Article

On an Identity of Wilf for the Euler-Mascheroni's Constant

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Abstract

Wilf (1997) proposed to show certain identity involving the Euler-Mascheroni's constant. Here we employ properties of the gamma function to exhibit an elementary proof of these identity.

Keywords: Constant of Euler-Mascheroni, gamma function, Newman-Weierstrass identity.

1. Introduction

Wilf [1] proposed to prove the identity [2-4]:

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\, e^{\gamma}\, \prod_{k=1}^{\infty} e^{-\frac{1}{k}} \left(1 + \frac{1}{k} + \frac{1}{2\,k^2}\right), \tag{1}$$

where γ =0.5772 1566 4901 5328 6060 … is the Euler-Mascheroni's constant [4, 5].

In Sec. 2 we use properties of the gamma function [4-9] to give an elementary deduction of (1).

2. Wilf's formula

In [10] we find the following relation involving an infinite product and the gamma function:

$$\prod_{k=m}^{\infty} \frac{(k+z)^2-b}{(k+z)^2-a} = \frac{\Gamma(z+m-\sqrt{a})\,\Gamma(z+m+\sqrt{a})}{\Gamma(z+m-\sqrt{b})\,\Gamma(z+m+\sqrt{b})}, \tag{2}$$

where we can employ $a = -b = \frac{1}{4}$, $m = 1$ and $z = \frac{1}{2}$ to obtain the expression:

$$\prod_{k=1}^{\infty} \frac{(2k+1)^2+1}{(2k+1)^2-1} \equiv \prod_{k=1}^{\infty} \left[1 + \frac{1}{2k\,(k+1)}\right] = \frac{1}{\Gamma\left(\frac{3-i}{2}\right)\Gamma\left(\frac{3+i}{2}\right)} = \frac{2}{\Gamma\left(\frac{1-i}{2}\right)\Gamma\left(\frac{1+i}{2}\right)}\,. \tag{3}$$

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On the other hand, we know the property [7]:

$$\frac{\pi}{\cosh(\pi\, x)} = \Gamma\left(\frac{1}{2} - i\, x\right)\Gamma(\frac{1}{2} + i\, x), \tag{4}$$

that we can apply with $x = \frac{1}{2}$ into (3) to deduce the identity:

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \prod_{k=1}^{\infty} [1 + \frac{1}{2k\,(k+1)}]. \tag{5}$$

Now we observe the relation:

$$\frac{1 + \frac{1}{k} + \frac{1}{2k^2}}{1 + \frac{1}{k}} = 1 + \frac{1}{2k\,(k+1)}\,, \tag{6}$$

then (5) is equivalent to:

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\; \frac{\prod_{k=1}^{\infty} e^{-\frac{1}{k}}\,(1 + \frac{1}{k} + \frac{1}{2k^2})}{\prod_{r=1}^{\infty} e^{-\frac{1}{r}}\,(1 + \frac{1}{r})}\,, \tag{7}$$

but the Newman (1848)-Weierstrass (1856) formula [7]:

$$z\, e^{\gamma z} \prod_{r=1}^{\infty} e^{-\frac{z}{r}}\, (1 + \frac{z}{r}) = \frac{1}{\Gamma(z)}\,, \tag{8}$$

with $z = 1$ gives the expression:

$$\prod_{r=1}^{\infty} e^{-\frac{1}{r}}\left(1 + \frac{1}{r}\right) = e^{-\gamma}, \tag{9}$$

whose application in (7) implies the Wilf's identity (1), q.e.d.

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