Article

On Sun's Identities for the Stirling Numbers

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Abstract

We show that the formulas of Zhi-Hong Sun for the Stirling numbers are consequences of the known identities of Schläfli and Gould.

Keywords: Gould, Schläfli, formulas, Stirling numbers.

1. Introduction

Sun [1] used the inversion technique to obtain the following expressions:

$$S_{m+n}^{[n]} = \sum_{k=0}^{m} \binom{m-n}{m-k} \binom{m+n}{m+k} S_{m+k}^{[k]}\,, \tag{1}$$

$$S_{m+n}^{(n)} = \sum_{k=0}^{m} \binom{m-n}{m-k} \binom{m+n}{m+k} S_{m+k}^{(k)}\,, \tag{2}$$

for the Stirling numbers $S_r^{(j)}$ and $S_q^{[t]}$ of the first and second kind, respectively [2].

In Sec. 2 we show that the Schläfli's identity [2-5] implies (1); similarly, we exhibit that the Gould's relation [2, 5, 6] generates the property (2).

2. Sun's formulas

Knuth [7] comments that Gould was the first to extend the domain of Stirling numbers to negative values of the indices, in fact [2]:

$$S_{-n}^{(-N)} = (-1)^{N+n}\ S_N^{[n]}\,, \tag{3}$$

$$S_{-n}^{[-N]} = (-1)^{N+n}\ S_N^{(n)}\,, \tag{4}$$

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On the other hand, we have the Schläfli's relation [2-5]:

$$S_n^{(n-m)} = (-1)^m \sum_{k=0}^{m} \binom{m+n}{m-k} \binom{m-n}{m+k} S_{m+k}^{[k]} , \tag{5}$$

where we realize the change $n \to -n$ to obtain:

$$S_{-n}^{(-(m+n))} = (-1)^m \sum_{k=0}^{m} \binom{m-n}{m-k} \binom{m+n}{m+k} S_{m+k}^{[k]} ,$$

which implies (1) through (3).

The Gould's identity is given by [2, 5, 6]:

$$S_n^{[n-m]} = (-1)^m \sum_{k=0}^{m} \binom{m+n}{m-k} \binom{m-n}{m+k} S_{m+k}^{(k)} , \tag{6}$$

where we make the change $n \to -n$ to deduce:

$$S_{-n}^{[-(m+n)]} = (-1)^m \sum_{k=0}^{m} \binom{m-n}{m-k} \binom{m+n}{m+k} S_{m+k}^{(k)} ,$$

which is equivalent to (2) due to (4).

Thus, we see that the expressions (1) and (2) found by Sun [1] via inversion technique, are implied by the formulas of Schläfli and Gould.

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