Article

An Elementary Proof of the Kauers-Schneider's Identity

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Abstract

Kauers-Schneider employs the Wilf-Zeilberger (WZ) algorithm to deduce an identity between the harmonic and Stirling numbers. Here we show that this identity can be deduced without the application of the WZ process.

Keywords: Stirling number, harmonic numbers, convolution formula.

1. Introduction

Kauers-Schneider [1] use the Wilf-Zeilberger (WZ) algorithm [2-5] to obtain the identity:

$$\frac{(n+1)(n+2)}{2} S_m^{(n+2)} = \sum_{k=1}^{m-1} (-1)^{m-k+1} (m-k)! \binom{m}{k-1} H_{m-k}\, S_{k-1}^{(n)}, \quad m \geq 2, \quad n = 0, \dots, m-2, \tag{1}$$

between the harmonic numbers $H_n \equiv \sum_{j=1}^{n} \frac{1}{j}$ [6-9] and the Stirling numbers of the first kind $S_r^{(k)}$ [6,8,10,11]. For $n = m-1, m, m+1, \dots$ the expression (1) gives $0 = 0$.

In Section 2 we employ a convolution formula [10] for $S_r^{(k)}$ to exhibit an elementary proof of (1) independent of the WZ technique.

2. The Kauers-Schneider's Relation

From [10] we have the following convolution identity for Stirling numbers of the first kind:

$$\binom{n+j}{n} S_m^{(n+j)} = \sum_{r=0}^{m} \binom{m}{r} S_{m-r}^{(j)}\, S_r^{(n)}, \tag{2}$$

where we can use $j = 2$ to obtain:

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$$\binom{n+2}{n} S_m^{(n+2)} = \sum_{r=0}^{m-2} \binom{m}{r} S_{m-r}^{(2)} S_r^{(n)}. \tag{3}$$

On the other hand, we know the expression [10]:

$$H_k = \frac{(-1)^{k+1}}{k!} S_{k+1}^{(2)} , \tag{4}$$

then (3) and (4) imply the identity:

$$\frac{(n+1)(n+2)}{2} S_m^{(n+2)} = \sum_{r=0}^{m-2} (-1)^{m-r} (m-r-1)! \binom{m}{r} H_{m-r-1} S_r^{(n)} ,$$

which is equivalent to (1), Q.E.D.

Our proof of the Kauers-Schneider formula is very simple, hence it shows that the WZ algorithm is not necessary.

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