Article

A Note on He-Ricci's Identity

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Abstract
We employ a formula of Nörlund's to give an elementary proof of He-Ricci's identity for the Bernoulli polynomials.

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1. Introduction

He-Ricci [1, 2] used the factorization method to obtain the property:

$$B_n(x) = \left(x - \frac{1}{2}\right) B_{n-1}(x) - \frac{1}{n} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k} \cdot B_k(x), \qquad n \geq 1, \tag{1}$$

where $B_n(x)$ and $B_n = B_n(0)$ are the polynomials and numbers of Bernoulli [3], respectively.

In the next section, we indicate that (1) is a particular case of an identity of Nörlund [4-6].

2. He-Ricci's Formula

Nörlund [4] proved the expression:

$$\sum_{k=0}^{n} \binom{n}{k} B_{n-k}(y)\, B_k(x) = n(x + y - 1)\, B_{n-1}(x + y) - (n - 1)\, B_n(x + y), \tag{2}$$

then (1) comes immediately from (2) for $y = 0$. From (2) with $x = y = 0$, we deduce the known convolution identity on Bernoulli numbers [7]:

$$\sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, B_k = -n\, B_{n-1} - (n - 1)\, B_n, \qquad n \geq 1. \tag{3}$$

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On the other hand, we have [3] the property $\frac{d}{dx} B_k(x) = k\, B_{k-1}(x)$, therefore [8]:

$$\binom{n}{n-r} B_{n-r}(x) = \frac{1}{r!} \frac{d^r}{dx^r} B_n(x), \tag{4}$$

whose application into (1) implies that $B_n(x)$ satisfies the differential equation [1]:

$$\frac{B_n}{n!} y^{(n)} + \frac{B_{n-1}}{(n-1)!} y^{(n-1)} + \cdots + \frac{B_2}{2!} y'' + \left(\frac{1}{2} - x\right) y' + n\, y = 0, \qquad n \geq 2. \tag{5}$$

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