delta^\alpha A}{\delta u^\alpha} + \frac{\delta^\alpha B}{\delta u^\alpha}$$

$$\therefore \frac{\delta^\alpha}{\delta u^\alpha}(A+B) = \frac{\delta^\alpha A}{\delta u^\alpha} + \frac{\delta^\alpha B}{\delta u^\alpha}$$

4.2 Conformable Fractional Covariant Derivative

Consider a contarvariant vector field T^r defined in the region R^α of V_N^α . The conformable fractional contravariant derivative of T^r along any curve C^α in the region R^α is defined as

$$ {}^\alpha T^r;u = \frac{\partial^\alpha T^r}{\partial x^{u\alpha}} + T^m \ {}^\alpha\Gamma^r_{mu} \tag{27}$$

${}^\alpha T^r;u$ is a mixed tensor and it is called as conformable fractional covariant derivative of T^r with respective fundamental metric tensor g_{mn} .

For a covariant vector field we have

$$ {}^\alpha T_r;u = \frac{\partial^\alpha T^r}{\partial x^{u\alpha}} - T_m \ {}^\alpha\Gamma^m_{ru} \tag{28}$$

Similarly we can define the conformable fractional contravariant derivatives of higher order tensors.

$$ {}^\alpha T^{rs};u = \frac{\partial^\alpha T^{rs}}{\partial x^{u\alpha}} + T^{ms} \ {}^\alpha\Gamma^r_{mu} + T^{rm} \ {}^\alpha\Gamma^s_{mu} \tag{29}$$

$$ {}^\alpha T_{rs};u = \frac{\partial^\alpha T_{rs}}{\partial x^{u\alpha}} - T_{ms} \ {}^\alpha\Gamma^m_{ru} - T_{rm} \ {}^\alpha\Gamma^m_{su} \tag{30}$$

$$ {}^\alpha T^r_s;u = \frac{\partial^\alpha T^r_s}{\partial x^{u\alpha}} + T^m_s \ {}^\alpha\Gamma^r_{mu} - T^r_m \ {}^\alpha\Gamma^m_{su} \tag{31}$$

The conformable fractional covariant derivative describes how fast vector field varies in all directions. It also increases the order of tensor by one in covariant place.

Theorem: The Conformable fractional covariant derivative of g_{mn} and g^{mn} vanishes.

Proof: The conformable fractional covariant derivative of g_{mn} is given by

$$ {}^\alpha g_{mn;r} = \frac{\partial^\alpha}{\partial x^r}\left(g_{mn}\right) - {}^\alpha\Gamma^s_{mr} \ {}^\alpha g_{sn} - {}^\alpha\Gamma^s_{nr} \, g_{ms}$$

$$ = \left[mr,n\right]^\alpha + \left[nr,m\right]^\alpha - \left[mr,n\right]^\alpha - \left[nr,m\right]^\alpha = 0$$

$$ \therefore {}^\alpha g_{mn;r} = 0$$

The conformable fractional covariant derivative g^{mn} is given by

$$^{\alpha}g^{mn}{}_{;r}=\frac{\partial^{\alpha}}{\partial x^{r}}\left(g^{mn}\right)+{}^{\alpha}\Gamma^{m}_{sr}\,g^{sn}+{}^{\alpha}\Gamma^{n}_{sr}\,g^{ms}$$

$$=-{}^{\alpha}\Gamma^{m}_{sr}\,g^{sn}-{}^{\alpha}\Gamma^{n}_{sr}\,g^{ms}+{}^{\alpha}\Gamma^{m}_{sr}\,g^{sn}+{}^{\alpha}\Gamma^{n}_{sr}\,g^{ms}=0$$

$$\therefore {}^{\alpha}g^{mn}{}_{;r}=0$$

Theorem: Let A and B be any tensors then $^{\alpha}\left(A+B\right);n={}^{\alpha}\left(A\right);n+{}^{\alpha}\left(B\right);n$

Proof:

$$^{\alpha}\left(A+B\right);n=\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left(A+B\right)+{}^{\alpha}\Gamma^{r}_{mn}\left(A+B\right)$$

$$=\left(\frac{\partial^{\alpha}A}{\partial x^{n\alpha}}+{}^{\alpha}\Gamma^{r}_{mn}\,A\right)+\left(\frac{\partial^{\alpha}B}{\partial x^{n\alpha}}+{}^{\alpha}\Gamma^{r}_{mn}\,B\right)={}^{\alpha}\left(A\right);n+{}^{\alpha}\left(A\right);n$$

$$\therefore {}^{\alpha}\left(A+B\right);n={}^{\alpha}\left(A\right);n+{}^{\alpha}\left(B\right);n$$

4.3 Conformable Fractional Divergence

Let T^{r} be a contravariant vector field defined in a region R^{α}, the conformable fractional divergence of T^{r} is the contraction of the conformable fractional covariant derivative of T^{r} and it is denoted by $div^{\alpha}\left(T^{r}\right)$

$$\therefore div^{\alpha}\left(T^{r}\right)={}^{\alpha}T^{r};r$$

$$=\frac{\partial^{\alpha}T^{r}}{\partial x^{r\alpha}}+{}^{\alpha}\Gamma^{r}_{rm}\,T^{m}$$

$$=\frac{\partial^{\alpha}T^{r}}{\partial x^{r\alpha}}+\frac{1}{2}g^{rs}\left\{\frac{\partial^{\alpha}g_{rs}}{\partial x^{m\alpha}}+\frac{\partial^{\alpha}g_{ms}}{\partial x^{r\alpha}}-\frac{\partial^{\alpha}g_{rs}}{\partial x^{s\alpha}}\right\}T^{m}$$

$$=\frac{\partial^{\alpha}T^{r}}{\partial x^{r\alpha}}+\frac{1}{2}\left\{g^{rs}\frac{\partial^{\alpha}g_{rs}}{\partial x^{m\alpha}}+g^{rs}\frac{\partial^{\alpha}g_{ms}}{\partial x^{r\alpha}}-g^{rs}\frac{\partial^{\alpha}g_{rm}}{\partial x^{s\alpha}}\right\}T^{m}$$

$$\therefore div^{\alpha}\left(T^{r}\right)=\frac{\partial^{\alpha}T^{r}}{\partial x^{r\alpha}}+\frac{1}{2}g^{rs}\left\{\frac{\partial^{\alpha}g_{rs}}{\partial x^{m\alpha}}\right\}T^{m} \qquad (32)$$

Above equation gives the conformable fractional divergence of the contravariant vector

Theorem: $div^{\alpha} T^{r} = \frac{1}{\sqrt{g}} \frac{\partial^{\alpha}}{\partial x^{r\alpha}} \left(T^{r} \sqrt{g} \right)$

Proof:

$$div^{\alpha} T^{r} = \frac{\partial^{\alpha} T^{r}}{\partial x^{r\alpha}} + {}^{\alpha}\Gamma^{r}_{rm} T^{m}$$

$$= \frac{\partial^{\alpha} T^{r}}{\partial x^{r\alpha}} + \frac{\partial^{\alpha}}{\partial x^{m\alpha}} \left(\ln \sqrt{g} \right) T^{m} \qquad = \frac{\partial^{\alpha} T^{r}}{\partial x^{r\alpha}} + \frac{\partial^{\alpha}}{\partial x^{r\alpha}} \left(\ln \sqrt{g} \right) T^{r}$$

$$= \frac{\partial^{\alpha} T^{r}}{\partial x^{r\alpha}} + \left(\frac{1}{\sqrt{g}} \frac{\partial^{\alpha} \sqrt{g}}{\partial x^{r\alpha}} \right) T^{r}$$

$$\therefore div^{\alpha} T^{r} = \frac{1}{\sqrt{g}} \frac{\partial^{\alpha}}{\partial x^{r\alpha}} \left(T^{r} \sqrt{g} \right) \tag{33}$$

5. CONCLUSIONS

In this work we generalized the tensor calculus by using the conformable fractional derivative. The conformable fractional christoffel index symbols are suggested and its properties are studied, the transformations of conformable fractional christoffel index symbols are obtained. We define the conformable fractional absolute and covariant derivative of tensor and verified the summation rule of differentiation. The conformable fractional divergence is defined.

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References

1. S.G. Samka, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives; Theory and Applications Gordon and Breah. Y. Verdon et. Alibi, 1993.
2. K. Oldham, Spanier, The Fractional Calculus, Theory and Applications of Differentiation and Integration of Arbitatary Order, Academic press U.S.A. (1974).
3. K.S.Miller, An Introduction to Fractional Calculus and Fractional Differential Equations, J.Wiely and Sons. New York (1993).
4. I.podlubny, Fractional Differential Equations, Academic Press, USA (1999).
5. A. El- Nabulsi, R, Fractional Unstable Euclidean Universe, EJTP 8 p.111(2005),
6. E.Rabei, B. Ababneh, Hamiltonian Jacobi Fractional Mechanics, J, math Anal. Appl. 334, 799-805(2008).
7. El. Nabulsi Ahmed Rami, Rexamined Friedman Robertson -Walker Universe in the Context of Fractionally Differentiated Lagrangian Function Depending on Parameter α
8. El. Nabulsi Ahmed Rami, Cosmology with Fractional Action Principle. Romanian Reports in Physics. vol.59,No.3,763-771(2007)

9. El. Nabulsi Ahmed Rami, Differential Geometry and Modern Cosmology with Fractionally Differentiated Lagrangian Function and Fractional Decaying Force Term. Rom. Journ .phys. vol.52 No. 3-4 467-481, Bucharest(2007)
10. V.K. Shchigolev, Cosmological Models with Fractional Derivatives and Fractional action Functional, Commun. Theo. Physics. Vol.56,No-2,389-396(2011)
11. F. Riewe, Neoconservative Lagrangian and Hamiltonian Mechanics.phys. Rev. E53(1996) 1890
12. M.Klimek. Fractional Sequential Mechanics – Models With Symmetric Fractional Derivative. Czeh J.phys.51, 1348 (2001)
13. M.Naber. Time Fractional Schrodinger Equation. Journal of Mathematical physics.45, 3339 (2004)
14. Alireza K. Golmankhaneh, Xiao-Jun Yang and D.Baleanu, Einstein Field Equations within Local Fractional Calculus, Rom. Journ.Phys. vol.60,No. 1-2, 22-31, Bucharest(2015)
15. Khalial R, Al Horani M, Yousef A.Sababheh M.A , A New Definition of Fractional Derivative J. Comput, Appl.math. 264, 65-70 (2014)
16. A.Atangana et.al. New Properties of Conformable Derivative. Central European Journal of Mathematics.(2015)
17. T.Abdeljawad, On Conformable Fractional Calculus journal of computational and applied mathematics 279, 57-66 (2015)
18. A. Tallafha , S.Al Hihi, Total and Dirctional Fractional Derivatives. International Journal of Pure and Applied Mathematics 107, 1037-1051(2016)
19. Matheus J. Lazo and Delfim F.M. Torres, Variational Calculus with Conformable Fractional Derivatives. IEEE/CAA Journal of Automatica Sincia, vol.4, No.2 (2017)
20. W.S. Chung, Fractional Newtonian Mechanics with Conformable Fractional Derivative. Journal of computational and applied mathematics. 290 ,150-158 (2015)
21. R C Tolman : Relativity, Thermodynamics and Cosmology
22. Adler, Bazin and Schiffer: Introduction to General relativity.