On an Identity of Wilf for the Euler-Mascheroni’s Constant

J. López-Bonilla* & R. López-Vázquez

ESIME-Zacatenco, Instituto Politécnico Nacional, CDMX, México

Abstract
Wilf (1997) proposed to show certain identity involving the Euler-Mascheroni’s constant. Here we employ properties of the gamma function to exhibit an elementary proof of these identity.

Keywords: Constant of Euler-Mascheroni, gamma function, Newman-Weierstrass identity.

1. Introduction
Wilf [1] proposed to prove the identity [2-4]:
\[
\cosh \left( \frac{\pi}{2} \right) = \pi e^\frac{\gamma}{2} \prod_{k=1}^{\infty} \left( 1 + \frac{1}{k + \frac{1}{2 k^2}} \right),
\]
(1)
where \( \gamma = 0.57721566490153286060 \ldots \) is the Euler-Mascheroni’s constant [4, 5].

In Sec. 2 we use properties of the gamma function [4-9] to give an elementary deduction of (1).

2. Wilf’s formula
In [10] we find the following relation involving an infinite product and the gamma function:
\[
\prod_{k=m}^{\infty} \frac{(k+z)^2-b}{(k+z)^2-a} = \frac{\Gamma(z+m-\sqrt{a})}{\Gamma(z+m+\sqrt{a})} \frac{\Gamma(z+m+\sqrt{b})}{\Gamma(z+m-\sqrt{b})},
\]
(2)
where we can employ \( a = -b = \frac{1}{4} \), \( m = 1 \) and \( z = \frac{1}{2} \) to obtain the expression:
\[
\prod_{k=1}^{\infty} \frac{(2k+1)^{2}+1}{(2k+1)^{2}-1} \equiv \prod_{k=1}^{\infty} \left[ 1 + \frac{1}{2k (k+1)} \right] = \frac{1}{\Gamma\left(\frac{3-i}{2}\right) \Gamma\left(\frac{3+i}{2}\right)} = \frac{2}{\Gamma\left(\frac{1-i}{2}\right) \Gamma\left(\frac{1+i}{2}\right)}. \]
(3)

*Correspondence: J. López-Bonilla. ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5, 1er. Piso, Col. Lindavista CP 07738, CDMX, México. Email: jlopezb@ipn.mx
On the other hand, we know the property [7]:

$$\frac{\pi}{\cosh(\pi x)} = \Gamma\left(\frac{1}{2} - i x\right) \Gamma\left(\frac{1}{2} + i x\right),$$

that we can apply with $x = \frac{1}{2}$ into (3) to deduce the identity:

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \prod_{k=1}^{\infty} \left[ 1 + \frac{1}{2k (k+1)} \right].$$

Now we observe the relation:

$$\frac{1 + \frac{1}{K} + \frac{1}{2K^2}}{1 + \frac{1}{K}} = 1 + \frac{1}{2k (k+1)} \ ,$$

then (5) is equivalent to:

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \frac{\prod_{k=1}^{\infty} e^{\frac{1}{K}} (1 + \frac{1}{K} + \frac{1}{2K^2})}{\prod_{r=1}^{\infty} e^{-\frac{1}{r}} (1 + \frac{1}{r^2})} \ ,$$

but the Newman (1848)-Weierstrass (1856) formula [7]:

$$z e^{rz} \prod_{r=1}^{\infty} e^{-\frac{x}{r}} (1 + \frac{x}{r^2}) = \frac{1}{\Gamma(z)} ,$$

with $z = 1$ gives the expression:

$$\prod_{r=1}^{\infty} e^{-\frac{1}{r}} (1 + \frac{1}{r}) = e^{-\gamma} ,$$

whose application in (7) implies the Wilf’s identity (1), q.e.d.

Received June 17, 2018; Accepted July 15, 2018
References

10. http://www-elsa.physik.uni-bonn.de/~dieckman/InfProd/InfProd.html#InfinitexProducts