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Characterizations of the Spacelike Curves in the 3-Dimentional Lightlike Cone

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Abstract. In this paper, m-curvature functions are defined for spacelike curve in the 3-dimensional null cone. Some characterizations to deal with the spacelike curve concept are obtained on this curvature functions.

Keywords: Null cone, m-curvature functions.

1. Introduction

The theory of lightlike hypersurfaces is one of the most interesting areas of in differential geometry as well as physics. Because the study of lightlike hypersurfaces has played a key role in the development of general relativity. They have extensive use in general relativty. Researchers use lightlike hypersurfaces in order to show a class of lightlike hypersurfaces came from the physically significant homegeneous spacetime manifolds of general relativity [1]. Studies of lightlike hypersurfaces have been essential in order to comprehend the casual structure of spacetimes, black holes, asymptotically flat systems and gravitational waves. This is the reason they were studied intensively by geometers and physicists.

On the other hand in special relativity, a null cone is the surface describing the temporal evolation of flash of light in Minkowski spacetime. An another research area is the characterizations of curves in the lightlike cone [2-4]. In this study, we look on the progress which in this direction and we give new characterizations of a curve in the 3-dimensional lightlike cone. Many studies on spherical curves have been done by many mathematicians. For example [5], [6] the authors gave characterizations related to these curves in E^n. Furthermore, in [7] Camcı et all. studied spherical curves in 3-dimensional Sasakian spaces. In recent years, many important and intensive studies are seen about timelike, spacelike and null curves in Minkowski space. Papers in [2-4], [6], [8-10] show how spherical curves are important field of interest.

The main purpose of this paper is to carry out some results which were given in [7] and [11], [12] to characterizations of a curve in 3-dimensional lightlike cone.

2. Curves in the Lightlike Cone

In the following we use the similar notations and concepts in [2-4].

The Minkowski 4-space E_1^4 is the Euclidean 4-space E^4 provided with the standart flat metric given by

$$\langle\ ,\ \rangle = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4 .

Let c be a fixed point in E_1^4 and $r > 0$ be a constant. The pseudo-Riemannian sphere is defined by

$$S_1^3(c,r) = \{x \in E_1^4 \ : \ < x - c, x - c > \ = r^2\};$$

the pseudo-Riemannian hyperbolic space is defined by

$$H_1^2(c,r) = \{x \in E_2^4 \ : \ < x - c, x - c > \ = -r^2\};$$

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the pseudo-Riemannian null cone (quadric cone) is defined by

$$Q^3(c) = \{x \in E_1^4 \ : \ < x - c, x - c > \ = 0\}.$$

When $c = 0$ and $q = 1$, we simply denote $Q_1^n(0)$ by Q^n and call it the lightlike cone (or simply the light cone).

Since $\langle , \rangle$ is an indefinite metric, it can be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and null (lightlike) if $\langle v, v\rangle = 0$ and $v \neq 0$.

Similarly, an arbitrary curve $x = x(s)$ can be locally spacelike, timelike or null (lightlike), if all of its velocity $x^{'}(s)$ are spacelike, timelike or null (lightlike), respectively.

Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. We put (x_1, x_2, x_3, x_4) and have

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0.$$

Then from

$$x_1^2 - (ix_2)^2 = - \left(x_3^2 - x_4^2\right)$$

we get

$$\frac{x_1 + ix_2}{x_3 + x_4} = - \frac{x_3 - x_4}{x_1 - ix_2} \ \text{ or } \frac{x_1 + ix_2}{x_3 + x_4} = - \frac{x_3 + x_4}{x_1 - ix_2} \tag{2.1}$$

without loss of generality for a curve $x = x(s) : I \to Q^3 \subset E_1^4$ with $x = x(s) = (x_1, x_2, x_3, x_4)$, we may assume that

$$\frac{x_1 + ix_2}{x_3 + x_4} = - \frac{x_3 - x_4}{x_1 - ix_2}$$

and putting

$$y(s) = - x^{''}(s) - \frac{1}{2} < x^{''}(s), x^{''}(s) > x(s) \tag{2.2}$$

we have

$$< y(s), y(s) >=< x(s), x(s) >=< y(s), x^{'}(s) >= 0, \ < x(s), y(s) >= 1. \tag{2.3}$$

Let's put $\alpha(s) = x^{'}(s)$ and choose $\beta(s)$ such that

$$\det \ (x(s), \beta(s), \ y(s) \ ) = 1.$$

Then from (2.2), we obtain

$$\alpha^{'}(s) = x^{''}(s) = -\frac{1}{2} < x^{''}(s), x^{''}(s) > x(s) - y(s) = k(s) \ x(s) - y(s). \tag{2.4}$$

Therefore, the Frenet Formulas of spacelike curve $x = x(s) : I \to Q^3 \subset E_1^4$ can be written as

$$\begin{aligned} x^{'}(s) &= \alpha(s) \\ \alpha^{'}(s) &= \kappa(s)x(s) - y(s) \\ \beta^{'}(s) &= \tau(s)x(s) \\ y^{'}(s) &= -\kappa(s)\alpha(s) - \tau(s)\beta(s). \end{aligned} \tag{2.5}$$

The frame field $\{x(s), \alpha(s), y(s), \beta(s)\}$ is called the cone frenet frame of the curve $x(s)$.
The functions $\kappa(s)$ and $\tau(s)$ are defined as

$$k(s) = -\frac{1}{2} < x^{''}(s), x^{''}(s) > \tag{2.6}$$

$$(\tau(s))^2 = \; < x^{''}(s), x^{''}(s) > -4\,(k(s))^2\,. \tag{2.7}$$

From [3], for any asymtotic orthonormal frame $\{x(s), \alpha(s), y(s), \beta(s)\}$ of the spacelike curve $x = x(s) : I \to Q^3 \subset E_1^4$ with

$$\begin{aligned} &< x(s), x(s) >=< y(s), y(s) >=< x(s), \alpha(s) >=< x(s), \beta(s) >= \\ &< y(s), \alpha(s) >=< y(s), \beta(s) >=< \alpha(s), \beta(s) >= 0 \end{aligned} \tag{2.8}$$

$$< x(s), y(s) >=< \alpha(s), \alpha(s) >=< \beta(s), \beta(s) >= 1 \tag{2.9}$$

the frenet Formulas read

$$\begin{aligned} x^{'}(s) &= \alpha(s) \\ \alpha^{'}(s) &= \kappa(s)x(s) + \lambda(s)\beta(s) - y(s) \\ \beta^{'}(s) &= \tau(s)x(s) - \lambda(s)\alpha(s) \\ y^{'}(s) &= -\kappa(s)\alpha(s) - \tau(s)\beta(s), \end{aligned} \tag{2.10}$$

we know $\lambda(s) \equiv 0$ if and if only $y(s)$ satisfies (2.2). Therefore some authors called the frame, satisfying (2.2), cartan frame we know (2.2) and (2.3) are true in any dimension.

Definition 2.1. The functions $\kappa(s)$ and $\tau(s)$ in (2.5) are called the (first) cone curvature and cone torsion (or second cone curvature) of the spacelike curve $x(s)$ in $Q^3 \in E_1^4$.

Definition 2.2. A curve $x(s)$ such that the functions $\frac{\kappa(s)}{\tau(s)} = \; const.$ is called a general helix.

If both $\kappa(s)$ and $\tau(s)$ are constants along $x(s)$, then $x(s)$ is called a circular helix.

If $\kappa(s) = \tau(s) = 0$, then $x(s)$ is called a null cubic.

3. The Characterization of Spacelike Curves In The Lightlike Cone

Theorem 3.1. Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. Let us suppose that if $c - x(s)$ is lightlike vector (c is a constant) in $Q^3 \subset E_1^4$, then $m_i(s)$ $(1 \leq i \leq 4)$ curvature functions of $x(s)$ are given by

$$m_1(s) = 0 \tag{3.1}$$

$$m_2(s) = \frac{-\left(\kappa^{'}(s)\right)^2}{\Delta(s)} \tag{3.2}$$

$$m_3(s) = \frac{(\tau(s))^2}{\Delta(s)} \tag{3.3}$$

$$m_4(s) = \frac{-2\;\kappa^{'}(s)\tau(s)}{\Delta(s)} \tag{3.4}$$

where

$$\Delta(s) = \left(\kappa^{'}(s)\right)^2 + 2\kappa(s)\tau(s) \neq 0. \tag{3.5}$$

Proof. Since $c - x(s)$ is a lightlike vector, we can write

$$< c - x(s), c - x(s) >= 0. \tag{3.6}$$

Differentiating the above equation, we have

$$< x^{'}(s), c - x(s) >= 0. \tag{3.7}$$

Hence using (2.5), we get

$$< \alpha(s), c - x(s) >= 0. \tag{3.8}$$

If we differentiate (3.8), we obtain

$$< \alpha^{'}(s), c - x(s) > - < \alpha(s), x^{'}(s) >= 0, \tag{3.9}$$

and using (2.5), we can write

$$\kappa(s) < x(s), c - x(s) > - < y(s), c - x(s) >= 1. \tag{3.10}$$

Since $c - x(s)$ is a lightlike vector in the three dimensional lightlike cone, it can be written as a linear combination of the frame in the form

$$c - x(s) = m_1(s)\alpha(s) + m_2(s)x(s) + m_3(s)y(s) + m_4(s)\beta(s), \tag{3.11}$$

where $m_i(s), 1 \leq i \leq 4$ are differentiable functions on $\mathbb{R}$. A direct computation shows that the values of the differentiable functions

$$\begin{aligned} m_1(s) &= 0, \\ m_2(s) &=< c - x(s), y(s) >, \\ m_3(s) &=< c - x(s), x(s) >, \\ m_4(s) &=< c - x(s), \beta(s) > . \end{aligned} \tag{3.12}$$

Considering (3.2) in (3.10), then

$$\kappa(s)m_3(s) - m_2(s) = 1. \tag{3.13}$$

Taking the derivative of (3.10) and considering also (3.12) and (2.5), then we have

$$\kappa^{'}(s)m_3(s) + \tau(s)m_2(s) = 0. \tag{3.14}$$

On the other hand, from (3.11) and (3.8), we get

$$2m_2(s)m_3(s) + m_4^2(s) = 0. \tag{3.15}$$

Using (3.13) and (3.14) in the (3.15), we get (3.1), (3.2), (3.3) and (3.4) equalities.
Corollary 3.1. If $x(s)$ be a circular helix, then we obtain

$$m_1(s) = m_2(s) = m_4(s) = 0 \text{ and } m_3(s) = const. \tag{3.16}$$

Proof. Let $x(s)$ be a circular helix, then from (3.1), (3.2), (3.3) and (3.4) , we can obtain easily (3.16).

Corollary 3.2. If the point c is a center of the three dimensional lightlike cone Q^3, then we obtain

$$m_1(s) = m_3(s) = m_4(s) = 0 \text{ and } m_2(s) = const. \tag{3.17}$$

The proof of Corollary 3.2 can show easily (3.1), (3.2), (3.3) and (3.4) .

Theorem 3.2. Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. If the following differential equation is valid

$$\left(\frac{d\kappa(s)}{ds}\right)^2 + 2\,(\tau(s))^2 = 0, \tag{3.18}$$

then we get

$$m_2(s) = m_3(s) \tag{3.19}$$

and

$$m_4(s) = \frac{-\kappa^{'}(s)}{\tau(s)\,(\kappa(s)-1)}, \tag{3.20}$$

where $\tau(s) \neq 0$ and $\kappa(s) \neq 1$.

Proof. Suppose that (3.18) is valid. Thus from (3.18) and (3.5), we get $m_2(s) = m_3(s)$. Further, putting (3.18) in equality of (3.4) can be written as

$$m_4(s) = \frac{-\kappa^{'}(s)}{\tau(s)\,(\kappa(s)-1)}.$$

The proof of the following theorem can be easily used the method of proof of Theorem 3.2.

Theorem 3.3. Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a space curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s.

1) If the following differential equation is valid

$$\frac{d\kappa(s)}{ds} + \tau(s) = 0 \tag{3.21}$$

then we get

$$m_3(s) = m_4(s)$$

and

$$m_2(s) = \frac{-1}{1+2\kappa(s)},$$

where $\kappa(s) \neq -\frac{1}{2}$.

2) If the following differential equation is valid

$$\frac{d\kappa(s)}{ds} - 2\tau(s) = 0, \tag{3.22}$$

then we get

$$m_2(s) = m_4(s)$$

and

$$m_3(s) = \frac{1}{2 + 2\kappa(s)},$$

where $\kappa(s) \neq -\frac{1}{2}$.

4. Conclusions and Final Remarks

We have used $m-$curvature functions to obtain some characterizations for a spacelike curve in the 3-dimensional lightlike cone. Each solution contains a new result that helps to characterize a curve. These characterizations might be useful for many researchers, for example, in physics lightlike cone has an extensive usage. Furthermore from the viewpoint of general relativity and related theories, the physical events space is repesented by a Minkowski space-time [13], [14]. Minkowski space-time has three causal types of curves; space-like, time-like and lightlike. As is well known these curves form a lightlike cone.

5. Appendix

1. Let us suppose that $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with arc length parameter s. Let us soppose that if c is a constant and

$$c - x(s) = m_1(s)(\alpha(s) + x(s) + \beta(s)) + m_2(s)y(s)$$

is lightlike vector in $Q^3 \subset E_1^4$, then the following statements hold.

i) The $m_i(s)$ $(1 \leq i \leq 2)$ curvature functions of $x(s)$ satisfy the following equality

$$m_1(s) = 0 \text{ and } m_2(s) = \frac{1}{\kappa(s)}$$

or

$$m_1(s) = const. \text{ and } m_2(s) = 0.$$

ii) If the $m_1(s) = 0$, then the cone curvatures of $x(s)$ is a constant or

If the $m_2(s) = 0$, then the cone torsion of $\tau(s) = 0$.

2. Let us suppose that $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with arc length parameter s. Let us soppose that if c is a constant and

$$c - x(s) = m_1(s)(\alpha(s) + y(s) + \beta(s)) + m_2(s)x(s)$$

is lightlike vector in $Q^3 \subset E_1^4$, then the following statements hold.

i) The $m_i(s)$ $(1 \leq i \leq 2)$ curvature functions of $x(s)$ satisfy the following equality

$$m_1(s) = 0 \quad \text{and} \quad m_2(s) = const$$

or

$$m_1(s) = \frac{1}{\kappa(s)} \text{ and } m_2(s) = 0.$$

ii) The cone curvatures is defined by $\kappa(s) = -2\int \tau(s)ds - C, C \in \mathbb{R}$.

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