Article

The Darboux Rotation Axis & Special Curves According to Rotation Minimizing Frame in Minkowski 3-Space

Ümit Ziya Savcı* & Süha Yılmaz**[1]

*Department of Mathematics, Dumlupınar University, 43050, Kütahya, Turkey
**Dept. of Math. Education, Dokuz Eylül University, 35150, Buca-Izmir, Turkey

Abstract

In this paper, we introduce a new Darboux vector and Darboux helix for timelike curves in Minkowski 3-space. We defined Darboux vector in term of type-2 Bishop frame in $\mathbb{R}_1^3$. We introduce new spherical images and call them as type-2 Bishop Darboux spherical images for timelike curves. Later, we introduce new Darboux helix and constant precession of the timelike curve. Moreover, we give characterization of the axis of the Darboux helices. Finally we illustrate one example of our main results.

Keywords: Bishop spherical image, Darboux helix, Bishop frame, Darboux vector, constant precession of the timelike curve.

1 Introduction

Bishop frame constructed by L.R. Bishop in 1975. Recently, many research papers related to this frame have been studied in Euclidean and Minkowski space. In [22] the authors introduced a new version Bishop frame using a common vector field as binormal vector field of a regular curve and called Type-2 Bishop frame in Euclidean 3-space. In [10] extended this frame to Euclidean 4-space and called parallel transport frame in Euclidean 4-space. It is well known that Bishop introduced the Relatively Parallel Adapted Frame of regular curves embedded in Euclidean 3-space and this frame was widely applied in the area of Biology and Computer Graphics [18].

Vectors Ω_1, Ω_2, B change while a point P on the curve drawing the curve. Hence these vectors constitute of spherical indicatrices of the curve. Suppose that Bishop trihedral $\{\Omega_1, \Omega_2, B\}$ of the curve makes an instantaneous helix motion about an axis at each s time. The axis is called Darboux Bishop axis corresponding s parameter at $\alpha(s)$ point. The vector giving oriented and direction of this axis is Darboux Bishop vector at point $\alpha(s)$ of the curve.

Some curves in Minkowski space (especially spherical indicatrices) are used a lot of area. For example, structure of the Higgs potential, cosmology on a brane, 4D gravity on a brane in 5D Minkowski space, vacuum tunneling of gauge theory, physical energy, the global stability, spherical indicatrix camera, spherical indicatrix by fish-eye conversion lens [2,7,8,11,14,15,16,17].

In this work, using common vector field as the binormal vector of Serret-Frenet frame, we introduce a new version of the Bishop frame in $\mathbb{R}_1^3$. We call it is “Type-2 Bishop frame” of regular curves. We defined new Darboux helix, thereafter, translating new frames vector fields to the center of unit sphere, we obtain new spherical images. We call them as ”Type-2 Bishop Darboux Spherical Image” of regular curves.

[1]Correspondence: E-mail: suha.yilmaz@deu.edu.tr

2 Preliminaries

The Minkowski three dimensional space $\mathbb{R}^3_1$ is a real vector space $\mathbb{R}^3$ endowed with the standard flat Lorentzian metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is rectangular coordinate system of $\mathbb{R}^3_1$. Since g is an indefinite metric. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary an vectors in $\mathbb{R}^3_1$, the Lorentzian cross product of u and v defined by

$$u \times v = -\det \begin{bmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3. \end{bmatrix}$$

Recall that a vector $v \in \mathbb{R}^3_1$ can have one of three Lorentzian characters it can be spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$ and null(lightlike) if $g(v, v) = 0$ for $v \neq 0$. Similarly, an arbitrary curve $\delta = \delta(s)$ in $\mathbb{R}^3_1$ can locally be spacelike, timelike or null (lightlike) if all of its velocity vector $\delta^{\shortmid}$ are respectively spacelike, timelike, or null (lightlike), for every $s \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in \mathbb{R}^3_1$ is given by

$$\|a\| = \sqrt{|g(a, a)|}.$$

The curve $\delta = \delta(s)$ is called a unit speed curve if velocity vector $\delta^{\shortmid}$ is unit i.e, $\|\delta^{\shortmid}\| = 1$. For vectors $v, w \in \mathbb{R}^3_1$ it is said to be orthogonal if and only if $g(v, w) = 0$. Denote by $\{T, N, B\}$ the moving Serret-Frenet frame along the curve $\delta = \delta(s)$ in the space $\mathbb{R}^3_1$.

For an arbitrary timelike curve $\delta = \delta(s)$ in $\mathbb{R}^3_1$, the following Serre-Frenet formulae are given in as follows

$$\begin{bmatrix} T^{\shortmid} \\ N^{\shortmid} \\ B^{\shortmid} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} . \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where $< T, T >= -1$, $< B, B >=< N, N >= 1$, $T(s) = \delta^{\shortmid}(s)$, $N(s) = \dfrac{T^{\shortmid}(s)}{\kappa(s)}$, $B(s) = T(s) \times N(s)$ and first curvature and second curvature (torsion) $\kappa(s)$, $\tau(s)$ respectively.$\kappa(s)$=$\|\delta^{\shortmid\shortmid}\|$, $\tau(s)$=$\dfrac{\det(\delta^{\shortmid}, \delta^{\shortmid\shortmid}, \delta^{\shortmid\shortmid\shortmid})}{\kappa^2}$ [8].

Theorem 2.1 Let $\alpha = \alpha(s)$ be timelike curve with a spacelike principal normal unit speed. If $\{\Omega_1, \Omega_2, B\}$ is adapted frame, then we have defined by "Type-2 Bishop Frame in $\mathbb{R}^3_1$"

$$\begin{bmatrix} \Omega_1^{\shortmid} \\ \Omega_2^{\shortmid} \\ B^{\shortmid} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 0 & \varepsilon_2 \\ \varepsilon_1 & \varepsilon_2 & 0 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{2.2}$$

where $g(\Omega_1, \Omega_1) = -1$, $g(\Omega_2, \Omega_2) = g(B, B) = 1$, and $g(\Omega_1, \Omega_2) = g(\Omega_1, B) = g(\Omega_2, B) = 0$. If Ω_1 timelike Ω_2 and B spacelike vectors. First curvature ε_1 and second curvature ε_2 of the curve are defined by

$$\varepsilon_1 =< \Omega_1^{\shortmid}, B >, \quad \varepsilon_2 =< \Omega_2^{\shortmid}, B > . \tag{2.3}$$

Theorem 2.2 Let $\{T, N, B\}$ and $\{\Omega_1, \Omega_2, B\}$ be Frenet and Bishop frames, respectively. There exists a relation between them as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sinh\theta(s) & -\cosh\theta(s) & 0 \\ \cosh\theta(s) & -\sinh\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix}$$

where θ is the angle between the vectors N and Ω_1.

Proof: We write the tangent vector according to frame $\{\Omega_1, \Omega_2, B\}$ as

$$T = \sinh\theta(s)\Omega_1 - \cosh\theta(s)\Omega_2 \tag{2.4}$$

and differentiate with respect to s

$$\begin{aligned} T' = \kappa N = \; & \theta'(s) \left[\cosh\theta(s)\Omega_1 - \sinh\theta(s)\Omega_2\right] \\ & + \sinh\theta(s)\Omega_1' - \cosh\theta(s)\Omega_2' \end{aligned} \tag{2.5}$$

substituting $\Omega_1' = \varepsilon_1 B$ and $\Omega_2' = \varepsilon_2 B$ into equation (2.5), we get

$$\begin{aligned} \kappa N = \; & \theta'(s) \left[\cosh\theta(s)\Omega_1 - \sinh\theta(s)\Omega_2\right] \\ & + [\sinh\theta(s)\varepsilon_1 - \cosh\theta(s)\varepsilon_2] B. \end{aligned} \tag{2.6}$$

In above equation let us take $\theta'(s) = \kappa(s)$. So immediately arrive at

$$N = \cosh\theta(s)\Omega_1 - \sinh\theta(s)\Omega_2$$

Considering the obtained equations, the relation matrix between Frenet-Serret and type-2 Bishop frames can expressed

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sinh\theta(s) & -\cosh\theta(s) & 0 \\ \cosh\theta(s) & -\sinh\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B. \end{bmatrix} \tag{2.7}$$

Since there is a solution for θ satisfying any initial condition, this show that locally relatively parallel normal fields exist. Besides equation (2.1) can also written as

$$B' = \tau N = \varepsilon_1 \Omega_1 + \varepsilon_2 \Omega_2$$

taking the norm of both sides, we have

$$\tau = \sqrt{\left|\varepsilon_2^2 - \varepsilon_1^2\right|} \tag{2.8}$$

$$1 = \sqrt{\left|\left(\frac{\varepsilon_1}{\tau}\right)^2 - \left(\frac{\varepsilon_2}{\tau}\right)^2\right|} \tag{2.9}$$

From equation (2.6) we get $\theta(s) = \arg\tanh(\frac{\varepsilon_2}{\varepsilon_1})$, from equation (2.9) we obtain

$\{\ \varepsilon_1 = \tau(s)\cosh\theta(s), \qquad \varepsilon_2 = \tau(s)\sinh\theta(s).$

The frame $\{\Omega_1, \Omega_2, B\}$ is properly oriented, and τ and $\theta(s) =_0^s \kappa(s)ds$ are polar coordinates for the curve $\alpha = \alpha(s)$. We shall call the set $\{\Omega_1, \Omega_2, B, \varepsilon_1, \varepsilon_2\}$ as type-2 Bishop invariants of the curve $\alpha = \alpha(s)$ in $\mathbb{R}_1^3$.

3 Bishop Darboux Vector in Type-2 Bishop Frame

Consider the rigid object moving smoothly along the regular curve. Once the translation "factored out", the object is seen to rotate the same way as it's Bishop frame in Minkowski 3 space. The total rotation of the Bishop frame is the combination of the rotations of each of the three Bishop vectors;

$$w = w_{\Omega_1} + w_{\Omega_2} + w_B$$

each Bishop vector moves about an "origin" which is the centre of the rigid object. The areal velocity of the tangent vector Ω_1, Ω_2 and B are

$$\omega_{\Omega_1} = \frac{\Omega_1(t) \wedge \Omega_1^{\prime}(t)}{2},$$

$$\omega_B = \frac{B(t) \wedge B^{\prime}(t)}{2},$$

$$\omega_{\Omega_2} = \frac{\Omega_2(t) \wedge \Omega_2^{\prime}(t)}{2}$$

Theorem 3.1: The rotation is determined by an angular velocity vector ω which is called the Bishop Darboux vector and given by

$$\omega = -\varepsilon_1 \Omega_2$$

Proof: Let be total rotation of the Bishop frame is the combination of the rotations of each Bishop vectors

$$\omega = \omega_{\Omega_1} + \omega_{\Omega_2} + \omega_B \tag{3.1}$$

The areal velocity of tangent vector is

$$\omega_{\Omega_1} = \underset{\Delta t \to 0}{Lim} \frac{\Omega_1(t) \wedge \Omega_1(t+\Delta t)}{2\Delta t} = \frac{\Omega_1 \wedge \Omega_1^{\prime}}{2} \tag{3.2}$$

substituting equations $(2.1)_1$ to equations (3.2) we have

$$\omega_{\Omega_1} = -\frac{1}{2}\varepsilon_1 \Omega_2 \tag{3.3}$$

similarly

$$\omega_{\Omega_2} = \underset{\Delta t \to 0}{Lim} \frac{\Omega_2(t) \wedge \Omega_2(t+\Delta t)}{2\Delta t} = \frac{\Omega_2 \wedge \Omega_2^{\prime}}{2} \tag{3.4}$$

substituting equations $(2.1)_2$ to equations (3.4) we have

$$\omega_{\Omega_2} = \frac{1}{2}\varepsilon_2 \Omega_1 \tag{3.5}$$

finally

$$\omega_B = \underset{\Delta t \to 0}{Lim} \frac{B(t) \wedge B(t+\Delta t)}{2\Delta t} = \frac{B \wedge B^{\prime}}{2} \tag{3.6}$$

substituting equations $(2.1)_3$ to equations (3.6) we have

$$\omega_B = -\frac{1}{2}\varepsilon_2\Omega_1 - \frac{1}{2}\varepsilon_1\Omega_2 \tag{3.7}$$

substituting equations (3.3), (3.5) and (3.7) to equations (3.1) we write

$$\omega = -\varepsilon_1\Omega_2 \tag{3.8}$$

The rotation is determined by an angular velocity ω which satisfies

$$\begin{aligned} \omega \wedge \Omega_1 &= \Omega_1^{'} \\ \omega \wedge \Omega_2 &= \Omega_2^{'} \\ \omega \wedge B &= B^{'}. \end{aligned} \tag{3.9}$$

Theorem 3.1.2: If B is binormal vector of α curve, then we can write following formulas
i) $B^{'} \wedge B^{''} = (\varepsilon_1^2 + \varepsilon_2^2)(\varepsilon_2^2\Omega_1 - \omega) + \varepsilon_2^2(\frac{\varepsilon_1}{\varepsilon_2})^{'}B$

ii) $det(B, B^{'}, B^{''}) = \varepsilon_2^2(\frac{\varepsilon_1}{\varepsilon_2})^{'}$

iii) $\theta^{'} = \mp\frac{det(B,B^{'},B^{''})}{\|B\wedge B^{'}\|^2} = \mp\kappa$

Proof: i) From $B^{'} = \varepsilon_1\Omega_1 + \varepsilon_2\Omega_2$ we get

$$B^{''} = \varepsilon_1^{'}\Omega_1 + \varepsilon_2^{'}\Omega_2 + (\varepsilon_1^2 + \varepsilon_2^2)B$$

Moreover one can easily find

$$\begin{aligned} &< B^{''}, B >= \varepsilon_1^2 + \varepsilon_2^2 \\ &< B^{''}, \Omega_1 >= -\varepsilon_1^{'} \\ &< B^{''}, \Omega_2 >= \varepsilon_2^{'}. \end{aligned}$$

From definition Darboux vector ω we write $\omega \wedge B = B^{'}$ so we have

$$\begin{aligned} B^{'} \wedge B^{''} &= -B^{''} \wedge (\omega \wedge B) \\ &=< B^{''}, B > \omega- < B^{''}, \omega > B \\ &= (\varepsilon_1^2 + \varepsilon_2^2)\omega- < B^{''}, -\varepsilon_1\Omega_2 > B \\ &= (\varepsilon_1^2 + \varepsilon_2^2)\omega + \varepsilon_1 < B^{''}, \Omega_2 > B \\ &= (\varepsilon_1^2 + \varepsilon_2^2)\omega + (\varepsilon_1\varepsilon_2^{'})B. \\ &= (\varepsilon_1^2 + \varepsilon_2^2)(\varepsilon_2^2\Omega_1 - \omega) + \varepsilon_2^2(\tfrac{\varepsilon_1}{\varepsilon_2})^{'}B. \end{aligned}$$

ii) Since

$$\begin{aligned} \det(B, B^{'} \wedge B^{''}) &= -\begin{bmatrix} 0 & 0 & 1 \\ \varepsilon_1 & \varepsilon_2 & 0 \\ \varepsilon_1^{'} & \varepsilon_2^{'} & \varepsilon_1^2+\varepsilon_2^2 \end{bmatrix} \\ &= -(\varepsilon_2^{'}\varepsilon_1 - \varepsilon_2\varepsilon_1^{'}) \end{aligned}$$

we get

$$< B, B^{\shortmid} \wedge B^{\shortmid\shortmid} >= \varepsilon_2^2 (\tfrac{\varepsilon_1}{\varepsilon_2})^{\shortmid}. \tag{3.10}$$

ii) From the equation $(3.9)_3$ $B^{\shortmid} = \omega \wedge B$ using equation $(2.1)_3$ we can write $-\omega = B \wedge B^{\shortmid}$ and then

$$\left\| B \wedge B^{\shortmid} \right\|^2 = \varepsilon_1^2 - \varepsilon_2^2$$

If we differentiating equality $\theta(s) = \arg\tanh(\frac{\varepsilon_2}{\varepsilon_1})$ we find

$$\theta^{\shortmid}(s) = \frac{(\frac{\varepsilon_2}{\varepsilon_1})^{\shortmid}}{1 - (\frac{\varepsilon_2}{\varepsilon_1})^2}$$

and using equation (3.10) we have

$$\begin{aligned} \theta^{\shortmid}(s) &= \mp \frac{det(B, B^{\shortmid}, B^{\shortmid\shortmid})}{\left\| B \wedge B^{\shortmid} \right\|^2} \\ &= \mp \kappa. \end{aligned}$$

4 Bishop Darboux Spherical Indicatrix Curve and Result

Let unit Darboux vector $\omega_0 = \dfrac{\omega}{\|\omega\|} = \dfrac{\varepsilon_1 \Omega_1}{|\varepsilon_1|}$, $\omega_0 = \mu\Omega_1$ (here $\mu = \mp 1$).

We obtain a spherical indicatrix curve which is called Bishop Darboux spherical indicatrix of the timelike curve α by translating the unit vector ω_0 the center of unit sphere H^2.

Let $\alpha_{w_0}(s) = \omega_0(s)$, with the arc-lenght parameter s_{w_0}, be Bishop Darboux spherical indicatrix of the timelike curve $\alpha(s)$. By differentiating α_{ω_0}, we get

$$\begin{aligned} \frac{d\alpha_{\omega_0}}{ds} &= \frac{d\omega_0}{ds} \\ \dot{\alpha}_{\omega_0} \frac{ds_{w_0}}{ds} &= \omega_0^{\shortmid} \end{aligned}$$

and derivative of ω_0 can be calculated as

$$\omega_0^{\shortmid} = (\mu_0)^{\shortmid} \Omega_1 = 0,$$

here ω_0 is constant.

Result: Since ω_0 is constant, tangent vector of Bishop Darboux spherical indicatrix is zero.

5 Darboux Helices

Definition 5.1: Let $\alpha = \alpha(s)$ be a unit speed timelike curve in $\mathbb{R}_1^3$ with $\dfrac{\varepsilon_2}{\varepsilon_1} \neq 0$ everywhere with nonzero curvature and ε_1 and ε_2 in $\mathbb{R}_1^3$. If unit Darboux vector of $\alpha = \alpha(s)$ makes constant angle a fixed direction d, then we say that $\alpha = \alpha(s)$ is Darboux helix.

Let $\alpha = \alpha(s)$ be a unit speed timelike curve in $\mathbb{R}_1^3$. We define the Darboux vector ω from Theorem 3.1.1 $\omega = -\varepsilon_1 \Omega_2$. If we take norm the Darboux vector, we find

$$\|\omega\| = \varepsilon_1$$

and $\omega \wedge \Omega_1 = \Omega_1^{'}$, $\omega \wedge \Omega_2 = \Omega_2^{'}$, $\omega \wedge B = B^{'}$.

Now we write the unit Bishop Darboux vector ω_0;

$$\omega_0 = \mu\Omega_1 \text{ (here } \mu = \mp 1).$$

or

$$\omega_0 = \cosh(\theta)\Omega_2.$$

If we take ω_0 as unit vector, then it defines a curve on the Lorentzian sphere H^2.

If we called the spherical image as β. $\beta(s_*) = \omega(s) = -\varepsilon_1\Omega_2$ where s_* parameter of β.

$$\frac{d\beta}{ds} = \frac{d\beta}{ds_*} \cdot \frac{ds_*}{ds}$$

By taking the derivative on the both sides with respect to s, we can write

$$\frac{d\beta}{ds} = \frac{d\beta}{ds_*} \cdot \frac{ds_*}{ds} = -\varepsilon_1^{'}\Omega_2$$

so

$$T_\beta = \dot{T}_\beta \cdot \frac{ds_*}{ds} = -\Omega_2 \tag{5.1}$$

where $\frac{ds_*}{ds} = -\varepsilon_1^{'}$. Now, we will find curvature κ_β of the curve $\beta(s_*)$

$$\kappa_\beta = \left\| \beta^{''} \right\| = \left\| \beta_{s_*}^{'} \right\|.$$

Hence by taking the derivative equation (5.1) with respect to s. We have

$$T_\beta^{'} = \dot{T}_\beta \cdot \tfrac{ds_*}{ds} = -\varepsilon_2 B \tag{5.2}$$

Since, we express

$$\kappa_\beta = \left\| \dot{T}_\beta \right\| = \frac{\varepsilon_2}{\varepsilon_1^{'}} \tag{5.3}$$

Hence by using the equations (5.2) and (5.3), we have the principal normal

$$N_\beta = -\varepsilon_1^{'} B \tag{5.4}$$

By the cross product of $T_\beta \wedge N_\beta$, we obtain the binormal vector field

$$B_\beta = \varepsilon_1^{'}\Omega_1 \tag{5.5}$$

Theorem 5.2 Let α is a regular curve with type-2 Bishop curvature $\varepsilon_1 \neq 0$ and $\varepsilon_2 \neq 0$. α is a general helix if and only if type-2 Bishop curvature of the curve satisfy

$$\frac{\varepsilon_1^2}{(\varepsilon_1^2 - \varepsilon_2^2)^{3/2}}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{'} = \text{constant}$$

Proof: Let $\alpha : I \to \mathbb{R}^3_1$ be regular timelike curve. Moreover the Frenet curvature of α are κ, τ and Bishop curvature of α are $\varepsilon_1, \varepsilon_2$. From equations (2.7) and (2.9) we get

$$\kappa = (\arg\tanh \frac{\varepsilon_2}{\varepsilon_1})' \text{ and } \tau = \sqrt{\left|\varepsilon_2^2 - \varepsilon_1^2\right|}$$

therefore we can calculate that $\dfrac{\tau}{\kappa} = \dfrac{\varepsilon_1^2}{(\varepsilon_1^2 - \varepsilon_2^2)^{3/2}} (\dfrac{\varepsilon_2}{\varepsilon_1})'$. If the α is general helix $\Leftrightarrow \dfrac{\tau}{\kappa}$ =constant. This completes the proof.

Corollary: The rotation motion of the Darboux axis can be separated into two rotation motions again. Here ω rotation vector is adulation of the rotation vector of the rotation motion. When continued in the similar way, the rotation motion of the Darboux axis is done in a consecutive manner. In this way the series of Darboux vectors are obtained. That is $\omega_0 = \omega, \omega_1, ...$

Theorem 5.3 Let α is a Darboux helix. If $< B, d >$ is constant, then α is a general helix.

Proof: We first assume that α is a Darboux helix. Let be d the vector field such that the function, $< B, d >= c$ is constant. There exists k_1 and k_2 such that

$$d = k_1\Omega_1 + k_2\Omega_2 + cB. \tag{5.6}$$

Then if we take the derivative of the equation (5.6) and using Bishop equations, we have

$$\begin{aligned} d' = \ & (k_1' + c\varepsilon_1)\Omega_1 + (k_2' + c\varepsilon_2)\Omega_2 \\ & - (k_1\varepsilon_1 + k_2\varepsilon_2)B. \end{aligned}$$

Since the system $\{\Omega_1, \Omega_2, B\}$ is linear independent, we get

$$\begin{array}{l} k_1' + c\varepsilon_1 = 0 \\ k_2' + c\varepsilon_2 = 0 \\ k_1\varepsilon_1 + k_2\varepsilon_2 = 0 \end{array} \tag{5.7}$$

So, from equation (5.6) and equations $(5.7)_1$, $(5.7)_3$, respectively

$$\begin{array}{l} k_1 = -\frac{\varepsilon_2}{\varepsilon_1} k_2 \\ < d, d >= -k_1^2 + k_2^2 + c^2. \end{array} \tag{5.8}$$

substituting equation $(5.8)_1$in the equation $(5.1.8)_2$ we have

$$-\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 k_2^2 + k_2^2 + c^2 = \text{const.} \tag{5.9}$$

or we arrangement to (5.9) we obtain

$$k_2 = \frac{c}{\sqrt{1 - \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2}}. \tag{5.10}$$

Taking the derivative in each part of the equation (5.10) and by using equation $(5.8)_2$, we obtain

$$k_2' = -\frac{\varepsilon_1^2 \cdot \varepsilon_2 \cdot c}{\left(\varepsilon_1^2 - \varepsilon_2^2\right)^{3/2}} (\frac{\varepsilon_2}{\varepsilon_1})' \tag{5.11}$$

substituting equation (5.11) to equation $(5.7)_2$, so we get

$$\frac{\varepsilon_1^2}{\left(\varepsilon_1^2-\varepsilon_2^2\right)^{3/2}}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{'}=\text{constant}, \tag{5.12}$$

from theorem 5.2, α is a general helix. This proof is completed.

Theorem 5.4: Let α is a Darboux helix. If $<B,d>$ is constant, then curvatures ε_1 and ε_2 of the timelike curve α satisfy the following non-linear equation system

$$\left(\frac{c\cdot\varepsilon_2}{\sqrt{\varepsilon_1^2\text{-}\varepsilon_2^2}}\right)^{'}+c\cdot\varepsilon_1=0 \quad \text{and} \quad \left(\frac{c\cdot\varepsilon_1}{\sqrt{\varepsilon_1^2\text{-}\varepsilon_2^2}}\right)^{'}+c\cdot\varepsilon_2=0$$

where $c\neq 0$, $c\in\mathbb{R}$, $(\varepsilon_1^2-\varepsilon_2^2)\neq 0$.

Proof: Since α is a Darboux helix, the axis of the α

$$d=\frac{c\cdot\varepsilon_2}{\sqrt{\varepsilon_1^2-\varepsilon_2^2}}\Omega_1-\frac{c\cdot\varepsilon_1}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}}\Omega_2+cB \tag{5.13}$$

where ε_1 and ε_2 are curvatures of α according to type-2 Bishop frame. Taking derivative in each part of the equation (5.6), we get

$$d^{'}=\left[\left(\frac{c\cdot\varepsilon_2}{\sqrt{\varepsilon_1^2\text{-}\varepsilon_2^2}}\right)^{'}c\cdot\varepsilon_1\right]\Omega_1-\left[\left(\frac{c\cdot\varepsilon_1}{\sqrt{\varepsilon_1^2\text{-}\varepsilon_2^2}}\right)^{'}c\cdot\varepsilon_2\right]\Omega_2=0. \tag{5.14}$$

Since the $\{\Omega_1,\Omega_2\}$ is linear independent, thus in equation (5.14) we have $\left(\frac{-\varepsilon_2}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}}\right)^{'}+c\cdot\varepsilon_1=0$ and $-\left(\frac{\varepsilon_1}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}}\right)^{'}+c\cdot\varepsilon_2=0$, where $\left(\varepsilon_1^2-\varepsilon_2^2\right)\neq 0$ and $c\neq 0$, $c\in\mathbb{R}$.

5.1 Constant Precession of the Curve

A unit constant precession of the curve is defined by the property that its Bishop Darboux vector revolves about fixed line is a space with angle and constant speed. A curve of constant precession is characterized by having

$$\varepsilon_1(s)=\varpi\sin(\mu(s))$$
$$\varepsilon_2(s)=\varpi\cos(\mu(s))$$

where $\varpi>0$ and μ are constant [11].

Recall that centroid axis of the Bishop frame is given by $d=-\frac{\varepsilon_2}{\sqrt{\varepsilon_1^2-\varepsilon_2^2}}\Omega_1+\frac{\varepsilon_1}{\sqrt{\varepsilon_1^2-\varepsilon_2^2}}\Omega_2+cB$ and

$$d=\frac{\omega}{\|\omega\|}+cB \tag{6.1}$$

where $\omega = -\varepsilon_1 \Omega_2$ from equation (6.1)

$$\|\omega\| d = \omega + \|\omega\| cB$$

by taking $\varpi = \|\omega\| = \sqrt{\varepsilon_1^2 - \varepsilon_2^2}$, $\varpi d = \delta$ and $\varpi c = \xi$, we have

$$\delta = \omega + \xi B.$$

If we take $\|\omega\|$ =constant, Darboux helix is constant precession.

Theorem 6.1 Let α is a timelike curve in $\mathbb{R}_1^3$. We assume that $\dfrac{\varepsilon_2}{\varepsilon_1}$ is not constant, where ε_1 and ε_2 Bishop curvature of α. Then α is general helix if and only if α is Darboux helix.

Proof: We assume that α is a general helix. From theorem 5.2 we can easy write $\sigma(s) = \dfrac{\varepsilon_1^2}{(\varepsilon_1^2 - \varepsilon_2^2)^{3/2}} (\frac{\varepsilon_2}{\varepsilon_1})'$, similarly if the curve α is Darboux helix from theorem 5.2

$$\sigma^*(s) = \frac{\varepsilon_1^2}{(\varepsilon_1^2 - \varepsilon_2^2)^{3/2}} (\frac{\varepsilon_2}{\varepsilon_1})' = \text{constant}.$$

Consequently, we have $\sigma(s) \cdot \sigma^*(s) = \text{constant}$

$$\sigma(s) = \text{constant} \Leftrightarrow \sigma^*(s) = \text{constant}.$$

6 Example

In this section,we illustrate one example of Frenet frame and new spherical images in $\mathbb{R}_1^3$.

Example 7.1: Next, let us consider the following unit speed timelike curve $\alpha(s)$ of $\mathbb{R}_1^3$ by $\alpha = \alpha(s) = (\sqrt{2}s, \ln(\sec h(s)), \arctan(\sinh(s))$. It is rendered in figure 1.

And this curves's curvature and torsion functions are expressed as in $\mathbb{R}_1^3$

$\{\ \kappa(s) = \sec h(s), \quad \tau(s) = -\sqrt{2} \sec h(s)$

The Serret-Frenet frame of the $\alpha = \alpha(s)$ may be written by the aid Mathematical program as follows

$$\begin{aligned}
T &= (\sqrt{2},\ -\tanh(s),\ \sec h(s)), \\
N &= (0,\ -\text{sech(s)},\ -\tanh(s)), \\
B &= (1,\ \text{-}\sqrt{2}\tanh(s),\ \sqrt{2}\sec h(s)), \\
\theta(s) &=_0^s \sec h(s) ds = \arctan(\sinh(s))
\end{aligned}$$

Using transformation matrix equation (2.6) we get $\{\Omega_1, \Omega_2, B\}$ type-2 Bishop frame of $\alpha = \alpha(s)$.

$$\Omega_1 = (-\sqrt{2}\sinh(\arctan(\sinh(\mathrm{s}))),\ \frac{\sinh(\arctan(\sinh(s)))\sinh(s) - \cosh(\arctan(\sinh(s)))}{\cosh(s)},$$
$$-\frac{\sinh(\arctan(\sinh(s))) + \cosh(\arctan(\sinh(s)))\sinh(s)}{\cosh(s)}),$$

$$\Omega_2 = (-\sqrt{2}\cosh(\arctan(\sinh(s))),\ \frac{\cosh(\arctan(\sinh(s)))\sinh(s) - \sinh(\arctan(\sinh(s)))}{\cosh(s)},$$
$$-\frac{\cosh(\arctan(\sinh(s))) + \sinh(\arctan(\sinh(s)))\sinh(s)}{\cosh(s)}),$$

$$B = (1,\ -\sqrt{2}\tanh(s),\ \sqrt{2}\sec h(s))$$

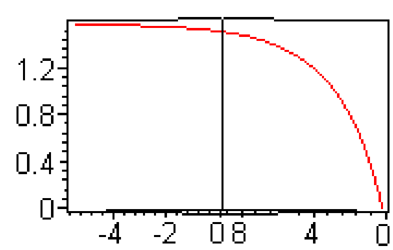

Figure 1

Darboux vector see figure 2 , Darboux helix is colored with red for $\mu = 1$ and colored with navy $\mu = -1$ see figure 3

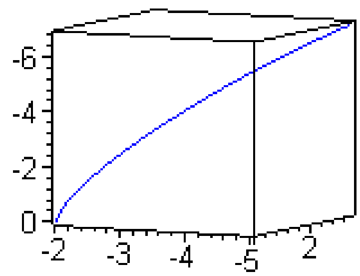

Figure 2

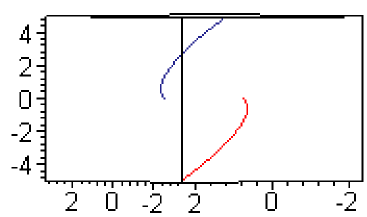

Figure 3

Received January 27, 2018; Accepted March 02, 2018

References

[1] L.R. Bishop: There is More Than One Way to Frame a Curve, *Amer. Math. Monthly*, **82** (12) 246-251 (1975).

[2] K.M.Bitar and S.J. Chang: Vacuum Tunneling of Gauge Theory in Minkowski Space, *Phys. Rev. D,* **17** (12) 486-497 (1978).

[3] W. Blaschke, in:Vorlesungen Über Differential Geometry I, Verlag von Julieus Springer, Berlin, 1930.

[4] B. Bükcü and M.K. Karacan: Bishop Frame of the Spacelike Curve with a Spacelike Principal Normal in Minkowski 3-Space, *Commun. Fac. Sci. Univ. Ank.* Ser, **57** 13-22 (2008).

[5] B. Bükcü and M.K. Karacan: The Bishop Darboux Rotation Axis of the Spacelike Curve in Minkowski 3- Space. *Ege University J.F.S*, **3** (1) 1-5 (2007).

[6] B. Bükcü and M.K. Karacan: An Alternative Moving Frame for Tubular Surface Around The Spacelike Curve With a Spacelike Binormal in Minkowski 3-Space, *Math. Morav*, **11** 73-80 (2007).

[7] C. Deffayet: Cosmology on a Brane in Minkowski Bulk. *Physics Letters B,* **502** 199–208 (2001).

[8] G. Dvali, G. Gabadadze and M. Porati: 4D Gravity on a Brane in 5D Minkowski Space, *Phys. Lett. B*, **485** 208-214 (2000).

[9] R.T. Farouki: Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable (Geometry and Computing), Springer, Newyourk, 2008.

[10] F. Gokçelik, Z. Bozkurt, I. Gok, F.N. Ekmekci and Y. Yaylı: Parallel Transport Frame in 4-dimensional Euclidean Space E^4, *Caspian Journal of Mathematical Sciences*, **3** (1) 91-102 (2014).

[11] I.P. Ivanov: Minkowski Space Structure of The Higgs Potential in 2HDM, *Phys. Rev. D*, **75** 1-20 (2007).

[12] S. Izumiya and N. Takeuchi: New Special Curves and Developable Surfaces, *Turkish J. Math*, **28** 531-537 (2004).

[13] L.Kula and Y. Yaylı: On Slant Helix and Its Spherical Indicatrix, *Appl. Math. Comput*, **169** (1) 600-607 (2005).

[14] S. Li: Full-view Spherical Image Camera, *in IEEE Int. Conf. Pattern Recognition*, 386–390 (2006).

[15] S. Li, M. Nakona and N. Chiba: Acquisition of Spherical Image by Fish-eye Conversion Lens, V*irtual Reality, Proceedings*, IEEE 235-236 (2004).

[16] H. Lindblad and I. Rodnianski: The Global Stability of Minkowski Space-time in Harmonic Gauge, *Annals of Mathematics*, **171** 1401–1477 (2010).

[17] H. Liu and D. Pei: Singularities of a Space Curve According to The Relatively Parallel Adapted Frame and Its Visualization, *Mathematical Problems in Engineering*, **2013** 1-12 (2013).

[18] H. Liu and D. Pei: Cusps of Bishop Spherical Indicatrixes and Their Visualizations, *Mathematical Problems in Engineering*, **2013** 1-11 (2013).

[19] J. Oprea: Differential Geometry and Its Applications, Mathematical Association of America Textbooks, MAA, Washington, 2007.

[20] E. Özyılmaz: Classical Differential Geometry of Curves According to Type-2 Bishop Frame, *Mathematical and Computational Applications*, **16** (4) 858-867 (2011).

[21] I. E. Segal: Mathematical Cosmology and Extragalactic Astronomy, Academic Press, Inc, New York, 1976.

[22] S. Yılmaz and M. Turgut: A New Version of Bishop Frame and An Application to Spherical Images, *Journal of Mathematical Analysis and Applications*, **371** 764-776 (2010).