On the Langer Transformation

J. Morales¹, G. Ovando¹ & J. López-Bonilla²

¹CBI-Área de Física Atómica Molecular Aplicada, UAM-Azcapotzalco, Av. San Pablo 180, Col. Reynosa-Tamaulipas CP 02200, CDMX, México
²ESIME-Zacatenco, Instituto Politécnico Nacional, México

Abstract
We exhibit a simple procedure to deduce the Langer and Bateman et al-Mavromatis transformations to map the Coulomb problem into the Morse and simple harmonic oscillators, respectively.

Keywords: Langer mapping, Morse potential, Coulomb interaction.

1. Introduction

The Schrödinger equation for the Coulomb potential is given by [1-3]:

\[
\frac{1}{2} \left[ \frac{d^2 R}{dr^2} - \frac{l(l+1)}{r^2} R \right] + \frac{\alpha}{r} R = \beta R, \quad l = 0, \ldots, n - 1,
\]

where:

\[
\alpha = \frac{Z q^2}{4\pi \varepsilon_0}, \quad \beta = \frac{Z^2 q^4}{32 \pi^2 \varepsilon_0^2 n^2}, \quad n \geq 1.
\]

Here we exhibit how changes (1) under the mapping:

\[
R = A g(u) \psi(u), \quad r = B f(u), \quad A, B \text{ are constants},
\]

that is, the independent variable \( r \) and the Coulomb radial wave function \( R \) are transformed trying to obtain a new Schrödinger-like equation for \( \psi \) in the variable \( u \). Hence the aim is find \( f \) and \( g \) implying another potential of physical interest associated to \( \psi \).

In Sec. 2 we show that our approach gives, in natural manner, the Langer transformation [4] which allows to relate [5] the Coulomb potential with the Morse interaction [1, 2, 6, 7] for the vibrational motion of a diatomic molecule. Besides, it is also possible to deduce the Bateman et al [8]-Mavromatis [9] mapping which has been investigated in the context of connecting (1) and oscillator systems.

* Correspondence: J. López-Bonilla, ESIME-Zacatenco-IPN, Edif. 5, Col. Lindavista CP 07738, CDMX, México
E-mail: jlopezb@ipn.mx
2. Mappings between Schrödinger equations

We apply (3) into (1) to obtain:

$$\frac{1}{2} \left[ \psi'' + \left( \frac{2 \frac{g''}{g} - \frac{f''}{f} \right) \psi' + \left( \frac{g''}{g} - \frac{f''}{f} \right) - \frac{l(l + 1)}{f^2} \right] \psi + \alpha B \frac{\alpha^2}{f} \psi = \beta B^2 f'^2 \psi, \quad (4)$$

such that $\psi' = \frac{d\psi}{du}$; then (4) may be a Schrödinger-like equation if into it we eliminate the coefficient of $\psi'$, thus:

$$\frac{f''}{f'} = 2 \frac{g'}{g} \quad \therefore \quad f' = c \ g^2, \quad c = \text{constant}, \quad (5)$$

and (4) acquires the structure:

$$\frac{1}{2} \left[ \psi'' + \left( \frac{g''}{g} - 2 \frac{g'^2}{g^2} - c^2 \frac{\frac{l(l + 1)}{g^2}}{f^2} \right) \psi \right] + c^2 \alpha B \frac{\alpha^4}{f} \psi = c^2 \beta B^2 g^4 \psi. \quad (6)$$

Now (6) offers several options, for example, into it we can make the coefficient of $-c^2 l(l + 1)\psi$ equals to one, that is:

$$f = g^2, \quad (7)$$

then (5) implies the Langer transformation [4]:

$$f = e^{cu}, \quad g = e^{\frac{c}{2}u}, \quad (8)$$

and (6) takes the form:

$$-\frac{1}{2} \psi'' + \frac{\alpha B c^2}{2} \left( \frac{2 \beta B}{\alpha} e^{2cu} - 2 e^{cu} \right) \psi = -\frac{c^2}{2} \frac{1}{(l + \frac{1}{2})^2} \psi, \quad (9)$$

which is very attractive because permits to introduce the Morse potential [1, 2, 6, 7] if we select the values $c = -\alpha$ and $\frac{2\beta B}{\alpha} = 1$ with the notation $D = \frac{abc^2}{2}$, then from (2):

$$B = \frac{4\pi e_0}{2q^2} \ n^2, \quad D = \frac{a^2}{2} \ n^2, \quad a > 0, \quad (10)$$

therefore (9) becomes the vibrational Schrödinger equation for the Morse interaction:

$$-\frac{1}{2} \psi'' + D(e^{-2au} - 2 e^{-au}) \psi = E \psi, \quad E = -\frac{a^2}{2} \frac{1}{(l + \frac{1}{2})^2}, \quad (11)$$
where $a$ is a range parameter (associated with the width of the potential well) and $D$ is the energy of dissociation (well-depth) [10, 11].

Finally, from (3), (8) and (10) we obtain [5]:

$$ R = A e^{-\frac{a}{2}u} \psi(u), \quad r = \frac{4\pi\varepsilon_0}{z_q^2} n^2 e^{-au}, \quad (12) $$

where, by normalization, we can employ the value $A^2 = \frac{z_q^2}{4\pi\varepsilon_0 n(l+\frac{1}{2})}$. Now we back to the equation (6) seeking an alternative to (7), for example, to make the coefficient of $c^2\alpha\beta\psi$ equals to one, that is:

$$ f = g^4, \quad (13) $$

then (5) gives the Bateman et al [8]-Mavromatis [9] transformation:

$$ f = \frac{c^2}{4} u^2, \quad g = \sqrt[4]{\frac{c}{2}u}, \quad (14) $$

and (6) adopts the form:

$$ -\frac{1}{2} \left[ \psi'' - \frac{(4l+1)(4l+3)}{4u^2} \psi \right] + \frac{c^4\beta B^2}{4} u^2 \psi = c^2\alpha B \psi, \quad (15) $$

being achieved the mapping of the hydrogen-like atom into the 3-dimensional simple harmonic oscillator with certain parameters.

It is clear that in (6) we can try different connections between $f$ and $g$, respecting the constraint (5), to deduce Schrödinger-like equations associated to several potentials.

References

6. P. Morse, Diatomic molecule according to wave mechanics. Vibrational levels, Phys. Rev. 34, No. 7 (1929) 57-64